Physics Letters B 755 (2016) 452–455

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Generalization of the model-independent Laurent–Pietarinen single-channel pole-extraction formalism to multiple channels

Alfred Švarc [a,*], Mirza Hadžimehmedović [b], Hedim Osmanović [b], Jugoslav Stahov [b], Lothar Tiator [c], Ron L. Workman [d]

[a] Rudjer Bošković Institute, Bijenička cesta 54, P.O. Box 180, 10002 Zagreb, Croatia
[b] University of Tuzla, Faculty of Science, Univerzitetska 4, 75000 Tuzla, Bosnia and Herzegovina
[c] Institut für Kernphyik, Universität Mainz, D-55099 Mainz, Germany
[d] Data Analysis Center at the Institute for Nuclear Studies, Department of Physics, The George Washington University, Washington, DC 20052, United States

ARTICLE INFO

Article history:
Received 29 December 2015
Received in revised form 3 February 2016
Accepted 24 February 2016
Available online 2 March 2016
Editor: J.-P. Blaizot

Keywords:
Partial wave analysis
Pole parameters
Model independent extraction

ABSTRACT

A method to extract resonance pole information from single-channel partial-wave amplitudes based on a Laurent (Mittag-Leffler) expansion and conformal mapping techniques has recently been developed. This method has been applied to a number of reactions and provides a model-independent extraction procedure which is particularly useful in cases where a set of amplitudes is available only at discrete energies. This method has been generalized and applied to the case of a multi-channel fit, where several sets of amplitudes are analyzed simultaneously. The importance of unitarity constraints is discussed. The final result provides a powerful, model-independent tool for analyzing partial-wave amplitudes of coupled or connected channels based entirely on the concepts of analyticity and unitarity.

The Particle Data Group (PDG) [1] has begun to include and emphasize the importance of pole-related quantities, de-emphasizing and eliminating many Breit–Wigner parameters, as the link between experiment and QCD. As a result, the analytic structure of theoretical and experimental partial-wave amplitudes in the complex energy plane has become increasingly important. A common approach involves the construction and solution of elaborate theoretical models, with free parameters fitted to available sets of experimental data. These can then be analytically continued into the complex energy plane. As a typical model is extremely complex and very difficult to solve, simpler single-channel pole extraction methods such as the speed plot [2], time delay [3], the N/D method [4], regularization procedures [5], and Pade approximants [6] have been used. However, success has been limited. As a step forward, a simple but quite reliable, model-independent single-channel pole-extraction formalism has been constructed, based entirely on principles of analyticity and unitarity. This method was named the Laurent+Pietarinen (L+P) expansion [7], and is based on an early application of these principles in the analysis of pion-nucleon scattering data [8–11].

In spite of the fact that this single-channel L+P method is now generally applicable, extensively used in a wide array of problems [12–14], and already recognized by PDG as a confident tool for extracting pole positions of most baryon resonances [1], all applications in which one pole couples to several correlated quantities are still beyond its reach. For example, correlated multipoles in π and η photoproduction, and partial wave amplitudes in coupled-channel models can only be treated in a sequence of independent single-channel procedures, missing the constraint that poles in all such situations must be the same. Also, in some cases, all existing poles may not be recognized in each individual process, and that in particular happens if a resonance coupling to a particular channel is weak. Thus, the main purpose of this paper is to create a new method which enables the treatment of all connected channels simultaneously. We have generalized the existing single-channel L+P formalism (SC L+P) to the multi-channel case (MC L+P) in such a way that pole positions are unique, but with differing residua which are to be related to branching fractions. This also allows the analysis of photo- and electro-production in which a single pole contributes to two or three multipoles. Just as in the single-channel L+P method, the most important application of the method would be the analysis of partial wave data (discreet quantities obtained directly from experiment, with very few stabilizing theoretical assumptions), rather than treating the partial wave am-

* Corresponding author.
E-mail address: alfred.svarc@irb.hr (A. Švarc).

http://dx.doi.org/10.1016/j.physletb.2016.02.058

plitudes which are coming from theoretical models. Therefore, this method as such represents the first model-independent way to treat multi-channel experimental data directly, and is extremely important for precise and rapid analysis of new ongoing experimental programs.

