# Hölder continuity of Oseledets splitting for semi-invertible operator cocycles

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# Crash course on Lyapunov exponents

Let $(X, \mathcal{B}, \mu)$ be a probability space and let $f : X \to X$ be an invertible transformation that preserves measure $\mu$. We recall that this means that $\mu(f^{-1}(B)) = \mu(B)$ for each $B \in \mathcal{B}$. We say that $\mu$ is *ergodic* if $f^{-1}(B) = B$ for $B \in \mathcal{B}$ implies that $\mu(B) \in \{0, 1\}$.

Let $M_d$ denote the set of all real matrices of order $d$. A *cocycle* is any measurable map $A : X \to M_d$.

**Example**

Let $X$ be a compact Riemannian manifold, $\mathcal{B}$ a Borel $\sigma$-algebra, $f : X \to X$ a diffeomorphism and $\mu$ any ergodic $f$-invariant measure. Then, the map $A : X \to M_d$ given by $A(x) = Df(x)$, $x \in X$ is the so-called *derivative cocycle*.

Let $A$ be a cocycle. We consider the product

$$A^{(n)}(x) := A(f^{n-1}(x)) \cdots A(f(x)) \cdot A(x),$$

for $n \in \mathbb{N}$ and $x \in X$. Can we say anything about the asymptotic behaviour of $\|A^{(n)}(x)\|$ when $n \to \infty$ for "typical" $x \in X$?

**Theorem (Furstenberg-Kesten, 1960)**

*If*

$$\int_X \log^+ \|A\| \, d\mu < \infty,$$

*then there exists $\lambda \in [-\infty, \infty)$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \log\|A^{(n)}(x)\| = \lambda,$$

*for $\mu$-a.e. $x \in X$.*

How about the asymptotic behaviour of $\|A^{(n)}(x)v\|$ for $v \in \mathbb{R}^d$?

## Theorem (Oseledets, 1968)

*Assume that $A$ is a cocycle with values in $GL_d$ and such that*

$$\log^+\|A\|,\ \log^+\|A^{-1}\| \in L^1(\mu).$$

*Then, there exist numbers (Lyapunov exponents of $\mathcal{A}$ w.r.t. $\mu$)*

*$\infty > \lambda_1 > \ldots > \lambda_k > -\infty$ and for $\mu$-a.e. $x \in X$ an decomposition*

$$\mathbb{R}^d = E_1(x) \oplus \ldots \oplus E_k(x)$$

*such that $A(x)E_i(x) = E_i(f(x))$ and*

$$\lim_{n\to\infty} \frac{1}{n} \log\|A^{(n)}(x)v\| = \lambda_i,$$

*for $v \in E_i(x) \setminus \{0\}$ and $i \in \{1, \ldots, k\}$.*

# Crash course on hyperbolicity

Let $X$ be a compact Riemannian manifold and let $f\colon X \to X$ be a diffeomorphism. An $f$-invariant set $\Lambda \subset X$ is *hyperbolic* if there exists $C > 0$ and $\lambda \in (0,1)$ and for every $x \in \Lambda$ an $Df$-invariant splitting

$$T_x X = E^s(x) \oplus E^u(x)$$

such that

$$\|Df^n(x)v\| \le C\lambda^n \|v\|, \quad \text{for } v \in E^s(x) \text{ and } n \in \mathbb{N},$$

$$\|Df^n(x)v\| \ge \frac{1}{C}\lambda^{-n} \|v\|, \quad \text{for } v \in E^u(x) \text{ and } n \in \mathbb{N}$$

and

$$\angle(E^s(x), E^u(x)) \ge \frac{1}{C}.$$

If $X$ is a hyperbolic set for $f$, we say that $f$ is *Anosov*.

**Example**

Let $X = \mathbb{R}^2/\mathbb{Z}^2$ and define $f \colon X \to X$ by

$$f((x_1, x_2) + \mathbb{Z}^2) = (2x_1 + x_2, x_1 + x_2) + \mathbb{Z}^2.$$

Then, $f$ is Anosov.

**Example**

Let $f$ be a diffeomorphism of $X$ and let $x \in X$ be a hyperbolic fixed point. Then, $\Lambda = \{x\}$ is hyperbolic.

In the continuous time case: geodesic flows on compact manifolds of negative curvature are Anosov.

