Map Projection Aspects

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Abstract

A projection aspect is usually defined in references as the relation to the so-called auxiliary surface. However, such surfaces do not usually exist in map projection theory, which raises the issue of defining projection aspects without reference to auxiliary surfaces. This paper explains how projection aspects can be defined in two ways which are not mutually exclusive. According to the first definition, the aspect is the position of a projection axis in relation to the axis of geographic parameterization of a sphere. The projection axis is the axis of the pseudogeographic parameterization of a sphere, based on which the basic equations of map projection are defined. The basic equations of map projection are selected according to agreement and/or custom. According to this definition, aspects can be normal, transverse or oblique. According to the second definition, an aspect is the representation of the area in the central part of a map, and can be polar, equatorial or oblique. Therefore, it is possible for a map projection to have a normal and polar aspect, but it can also have a normal and equatorial aspect. The second definition is not recommended for use, due to its ambiguity.

Key words: projection aspect, normal aspect, transverse aspect, oblique aspect, polar aspect, equatorial aspect

1. Introduction

The issue of map projection aspects was analysed by scientists a long time ago (e.g. Herz 1885, Hammer 1889, Vital 1903, Zöppritz and Bludau 1912). Hans Maurer was one of the first cartographers who tried systematizing map projections (Maurer 1935). According to him, map projections define the relation between the coordinate network on the Earth and corresponding coordinate networks in various map projections on a plane. Maurer used a sphere with a radius equal to that of the original, i.e. definition surface. As the main characteristic of a particular map projection, he took the properties of the coordinate network as the image of a network of circles on the unit sphere.

Maurer used the term *polar network* (*Polnetz*) to refer to the map image of the unit sphere graticule. Considering all points and great circles on a sphere are equal in terms of all geometric properties, we can replace meridians and parallels, i.e. great circles (*Großkreise*) through the poles and collateral circles perpendicular to them (*Nebenkreise*) with an equally good network of great and collateral circles, by selecting arbitrarily two diametrically opposite points. An infinite number of coordinate networks on a sphere is available. However, the images of those networks mapped on a plane are not at all the same, and have very different geometric properties. There are also certain properties independent of the represented coordinate system. These properties are defined using differential calculus and are valid for the entire map. Conformal and equal-area projections are obtained in this way. Therefore, a mathematician would group all conformal projections into one large group and all equal-area projections into another. However, map projections can be grouped in other ways in map projection theory. Maurer (1935) made his classification according to network systems on maps.

In general, he used the polar network as the map projection network. However, this does not have to be the case if another network of great and collateral circles of a sphere is significantly geometrically simpler. Otherwise, the polar network must be retained as a network property, even though there may be a simpler one, on condition that the map projection in question is related to the polar network. For example, this is the case for loxodrome maps or maps which preserve azimuths, where these lines are straight lines. For the sake of the generality of his classification, Maurer did not base the network properties in a projection on the polar network, but on a network of circles defined on an arbitrary pair of diametrically opposite points on a sphere. He called it the *basic network* (*Grundnetz*).

In the chapter entitled *Network Lines* (*Die Netzlinien*), Maurer defined basic terms related to coordinate systems on a sphere and in a plane. The *coordinate network on a sphere* (*Koordinatennetz auf der Kugel*) consists of two families of circles. These are *main circles* (*Hauptkreise*), halves of great circles, which share two points of one diameter of a sphere, and *collateral circles* (*Nebenkreise*), which intersect as whole circles with main circles at right angles. The common points of all main circles are called the *main points of a sphere* (*Kugelhauptpunkte*), and the straight line connecting them is called the *main axis* (*Hauptachse*). One of the main circles is called the *central main circle* (*Mittelhauptkreis*). An angle whose centre is at the main point of any main circle and the central main circle is marked λ (with a value between $-\pi$ and π).