The driving concept behind the single-channel L+P approach was to replace solving an elaborate theoretical model and analytically continuing its solution into the full complex energy plane, with a local power-series representation of partial wave amplitudes given on the real energy axis. In such a way, the global complexity of a model is replaced by much simpler model-independent expansion limited to the regions near the real energy axis which is sufficient to obtain poles and their residues. Formally, the introduced L+P method was based on the Mittag-Leffler expansion[1] of partial wave amplitudes near the real energy axis, representing the regular, but unknown, background term by a conformal-mapping-generated, rapidly converging power series called a Pietarinen expansion.[2] In practice we have represented the regular background part with three Pietarinen expansion series, and fitted all free parameters in our approach to the chosen channel input. The first Pietarinen expansion with branch-point x_P was restricted to an unphysical energy range and represented all left-hand cut contributions, and next two Pietarinen expansions described background in the physical range with branch-points x_Q and x_R defined by the analytic properties of the analyzed partial wave. A second branch-point was usually fixed to the elastic channel branch-point, and the third one was either fixed to the dominant channel threshold value or left free. Thus, solely on the basis of general physical assumptions about analytic properties of the fitted process (number of poles and number and position of conformal mapping branch-points) the pole parameters in the complex energy plane are obtained. In such a way, the simplest analytic function with a set of poles and branch-points which is fitting the input is actually constructed. This method is equally applicable to both theoretical and experimental input,[3] and represents the first reliable procedure to extract pole positions from experimental data, with minimal model bias.

The generalization of L+P method to MC L+P is performed in the following way: i) we have made separate Laurent expansions for each channel (coupled quantity); ii) we have kept pole positions fixed for all channels (quantities), iii) we have left all residua and all Pietarinen coefficients free; iv) we have chosen the branch-points exactly as we would for the single-channel model; v) we have generalized the single-channel discrepancy function D^a_{dp} (see Eq. (5) in Ref. [13]) which quantifies the deviation of the fitted function from employed input to a multi-channel quantity D_{dp} by summing up all single-channel contributions, and vi) the minimization is performed for all channels of the input in order to obtain the final solution.

The final model can be summarized by the following set of formulae for k resonances:

$$T^a(W) = \sum_{i=1}^{k} \frac{x_i^a + \imath\ y_i^a}{W_i - W} + \sum_{l=0}^{L^a} c_l^a\, X^a(W)^l + \sum_{m=0}^{M^a} d_m^a\, Y^a(W)^m + \sum_{n=0}^{N^a} e_n^a\, Z^a(W)^n$$

$$X^a(W) = \frac{\alpha^a - \sqrt{x_P^a - W}}{\alpha^a + \sqrt{x_P^a - W}}; \quad Y^a(W) = \frac{\beta^a - \sqrt{x_Q^a - W}}{\beta^a + \sqrt{x_Q^a - W}};$$

$$Z^a(W) = \frac{\gamma^a - \sqrt{x_R^a - W}}{\gamma^a + \sqrt{x_R^a - W}}$$

$$D_{dp} = \sum_a^{all} D^a_{dp}$$

$$D^a_{dp} = \frac{1}{2\,N_{data}} \sum_{i=1}^{N_{data}} \left\{ \left[\frac{\mathrm{Re}\,T^a(W_i) - \mathrm{Re}\,T^a_{exp}(W_i)}{Err^{\mathrm{Re}}_{i,a}} \right]^2 + \left[\frac{\mathrm{Im}\,T^a(W_i) - \mathrm{Im}\,T^a_{exp}(W_i)}{Err^{\mathrm{Im}}_{i,a}} \right]^2 \right\} + \mathcal{P}^a + \mathcal{U}^a$$

$\mathcal{P}^a$ and $\mathcal{U}^a$ … Pietarinen and unitarity penalty functions

$Err^{\mathrm{Re,\,Im}}_{i,a}$ … minimization error of real and imaginary part respectively,

a … correlated quantity index ($\pi N \to \pi N$, $\pi N \to \eta N$, $E_{l_\pm}$, $M_{l_\pm}$ …)

$L^a,\ M^a,\ N^a \ldots \in \mathbb{N}$ number of Pietarinen coefficients in channel a

$W_i,\ W \in \mathbb{C}$

$x_i^a,\ y_i^a,\ c_l^a,\ d_m^a,\ e_n^a,\ \alpha^a,\ \beta^a,\ \gamma^a \ldots \in \mathbb{R}$