An $f$-invariant set $\Lambda \subset M$ is *nonuniformly hyperbolic* if there exists $\lambda \in (0,1)$, a measurable function $C \colon \Lambda \to (0,\infty)$ and for every $x \in \Lambda$ an $Df$-invariant splitting

$$T_x X = E^s(x) \oplus E^u(x)$$

such that

$$\|Df^n(x)v\| \le C(x)\lambda^n \|v\|, \quad \text{for } v \in E^s(x) \text{ and } n \in \mathbb{N},$$

$$\|Df^n(x)v\| \ge \frac{1}{C(x)}\lambda^{-n}\|v\|, \quad \text{for } v \in E^u(x) \text{ and } n \in \mathbb{N},$$

$$\angle(E^s(x), E^u(x)) \ge \frac{1}{C(x)}$$

and

$$\lim_{n\to\pm\infty} \frac{1}{n} \log C(f^n(x)) = 0.$$

Concept of hyperbolicity/nonuniform hyperbolicity can be introduced for arbitrary cocycles.

**Theorem (Pesin, 1977)**

*Let $\mathcal{A}$ be a cocycle over $f$. If all Lyapunov exponents of $\mathcal{A}$ with respect to some $f$-ergodic invariant measure $\mu$ are nonzero, then $\mathcal{A}$ is nonuniformly hyperbolic on a set $\Lambda$ of full $\mu$-measure.*

# Uniform vs. nonuniform hyperbolicity I

There are no topological obstructions to nonuniform hyperbolicity as a global phenomenon!

**Theorem (Dolgopyat-Pesin, 2002)**

*Let $M$ be a compact smooth Riemannian manifold of dimension $\geq 2$. Then, there exists a $C^\infty$ volume-preserving diffeomorphims $f : M \to M$ which has nonzero Lyapunov exponents in almost every point.*

# Uniform vs. nonuniform hyperbolicity II

While the uniform hyperbolicity is robust under $C^1$-perturbations, nonuniform hyperbolicity is very far from this!

**Theorem (Bochi 2002, Bochi-Viana 2005)**

*Let $f$ be a volume preserving $C^1$-diffeomorphism of a smooth compact Riemannian manifold $M$ which is not Anosov. Then, for every $\varepsilon > 0$ there exists a volume preserving $C^1$-diffeomorphism $g$ of $M$ such that:*

1. $d_{C^1}(f, g) < \varepsilon$;
2. *all Lyapunov exponents of $g$ are zero.*

Open question: What if we replace $C^1$ with $C^r$ for $r > 1$?

# Semi-invertible Oseledets theorem

**Theorem (Froyland, Lloyd, Quas, 2010)**

*Assume that $A$ is a cocycle over $f$ with values in $M_d$ and such that*

$$\log^+\|A\| \in L^1(\mu).$$

*Then, there exists numbers $\infty > \lambda_1 > \ldots > \lambda_k \geq -\infty$ and for $\mu$-a.e. $x \in X$ an decomposition*

$$\mathbb{R}^d = E_1(x) \oplus \ldots \oplus E_k(x)$$

*such that $A(x)E_i(x) \subset E_i(f(x))$ and*

$$\lim_{n\to\infty} \frac{1}{n} \log\|A^{(n)}(x)v\| = \lambda_i, \quad \text{for } v \in E_i(x) \setminus \{0\} \text{ and } i = 1, \ldots, k.$$

Prior to the publication of the previous theorem, all versions of Oseledets theorem including infinite-dimensional versions required injectivity of operators. Subsequently, Froyland, Lloyd and Quas (DCDS, 2013), Gonzalez-Tokman and Quas (ETDS 2014, JMD 2015) established versions of the previous theorem for cocycles acting on Banach spaces. Semi-invertible cocycles arise naturally:

1. transfer operator cocycles;
2. markov chains in random enviroment.

# Regularity of Oseledets subspaces

**Theorem (Araujo, Bufetov, Filip, JLMS, 2016)**

*Assume that $f$ is a Lipschitz invertible map on a compact space $X$. Furthermore, let $A\colon X \to GL_d$ be Hölder continuous cocycle. Then, for each $\varepsilon > 0$ there exists a compact set $\Lambda \subset X$ of measure $1 - \varepsilon$ on which maps $x \mapsto E_i(x)$ are Hölder continuous for $i = 1, \ldots, k$.*

**Theorem (D., Froyland, ETDS, accepted)**

*Previous theorem holds for semi-invertible cocycles including the case of infinite-dimension.*