The largest collateral circle is called the *base circle* (*Grundkreis*). Graduation on the main circle enables the definition of angle φ on both sides of the base circle to $\pi/2$ or angle δ from 0 to π from a main point, the *base point* (*Grundpunkt*) to another main point. Each collateral circle is defined by the value φ or δ, which corresponds to its intersection with the main circle.

In the chapter entitled *Projection Positions* (*Die Entwurfslagen*), Maurer (1935) differentiated projections according to position (*nach seiner Lage*):

- *Earth-axis* (*erdachsig*), when the main axis of the sphere network coincides with the Earth's axis
- *Transverse-axis* (*querachsig*), when the main axis of the sphere network falls in the equatorial plane
- *Oblique-axis* (*schiefachsig*), in all other cases.

In case of the Earth-axis position, the network of main and collateral lines coincides with the polar network. This is not the case with transverse-axis and oblique-axis positions. It should be noted that a single aspect/position of an orthographic projection can be interpreted in several ways. For example, an orthographic projection equidistant along parallels can be considered as an Earth-axis azimuthal projection. However, an orthographic transverse-axis projection would be an orthographic Earth-axis one if interpreted as a pseudocylindrical projection.

In 1922 and 1925, Maurer proposed the terms Earth-axis, transverse-axis and oblique-axis (*erdachsig*, *querachsig*, *schiefachsig*) projections and criticized other terms which could lead to misunderstandings (Maurer 1922, 1925, 1935). He also objected to using the foreign terms *normal* and *transversal*, since there were more suitable German words – *erdachsig* (Earth-axis) and *querachsig* (transverse-axis). He noted that the terminology was not uniform and illustrated this with the following examples:

- *Earth-axis* is sometimes referred to as normal (*normal*), pole-standing (*polständing*), polar-axis (*polachsig*), straight-axis (*rechtachsig*), polar projection (*Polar-Projektion*), equatorial projection (*Äquator-Projektion*)
- *Transverse-axis* is sometimes referred to as transverse (*transversal*), equator-standing (*äquatorständing*), equator-axis (*äquatorachsig*), meridian projection (*Meridian-Projektion*), equatorial projection (*Äquator-Projektion*)
- *Oblique-axis* is sometimes referred to as horizontal (*horizontal*), zenithal (*zenital*), inter-standing (*zwischenständing*), inclined-axis (*schrägachsig*), meridian projection (*Meridian-Projektion*), transversal (*transversal*), transverse-axis (*querachsig*).

In his frequently cited paper on map projection terms and classification, Lee (1944) wrote:

"The writer has felt the want of a term to describe the direct, transverse and oblique forms in which every projection may be used. The overworked word 'case' hardly answers the requirements, and the term *aspect* is here suggested. In the case of the general conical projections, the *direct aspect* is that in which the axis of the cylinder, cone or plane coincides with the polar axis of the sphere; the *transverse aspect* that in which the axis of the cylinder, cone or plane lies in the plane of the equator; and the *oblique aspect* that in which the axis has any other position. It is difficult to extend the definitions to cover the non-conical projections, but a simple convenient test is that the direct aspect is always the simplest mathematically.

A rather curious position arises in the case of some of those projections which fall into more than one of the graticule groups given above. When the stereographic, for example, is considered as an azimuthal projection, the direct aspect is that in which the point of tangency is the pole, and the transverse aspect that in which the point of tangency is at the equator; but when the projection is considered as a polyconic, the direct aspect is that in which the point of tangency is at the equator, and the transverse aspect that in which the point of tangency is the pole. Similar remarks apply to the orthographic. The difficulty could be overcome if it were agreed that the terms, direct and transverse, applied to these projections should refer to them as perspectives, as they are more generally regarded. This would make them conform to the test that the direct aspect is the simplest mathematically.

The aspects of the azimuthal projections have received other names, but unfortunately confusion has arisen in regard to their use. According to whether the origin of the projection is a pole, a point on the equator or some other point, they have been called *polar*, *equatorial*, and *oblique*; while according to whether the plane of projection is (or is parallel) to the equator, a meridian or the horizon of some point, they have been called *equatorial*, *meridian*, and *horizon*. Thus 'equatorial' is used in two conflicting senses. Since the terms direct, transverse and oblique can be applied to these projections as to all others, it would seem advisable to abandon the use of terms about which there is no general agreement.