Here $x_i^a + \imath\ y_i^a$ are the channel residua and $W_i = M_i - \imath\ \frac{\Gamma_i}{2}$ are the pole positions of resonances "i". The Pietarinen expansions formalize the most general form of functions having a branch-point at the Pietarinen-expansion parameters, and in this paper we use three Pietarinen expansions with expansion parameters x_P^a, x_Q^a, and x_R^a to represent the full analytic structure of the non-resonant background. The first coefficient x_P^a is restricted to the unphysical range and represents all left hand cuts, including a circular cut. The second parameter x_Q^a is usually fixed to the elastic channel branch-point, and the third branch-point x_R^a represents all inelastic-channel openings in the physical domain. If inelastic channels consist of particles having weak or electromagnetic decays ($\pi N \to \eta N$, $\pi N \to K\Lambda$, etc.) the coefficient is real, and if it contains quasi-two-body final states (such as $\pi N \to \pi\Delta$, $\pi N \to \rho N$, etc.) it becomes complex. In principle, one should have a Pietarinen expansion for each branch-point, producing at least 4–5 Pietarinen expansions in the physical range. However, to avoid over-parametrizing our model, we have retained only a single Pietarinen expansion in the physical range, effectively representing all physical branch-points. The first and third branch points x_P^a and x_R^a are usually free, while the second one x_Q^a is fixed to the value of the πN elastic threshold. The initial value for the third branch-point x_R^a is chosen close to the branch-point dominant for the analyzed partial wave. This point varies depending on the partial wave. Coefficients α^a, β^a, and γ^a are Pietarinen-expansion strength parameters. The inclusion of additional Pietarinen expansions, while legitimate, has not yet been implemented.

In the MC L+P formalism, the Pietarinen single-channel penalty function $\mathcal{P}$ is generalized to:

[1] Mittag-Leffler expansion [15] is the generalization of a Laurent expansion to a more-than-one pole situation. For simplicity, we will simply refer to this as a Laurent expansion.

[2] A conformal mapping expansion of this particular type was introduced by Ciulli and Fisher [8,9], was described in detail and used in pion-nucleon scattering by Esco Pietarinen [10,11]. The procedure was denoted as a Pietarinen expansion by G. Höhler in [16].

[3] Observe that fitting partial wave data coming from experiment is even more favorable.

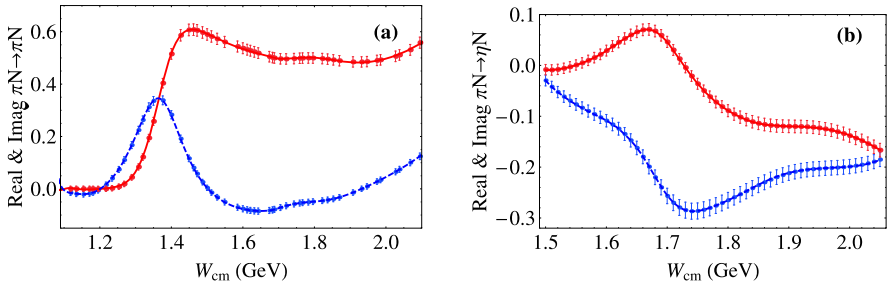

Fig. 1. (Color online.) The SC L+P result for BG2011-2 [17,18] $\pi N \to \pi N$ and $\pi N \to \eta N$ P_{11} PW amplitudes is shown in (a) and (b) respectively. Blue and red full and dashed lines give the real and imaginary parts respectively.

$$\mathcal{P}^a = \lambda_c^a \sum_{l=0}^{L^a} (c_l^a)^2 l^3 + \lambda_d^a \sum_{m=0}^{M^a} (d_m^a)^2 m^3 + \lambda_e^a \sum_{n=0}^{N^a} (e_n^a)^2 n^3$$

The unitarity single-channel penalty function $\mathcal{U}$ is also generalized. In the SC L+P formalism, the unitarity penalty function $\mathcal{U}$ ensures that the reaction is unitary from the elastic threshold up to the first inelastic threshold. For all energies lower than x_R we introduce a unitarity penalty factor $\mathcal{U}^{el}$ as:

$$\mathcal{U}^{el} = \sum_{j=1}^{N_{pts}^{el}} (1 - S^{el}(W_j) S^{el}(W_j)^{\dagger})^2$$

where the summation is over all energy points from threshold ($j = 1$) up to first inelastic threshold at $W = x_R$ ($j = N_{pts}^{el}$). In the MC L+P, we enforce this condition only for the elastic reaction, and only up to the energy where the first inelastic threshold opens. We have not yet attempted to impose many-body unitarity for energies above $W = x_R$.

