* University of Split
Faculty of EE, ME and Naval Arch.
www.fesb.hr

** The Pennsylvania State University
Department of CS & E
www.cse.psu.edu

IWASEP 10, June 2 - 5, 2014, Dubrovnik, Croatia

Accurate eigenvalue decomposition of real symmetric arrowhead (AH) and diagonal-plus-rank-one (DPR1) matrices and applications

N. Jakovčević Stor *, I. Slapničar *, J. Barlow **

Summary

- Our algorithms are forward stable – given a matrix of floating-point numbers each eigenvalue and each component of each eigenvector are computed with error in few least significant digits in $O(n)$ operations.
- Algorithms are based on shift-and-invert technique.
- Only a single element of the inverse of the shifted matrix eventually needs to be computed with double the working precision.
- Each eigenpair is computed independently of the others.

Structure of inverses

$$\left(\begin{bmatrix} \times & & & & \\ & \times & & & \\ & & 0 & & \\ & & & \times & \\ & & & & \times \end{bmatrix} + \rho z z^T\right)^{-1} = \begin{bmatrix} \times & & \times & & \\ & \times & \times & & \\ \times & \times & \times & \times & \times \\ & & \times & \times & \\ & & \times & & \times \end{bmatrix}$$

$$\left(\begin{bmatrix} \times & & & & \\ & \times & & & \\ & & \times & & \\ & & & \times & \\ & & & & \times \end{bmatrix} + \rho z z^T\right)^{-1} = \begin{bmatrix} \times & & & & \\ & \times & & & \\ & & \times & & \\ & & & \times & \\ & & & & \times \end{bmatrix} + \gamma u u^T.$$

Introduction

Let

$$A = D + \rho z z^T$$

$$D = \operatorname{diag}(d_1, \ldots, d_n), \quad z = \begin{bmatrix} \zeta_1 & \cdots & \zeta_n \end{bmatrix}^T, \quad \rho \in \mathbb{R},$$

be a $n \times n$ real symmetric DPR1 matrix (irreducible and ordered).

Let $A = V \Lambda V^T$ be the EVD of A. The interlacing implies

$$\lambda_1 > d_1 > \lambda_2 > d_2 > \cdots > d_{n-1} > \lambda_n > d_n,$$

The eigenvalues of A are the zeros of the function

$$\varphi_A(\lambda) = 1 + \rho \sum_{i=1}^{n} \frac{\zeta_i^2}{d_i - \lambda} = 1 + \rho z^T (D - \lambda I)^{-1} z = 0.$$

The corresponding eigenvectors are equal to

$$v_i = \frac{x_i}{\|x_i\|_2}, \quad x_i = (D - \lambda_i I)^{-1} z, \quad i = 1, \ldots, n. \qquad (1)$$

Problem: λ_i is not accurate $\Longrightarrow$ v_i may not be orthogonal.

Main idea

Let d_i be closest to λ. Then (interlacing)

$$\lambda = \lambda_i \quad \text{or} \quad \lambda = \lambda_{i+1}.$$

Let $A_i = A - d_i I$ be the shifted matrix,

$$A_i \equiv \begin{bmatrix} D_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_2 \end{bmatrix} + \rho \begin{bmatrix} z_1 \\ \zeta_i \\ z_2 \end{bmatrix} \begin{bmatrix} z_1^T & \zeta_i & z_2^T \end{bmatrix},$$

Then

$$A_i^{-1} = \begin{bmatrix} D_1^{-1} & w_1 & 0 \\ w_1^T & b & w_2^T \\ 0 & w_2 & D_2^{-1} \end{bmatrix},$$

$$w_1 = -D_1^{-1} z_1 \frac{1}{\zeta_i}, \quad w_2 = -D_2^{-1} z_2 \frac{1}{\zeta_i},$$

$$b = \frac{1}{\zeta_i^2} \left(\frac{1}{\rho} + z_1^T D_1^{-1} z_1 + z_2^T D_2^{-1} z_2 \right).$$

If λ is closest to d_i, then $\mu = \lambda - d_i$ is the eigenvalue of A_i closest to zero. Then $1/|\mu| = \left\|A_i^{-1}\right\|_2$, and $\nu = 1/\mu$ is computed accurately according to the standard perturbation theory (by bisection). Finally, v is computed by applying (1) to A_i.

Example

Let

$$A = D + z z^T$$

where

$$D = \operatorname{diag}(10^{10}, 5, 4 \cdot 10^{-3}, 0, -4 \cdot 10^{-3}, -5),$$

$$z = \begin{bmatrix} 10^{10} & 1 & 1 & 10^{-7} & 1 & 1 \end{bmatrix}^T.$$

The eigenvalues computed by Matlab routine eig, LAPACK routine $dlaed9.f$, our algorithm $dpr1eig$ and Mathematica with 100 digits precision (properly rounded to 16 decimal digits), are, respectively:

$\lambda_{(eig)}$	$\lambda_{(dlaed9)}$	$\lambda_{(dpr1eig, Math)}$
$1.000000000100000 \cdot 10^{20}$	$1.000000000100000 \cdot 10^{20}$	$1.000000000100000 \cdot 10^{20}$
5.000000000099998	5.000000000100000	5.000000000100000
$4.000000099999499 \cdot 10^{-3}$	$4.000000100000001 \cdot 10^{-3}$	$4.000000100000001 \cdot 10^{-3}$
$1.665334536937735 \cdot 10^{-16}$	$1.000000023272195 \cdot 10^{-24}$	$9.99999999899999(7, 9) \cdot 10^{-25}$
0	$-3.999999900000001 \cdot 10^{-3}$	$-3.999999900000001 \cdot 10^{-3}$
-25.00000000150000	-4.999999999900000	-4.999999999900000

Details

- All elements of A_i^{-1} are computed with high relative accuracy except possibly b – if b is inaccurate, it needs to be computed in double the working precision.
- For example, if all components of z are of the same order of magnitude, double precision is not needed.
- If λ is not closest to d_i (within some tolerance), we shift between λ and the eigenvalue on the other side of d_i – inverse is a DPR1 matrix.
- For only one λ, it can happen that the computation $\lambda = \mu + d_i$ is inaccurate – in this case λ is recomputed from the inverse of the original (unshifted) matrix (again DPR1 matrix).
- Implementation of the double the working precision
  - * Matlab command sym with parameter $'f'$,
  - * extended precision routines
  - * Intel FORTRAN compiler $ifort$

AH matrices

Let

$$A = \begin{bmatrix} D & z \\ z^T & \alpha \end{bmatrix},$$

where D is diagonal, z is a vector, and α is a scalar, be a $n \times n$ real symmetric AH matrix (irreducible and ordered).
The eigenvalues of A are the zeros of

$$f(\lambda) = \alpha - \lambda - \sum_{i=1}^{n-1} \frac{\zeta_i^2}{d_i - \lambda} = \alpha - \lambda - z^T (D - \lambda I)^{-1} z,$$

and the corresponding eigenvectors are given by

$$v_i = \frac{x_i}{\|x_i\|_2}, \quad x_i = \begin{bmatrix} (D - \lambda_i I)^{-1} z \\ -1 \end{bmatrix}, \quad i = 1, \ldots, n.$$

The algorithm and the results are similar as for DPR1 matrices.

Applications

- Our results extend to Hermitian case.
- The method can be used as a part of divide-and-conquer method for real symmetric tridiagonal matrices.
- For computing SVD of a triangular arrowhead matrix.
- For computing the zeros of the polynomials with distinct real roots.