The oblique aspect is the general case of every projection, and the direct and transverse aspects are limiting cases of the oblique aspect. Many text-books of projections go to the length of investigating each aspect independently, but labour and space can often be saved by making simple substitutions in the formulae for the oblique aspect. This method is particularly useful in the case of the perspective projections of the sphere, but is not perhaps so readily applicable in other instances. The problem is, of course, not so simple in the case of a spheroid."

Analysing conical projections in the wider sense (*conical*), Lee included those in the narrower sense (*conic*), along with cylindrical and azimuthal, as borderline cases. Conical projections in the wider sense also include pseudocylindrical, pseudoconical and polyconical projections. All other projections are non-conical. According to Lee (1944), the following can be defined:

- *Cylindric*: projections in which the meridians are represented as a system of equidistant parallel straight lines, and the parallels by a system of parallel straight lines at right angles to the meridians.
- *Conic*: projections in which the meridians are represented as a system of equally inclined concurrent straight lines, and the parallels by concentric circular arcs, the angle between any two meridians being less than their true difference of longitude.
- *Azimuthal*: projections in which the meridians are represented as a system of concurrent straight lines inclined to each other at their true difference of longitude, and the parallels by a system of concentric circles with their common centre at the point of concurrency of the meridians.

After Lee (1944), these definitions may be applied strictly to the normal or direct aspect, but may also be extended to the transverse and oblique aspects by changing the wording to "projections in which, in the direct aspect ..."Definitions could also be made to cover all aspects, though rather awkwardly, by replacing "meridians" and "parallels" with "great circles through a chosen point" and "small circles having that point as a pole", respectively.

Furthermore, according to the same author, "No reference has been made in the above definitions to cylinders, cones or planes. The projections are termed cylindric or conic because they can be regarded as developed on a cylinder or cone, as the case may be, but it is as well to dispense with picturing cylinders and cones, since they have given rise to much misunderstanding. Particularly is this so with regards to the conic projections with two standard parallels: they may be regarded as developed on cones, but they are cones which bear no simple relationship to the sphere. In reality, cylinders and cones provide us with convenient descriptive terms, but little else."

Based on the above citations, we can conclude that Lee's definition of aspect is not correct because it is based on the axes of the cylinder, cone and sphere, while it was stated in the same paper that those terms had led to misunderstandings. Furthermore, he used the property of tangenting when the projection aspect was not entirely clear, and the position of the projection plane in relation to the sphere when considering other terms for aspects. This approach was contradictory to Lee's own statement, "No reference has been made in the above definitions to cylinders, cones or planes". One has to wonder whether it is possible to define projection aspect in another way, i.e. without referring to auxiliary surfaces and their position in relation to the sphere.

Bugayevskiy and Snyder (1995) classified projections according to the orientation of the graticule on the map in relation to the pole position of the accepted coordinate system. According to them, map projections can be divided into direct, transverse and oblique projection aspects. The basis for this classification is the latitude φ_P of pole *Q* of the coordinate system used. If $\varphi_P = 90°$, it is the direct aspect; if $\varphi_P = 0°$ it is the transverse aspect, and if $0° < \varphi_P < 90°$ it is the oblique aspect. Thus, pole *Q* of the coordinate system coincides with the geographic pole in the direct aspect, and the graticule is the simplest, or normal. Snyder and Bugayevskiy neglected the option of selecting the pole in the southern hemisphere.