In the MC L+P generalization, we have also included subthreshold unitarity constraints for inelastic channels. Namely, we have:

1. subthreshold unitarity for the elastic channel

   $\mathrm{Im}\, T^{el}(W) = 0$ for $W < W_{el-thr}$, and

2. subthreshold unitarity for inelastic channels

   $\mathrm{Im}\, T^{inel}(W) = 0$ for $W < W_{inel-thr}$.

As we never fit below the elastic threshold, the final MC unitarity penalty function reduces to:

$$\mathcal{U}^a = \Lambda^{el} \cdot \mathcal{U}^{el} + \Lambda^{inel} \cdot \sum_{j=1}^{N_{pts}^{el}} \mathrm{Im}\; T^{a \neq el}(W_j),$$

where the summation also runs over all energy points from threshold ($j = 1$) to the a-channel inelastic threshold at $W = x_R$ ($j = N_{pts}^{el}$).

Observe that the additional inelastic unitarity condition can be used only if our input quantities (partial waves or multipoles) are also unitarized. Adding subthreshold unitarity, when possible, improved our model significantly. Introducing this constraint through a penalty function enabled us to study the importance of all unitarity restrictions explicitly by varying penalty function strength parameters Λ^{el} and Λ^{inel}.

Finally, we minimize the function D_{dp} where $T^a(W_i)$ is our MC L+P function and $T_{exp}^a(W_i)$ is our input function which can either originate in the form of a theoretical model, or be generated through a single-energy partial wave analysis, possibly with added theoretical constraints such as fixed-t analyticity.

Table 1
Two independent SC L+P analyses of πN elastic and $\pi N \to \eta N$ BG 2011-2 amplitudes. M_i and Γ_i are the resonance position and width; $|a_i^a|$ and θ_i^a give the residue in terms of modulus and phase.

Fitted channel	Resonance name	M_i	Γ_i	$\lvert a_i^a \rvert$	θ_i^a	D_{dp}^a
πN elastic	N(1440)1/2+	1368	193	49	−82	0.004
two poles	N(1880)1/2+	1857	321	15	179	
$\pi N \to \eta N$	N(1710)1/2+	1686	204	19	−27	0.002
two poles	N(1880)1/2+	1861	252	20	−95	

Comparing single- and multi-channel fits

We have tested the validity of MC L+P model, and analyzed which new insights can be gained in a comparison of the SC L+P and MC L+P analyses by applying it to Bonn–Gatchina BG 2011-2 P_{11} πN elastic and $\pi N \to \eta N$ amplitudes for which resonance pole parameters are provided through the analytic continuation of the Bonn–Gatchina model and published in [17,18]. We first show the problems which occur when two independent single-channel L+P analyses are performed on πN elastic and $\pi N \to \eta N$ amplitudes, then demonstrate how the MC L+P approach solves these problems. Finally, we confirm the validity and precision of our method by comparing our results with known and published pole parameter values. We do not expect to exactly reproduce the published pole results, as both the original SC L+P and new MC L+P methods are based on approximations to the analytic structure of the true non-resonant background functions. However, we do expect good agreement within the uncertainties of the various extraction methods.

We have first made two independent SC L+P analyses of πN elastic and $\pi N \to \eta N$ amplitudes. Results are shown in Fig. 1 and Table 1.

Here we see that the SC L+P fit for both reactions is very good with two poles only, but these poles are not identical. The πN elastic reaction can be fitted with the N(1440)1/2+ and N(1880)1/2+ while $\pi N \to \eta N$ can be fitted with the N(1710)1/2+ and N(1880)1/2+. In SC L+P fits there is no indication that the N(1710)1/2+ is needed in πN elastic scattering, nor the subthreshold N(1440)1/2+ in $\pi N \to \eta N$. Further, the presence of a fourth N(2100)1/2+ resonance is not indicated in either channel. Finally, numerical values for the second obtained resonance N(1880)1/2+ are different for each reaction.