The following terms can be found in contemporary English references: polar, equatorial, meridian, oblique, horizon and transverse aspect (Snyder 1987). Snyder and Voxland (1989) define aspect in a short glossary at the beginning of a map projection album as the "conceptual placement of a projection system in relation to the Earth's axis (direct, normal, polar, equatorial, oblique, and so on)". This definition is completely unclear. Furthermore, according to Snyder and Voxland (1989), the direct aspect is a form of projection which results in the simplest graticule and calculation. This is the polar aspect for azimuthal projections, as well as the aspect which represents the equator as a straight line in cylindrical and pseudocylindrical projections, and the aspect which represents straight meridians in conic projections. Snyder and Voxland referred to this aspect as *conventional* or *normal*. Even though they defined aspect in a particular way, they referred to types of aspects as forms of projection (?!).

According to Snyder (1987), the issue of aspect arises if the orientation or location is different from that intended by the author of the basic projection. However, he does not define the terms orientation, position, or basic projection.

Several authors mention the term projection axis (*die Entwurfsachse*), e.g. Maurer (1925), Hoschek (1969) and Beineke (1983). According to them, this is the rotational axis of the auxiliary surface onto which the Earth's sphere is mapped. The normal case (*Normalfall*), i.e the normal position (*normale Lage*) is recognized by the rotational axis of the auxiliary surface coinciding with the Earth's axis (*Erdachse*). It should be noted that introducing an axis into the definition of an aspect determines the projection up to rotation around this axis. For example, the central meridian of the mapped area can be any meridian in the normal/standing position.

According to more recent references, such as the *Encyclopaedic Dictionary of Cartography in 25 Languages* (Neumann 1997), terms related to projection aspects are still associated with mapping onto an auxiliary surface, which is subsequently developed onto a plane. This approach can also be found in most textbooks and monographs about map projections (e.g. Hoschek 1969, Kuntz 1983, Snyder 1987, Grafarend and Krumm 2006, Fenna 2007).

According to De Genst and Canters (1996), a projection aspect can be considered as a relationship with the way we "view" the Earth. In order to map the Earth onto a plane, we have to determine an adequate coordinate system on a globe. The usual way is to establish a geographic coordinate system characterized by a graticule. However, a similar grid can be defined in the same way, with great circles intersecting at two points which are not geographic poles. The parametric curves of the network can be called *pseudomeridians* and *pseudoparallels* to emphasize differences from the geographic system. Similarly, the poles of this *pseudonetwork* can be called *pseudopoles*.

If one applies a map projection starting with spherical coordinates defined on the pseudonetwork (pseudocoordinates) rather than a geographic network, a different representation of the Earth's sphere will be obtained. In fact, a pseudopole will replace the geographic pole, a pseudoequator the equator; a pseudoparallel corresponding to 30º latitude the 30º parallel, and so on. Such a change in aspect may initially be confusing, because the original meridians and parallels will be represented in an unusual way. The shape and position of other contents will also change. A change in the position of the geographic network in relation to projection distortion distribution enables an arbitrary area on the globe to be mapped with less distortion. This is the main reason for applying a change in aspect. According to De Genst and Canters (1996), there are three traditional projection aspects: normal or direct, transverse and oblique. Geographic and pseudogeographic networks coincide in the direct aspect, while they are mutually perpendicular in the transverse aspect. They form an arbitrary angle in the oblique aspect.

Čapek (1986) used the term construction axis position (*Position der Konstruktionsachse*) instead of map projection aspect. He also used the following related terms: construction coordinates (*Konstruktionskoordinaten*), construction network (*Konstruktionsnetz*), construction meridian (*Konstruktionsmeridian*), construction parallel (*Konstruktionsparallele*), etc. Čapek differentiated between the normal, transverse and oblique-axis positions (*normale Lage*, *trasversale Lage*, *schiefachsige Lage*).

In discussing the normal, oblique and transverse aspects, Grafarend and Krumm (2006) used the terms meta-equator, meta-North Pole, meta-spherical latitude, meta-spherical longitude, etc. Since the prefix *meta-* is used to mean *about* (e.g. *metadata* are data about data), we prefer the prefix *pseudo-*, proposed by De Genst and Canters (1996), though Canters (2002) adopted *meta-*. *Pseudo-* is already present in map projection theory, e.g. *pseudocylindrical* or *pseudoconic* projections.