In a coupled-channel analysis this is not permissible. If a resonance exists in a certain partial wave in one reaction, it should exist in all reactions which couple to this partial wave. Therefore, both reactions should be fitted with at least three resonances with consistent pole positions. This is achieved in the proposed MC L+P approach.

In preliminary MC L+P fits to the BG 2011-2 amplitudes [17], we began by using three poles. This corresponded to the number of poles reported in reference [17]. To our surprise, the fit failed

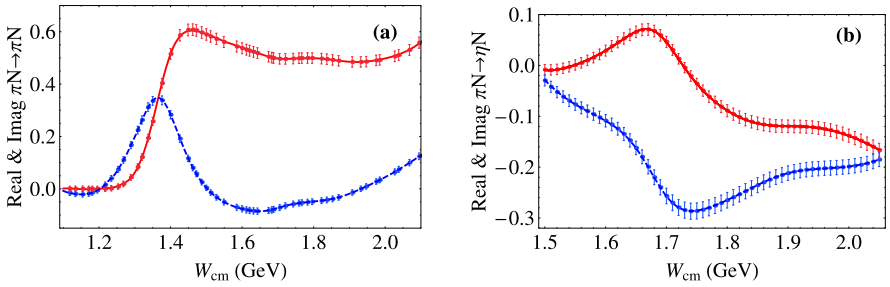

Fig. 2. (Color online.) The MC L+P result for BG2011-2 [17,18] $\pi N \to \pi N$ and $\pi N \to \eta N$ P_{11} PW amplitudes is shown in (a) and (b) respectively. Blue and red full and dashed lines give the real and imaginary parts respectively.

Table 2
Comparison of published theoretical BG2011-2 [17,18] pole parameters with MC L+P results. M_i and Γ_i are the resonance position and width; $|a_i^a|$ and θ_i^a give the residue in terms of modulus and phase.

Resonance name		PDG [1]	BG[17,18]	BG$^{\text{MC L+P}}$
N(1440)1/2+	M_1	1350–1380	1370(4) [17]	1368(3)
	Γ_1	160–220	190(7)	191(3)
	$\lvert a\rvert_1^{\pi N}$	40–52	48(3)	49(2)
	$\Theta_1^{\pi N}$	75–100	−78(4)	−82(3)
	$\frac{2\lvert a\rvert_1^{\eta N}}{\Gamma}$	–	–	0.1(0.1)%
	$\Theta_1^{\eta N}$	–	–	22(20)
N(1710)1/2+	M_2	1670–1770	1687(17) [17]	1686(8)
	Γ_2	80–330	200(25)	153(24)
	$\lvert a\rvert_2^{\pi N}$	6–15	6(4)	2(1)
	$\Theta_2^{\pi N}$	120–193	120(70)	155(21)
	$\frac{2\lvert a\rvert_2^{\eta N}}{\Gamma}$	–	12(4)%	14(3)%
	$\Theta_2^{\eta N}$	–	0(45)	21(7)
N(1880)1/2+	M_3	1860(35)	1860(35)[17]	1875(9)
	Γ_3	250(70)	250(70)	232(15)
	$\lvert a\rvert_3^{\pi N}$	6(4)	6(4)	3(1)
	$\Theta_3^{\pi N}$	80(65)	80(65)	107(16)
	$\frac{2\lvert a\rvert_3^{\eta N}}{\Gamma}$	–	11(7)%	6(1)%
	$\Theta_3^{\eta N}$	–	−75(55)	−131(26)
N(2100)1/2+	M_4	2120(40)	2100 [18]	2171(24)
	Γ_4	240(80)	500	210(48)
	$\lvert a\rvert_4^{\pi N}$	14(7)	–	15(5)
	$\Theta_4^{\pi N}$	35(25)	–	−50(8)
	$\frac{2\lvert a\rvert_4^{\eta N}}{\Gamma}$	–	–	16(4)%
	$\Theta_4^{\eta N}$	–	–	−139(19)

to obtain a good result with three resonances only. A good fit required the existence of a fourth state. This result turns out to be in perfect agreement with the fact that the Bonn–Gatchina group indeed does consider a further pole (Re(pole), -2Imag(pole) at (2100, 500) MeV which slightly improves the stability of the fit in their model [18]. However, the new pole is poorly determined, and they neither claim its existence nor rule it out. In effect, the MC L+P fit not only included all expected resonances in fitting both reactions, it predicted a fourth state as well which finally turned out to be allowed in the original BG 2011-2 model.

Final results of MC L+P fit are shown in Fig. 2 and Table 2.

The results presented in this table confirm that both, pole positions and residua, generally lie within one standard deviation intervals when compared with the published results. As discussed above, better agreement cannot be expected. Surprisingly, the weak and poorly determined N(2100)1/2+ resonance from the Bonn–Gatchina model is, in the MC L+P fit, not only well and confidently reproduced, but also necessary.

Finally, let us finish the discussion with stressing that the model dependence of the L+P method itself is fairly small, but the quality of obtained result (how close the obtained values are to the actual pole parameters) depends on the quality of input. L+P method will always give the correct pole positions of the input function, but how well the analyzed input function reproduces the reality is beyond our control. If input function reproduces the experiment nicely, our pole parameters will be close to actual values, if not we may miss quite a lot.

As a conclusion, we state that the generalization of the L+P model to a multi-channel case, as described in this paper, provides a powerful but simple and precise model independent method to extract pole positions from coupled processes (coupled-channel models) and correlated quantities ($E_{l_\pm}$ and $M_{l_\pm}$ in photoproduction), and that this is the first method which can be directly used to extract pole positions from partial-wave amplitudes extracted from experimental data.

References

[1] K.A. Olive, et al., Particle Data Group, Chin. Phys. C 38 (2014) 090001, and http://pdg.lbl.gov/.
[2] G. Höhler, πN Newsletter 9 (1993) 1.
[3] N.G. Kelkar, M. Nowakowski, Phys. Rev. A 78 (2008) 012709, and references therein.
[4] G.F. Chew, S. Mandelstam, Phys. Rev. 119 (1960) 467.
[5] S. Ceci, J. Stahov, A. Švarc, S. Watson, B. Zauner, Phys. Rev. D 77 (2008) 116007.
[6] Pere Masjuan, Jacobo Ruiz de Elvira, Juan José Sanz-Cillero, Phys. Rev. D 90 (2014) 097901.
[7] A. Švarc, M. Hadžimehmedović, H. Osmanović, J. Stahov, L. Tiator, R.L. Workman, Phys. Rev. C 88 (2013) 035206.
[8] S. Ciulli, J. in Fischer, Nucl. Phys. 24 (1961) 465.
[9] I. Ciulli, S. Ciulli, J. Fisher, Nuovo Cimento 23 (1962) 1129.
[10] E. Pietarinen, Nuovo Cimento Soc. Ital. Fis. 12A (1972) 522.
[11] E. Pietarinen, Nucl. Phys. B 107 (1976) 21.
[12] A. Švarc, M. Hadžimehmedović, R. Omerović, H. Osmanović, J. Stahov, Phys. Rev. C 89 (2014) 45205.
[13] A. Švarc, M. Hadžimehmedović, H. Osmanović, J. Stahov, R.L. Workman, Phys. Rev. C 91 (2015) 015207.
[14] A. Švarc, M. Hadžimehmedović, H. Osmanović, J. Stahov, L. Tiator, R.L. Workman, Phys. Rev. C 89 (2014) 065208.
[15] Michiel Hazewinkel, Encyclopaedia of Mathematics, Vol. 6, 31. 8, Springer, 1990, pg. 251.
[16] G. Höhler, Pion Nucleon Scattering, Part 2, Landolt-Bornstein: Elastic and Charge Exchange Scattering of Elementary Particles, vol. 9b, Springer-Verlag, Berlin, 1983.
[17] A.V. Anisovich, R. Beck, E. Klempt, V.A. Nikonov, A.V. Sarantsev, U. Thoma, Eur. Phys. J. A 48 (15) (2012).
[18] A.V. Anisovich, E. Klempt, V.A. Nikonov, A.V. Sarantsev, U. Thoma, Eur. Phys. J. A 47 (2011) 27;
A.V. Anisovich, E. Klempt, V.A. Nikonov, A.V. Sarantsev, U. Thoma, Eur. Phys. J. A 47 (2011) 153.