Although rarely used in practice, seven aspects of non-conic projections can be defined by modifying the position of the polar axis in relation to the fixed Cartesian coordinate system *X*, *Y*, *Z* with its origin in the centre of the spherical Earth (Wray 1974, Maling 1980, Canters 2002).

According to Lapaine and Usery (2014), a projection aspect is the appearance and position of a graticule image in the projection plane. Aspects can be polar, equatorial, normal, transverse and oblique. Accordingly, there are polar projections, normal projections, equatorial projections, transverse projections and oblique projections. These are the names of certain map projection sets, rather than systematic categorization (e.g. a projection can be simultaneously polar and normal). Theoretically, any projection can be in any aspect. However, most projections are almost exclusively used in a particular aspect, to emphasize special characteristics. For example, many factors such as temperature, biodiversity and pollution depend on climate, i.e. latitude. Latitude can be measured directly in the equatorial aspect for projections with a constant distance between parallels, which facilitates comparison.

According to Lapaine and Usery (2014), a map projection is a normal projection or in the *normal aspect* if the appearance of the graticule image is the most natural, usually determined by geometrical conditions. It is often determined by the simplest calculations, or the graticule appears to be the simplest. The polar aspect is normal in azimuthal projections, while the equatorial aspect is normal in cyclindrical projections. In azimuthal and conical projections, the graticule consists of straight line sections and circle arcs; the normal aspect of cylindrical projections only consists of straight lines which form a regular rectangular grid. A map projection is transverse or in *transverse aspect* if the appearance of the graticule image is derived by applying formulae for the normal aspect to a globe which has previously been rotated 90° around its centre in the direction of a meridian, so that all the poles are in the equatorial plane. A map projection is polar or in *polar aspect* if the pole image is in the map centre. This is often used as a synonym for the normal aspect of azimuthal projection. A map projection is equatorial or in *equatorial aspect* if the equator is in the map centre. The image of the equator is located in the direction of one of the main axes of the map, usually horizontally. The normal aspect of cylindrical projection is usually also the equatorial aspect. A map projection is oblique or in *oblique aspect* if it is not normal, transverse, polar or equatorial.

2. Discussion

The same projection can have various aspects. What do projections in different aspects have in common? The answer is a mapping principle depicted by, for example, the representation of main and collateral circles and distortion distribution. How do aspects of the same projection differ? They differ according to the representation of contents, for example graticule form and position, or orientation of representation elements (land, borders, etc.).

If we examine carefully Lee's definition of basic map projections (cylindrical, conical and azimuthal) and his comments on expanding them to all aspects of map projections, it is clear he implies the introduction of a spherical coordinate system, which is similar to the geographic coordinate system, though its poles do not coincide with geographic poles (also Maurer 1935, Snyder 1987, De Genst and Canters 1996). Thus, it is a spherical system consisting of main and collateral circles known in the Russian and Croatian literature as the system of verticals and almucantarats (Borčić 1955, Solov'ev 1969), which we will refer to as a *pseudogeographic system*, according to De Genst and Canters (1996). This pseudogeographic system is identical to the geographic coordinate system in one special case, when the pseudopoles coincide with the geographic poles.

In this paper, we would like to define projection aspects without referring to auxiliary surfaces and their axes. In fact, it is well known that mapping to auxiliary surfaces (cylinder or cone) which are to be developed onto a plane is only the case with a small group of map projections, e.g. perspective projections. All other projections map directly to the projection plane. Therefore, the use of general definitions which have terms which only refer to some cases should be avoided.

Thus, the issue of map projection aspects is not only a linguistic issue, but relates primarily to the definition of the term. How can a map projection aspect be defined unambiguously? Is it even possible?

We could start with the following assumptions:

1. We can define a geographic coordinate system on a sphere with radius R, i.e. define a sphere using geographic parameterization.
2. We can also define the same sphere using pseudogeographic parameterization, for which the poles do not have to coincide with poles of geographic parameterization. Of course, pseudomeridians and pseudoparallels will not generally coincide with meridians and parallels, but their relations will be preserved. This pseudogeographic coordinate system can be thought of as the usual geographic coordinate system rotated around its origin. Maurer (1935) referred to it as a basic network, De Genst and Canters (1996) as a pseudonetwork, and Borčić (1955) and Solov'ev (1969) as a network of verticals and almucantarats. It should also be mentioned that instead of pseudogeographic parameterization, which is a natural generalization of geographic parameterization, some authors introduce a spherical coordinate system with zenith distance instead of pseudogeographic latitude. This method does not clarify or simplify things, while the

formulae for transition from one system to another become more complex, as can be seen in e.g. Snyder's monograph (1987).

3. A projection aspect is a relationship with the way we "view" the Earth (De Genst and Canters 1996).
4. Changing a map projection aspect does not change the form of pseudonetwork.
5. Changing a map projection aspect does not change the distortion distribution of the projection.
6. Changing an aspect enables the selected area to be located on a globe with less distortion.
7. A map projection is primarily characterized by distortion distribution, along with the form of the graticule and other map content.
8. The normal (standard or standing) aspect of a projection should be the aspect in which distortion isolines are simultaneously parametric curves (e.g. parallels), whenever possible, of course.
9. We want to avoid the use auxiliary surfaces because they do not exist in most map projections.
10. We have to accept the fact that the aspect for a map projection cannot be defined before the basic equations of map projection or an equivalent representation of map projection are defined, which actually means defining the normal aspect, whether the definition is implicit or explicit.
11. Defining the aspect of mapping an ellipsoid is not simple, so authors frequently disregard it (e.g. … *much more difficult on the ellipsoid … general discussion is omitted*, Snyder 1987).

3. Geographic and Pseudogeographic Coordinate Systems

Mapping $(\varphi, \lambda) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [-\pi, \pi] \to (X, Y, Z) \in \mathrm{R}^3$ given with the formulae

$$X = R\cos\varphi\cos\lambda,\ Y = R\cos\varphi\sin\lambda,\ Z = R\sin\varphi \tag{1}$$

is called the (*basic*) *geographic parameterization of a sphere* with the radius *R*. R is the set of all real numbers. It is not difficult to demonstrate that the formulae (1) actually define a sphere with the radius *R*. It is sufficient to show that

$$X^2 + Y^2 + Z^2 = R^2.$$

The point with coordinates (0, 0, *R*) is called the *North Pole*. The diametrically opposite point with coordinates (0, 0, –*R*) is called the *South Pole*. The straight line intersecting the North and South Poles and the coordinate system origin and centre of the sphere (axis *Z*) is called the sphere axis parameterized by geographic parameterization, or the *geographic parameterization axis* for short. The parametric curves $\lambda = const.$ are semicircles connecting the North and South Poles and are called *meridians*. The parametric curves $\varphi = const.$ are circles in a plane perpendicular to the sphere axis and are called *parallels*. The formulae (1) define the coordinate system on a sphere; we could call it the *geographic coordinate system* (Fig. 1).

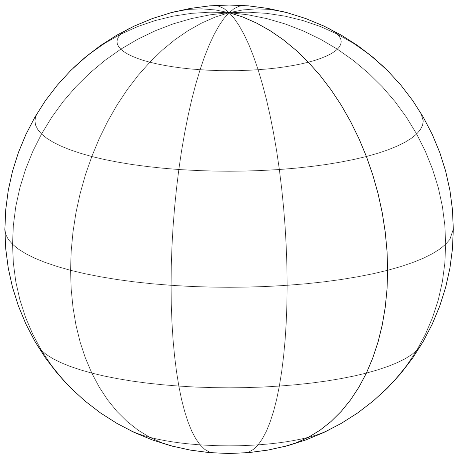


Figure 1. Geographic parameterization of a sphere

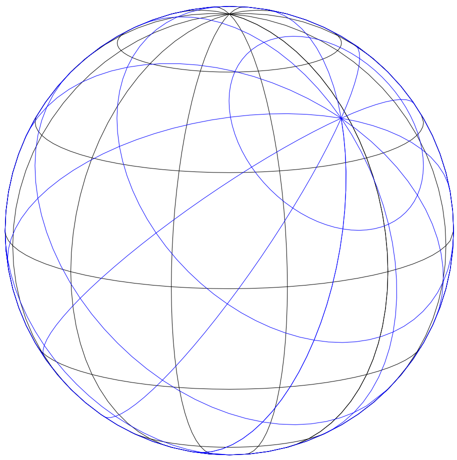


Figure 2. Geographic (black) and pseudogeographic (blue) parameterization of a sphere

Mapping $(\varphi', \lambda') \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [-\pi, \pi] \to (X', Y', Z') \in \mathrm{R}^3$ given with the formulae

$$X' = R\cos\varphi'\cos\lambda',\ Y' = R\cos\varphi'\sin\lambda',\ Z' = R\sin\varphi' \qquad (2)$$

is called the *pseudogeographic parameterization of a sphere* with radius *R*. The point with coordinates (0, 0, *R*) will be referred to as the North Pole of pseudogeographic parameterization, or the *North pseudopole* for short. The diametrically opposite point with coordinates (0, 0, –*R*) will be referred to as the South Pole of pseudogeographic parameterization, or the *South pseudopole* for short. We will refer to the straight line intersecting the North and South pseudopoles (*Z*' axis) as the sphere axis parameterized by pseudogeographic parameterization, or the *pseudogeographic parameterization axis* for short. The parametric curves $\lambda' = const.$ are semicircles connecting the North and South poles of pseudogeographic parameterization and are called *pseudomeridians*. The parametric curves $\varphi' = const.$ are circles in the plane perpendicular to the sphere axis and are called *pseudoparallels*. The formulae (2) define a spherical coordinate system we will refer to as the *pseudogeographic coordinate system*. The system is completely analogous to that defined by (1), but it does not necessarily coincide with it (Fig. 2).

We may say that the pseudogeographic parameterization of a sphere is the generalization of geographic parameterization of the same sphere, or that geographic parameterization of a sphere is a special case of pseudogeographic parameterization. In fact, if we position two 3D rectangular coordinate systems X, Y, Z and X', Y', Z' so that they have a mutual origin (0, 0, 0), then their coordinate axes are generally going to be placed at certain angles, and the relation between the two systems can be described using a rotation matrix (Bronštejn et al., 2004):

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}, \tag{3}$$

or, inversely

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \tag{4}$$

where $l_i, m_i, n_i, i = 1, 2, 3$, are cosines of directions of the new axes in relation to the old axes. The position of the new coordinate system in relation to the old one can also be determined using three Euler's angles (Bronštejn et al., 2004) and in various other ways. Important papers about the rotation of the spatial coordinate system were published by Cayley (1843, 1846 and 1849), and cartographic applications were analysed by Kavrayskiy (1958), Wray (1974), Barton (1976) and Beineke (1983). In any case, three pieces of information are needed for transitioning from one rectangular coordinate system to another. However, in some monographs on map projections (Kavrayskiy 1958, Hoschek 1969, Kuntz 1983, Yang et al. 2000, Grafarend, Krumm 2006), we find the following, or very similar formulae for the relation between geographic coordinates φ, λ and pseudogeographic coordinates φ', λ':

$$\sin\varphi' = \sin\varphi_P \sin\varphi + \cos\varphi_P \cos\varphi \cos(\lambda - \lambda_P), \tag{5}$$

$$\tan\lambda' = \frac{\sin(\lambda - \lambda_P)}{\cos\varphi_P \tan\varphi - \sin\varphi_P \cos(\lambda - \lambda_P)}, \tag{6}$$

where φ_P, λ_P are the geographic coordinates of the new North pole in the φ, λ system. This means only two pieces of information are necessary for transitioning from one system to another. How can this be explained?

Let us assume that the coordinate system X, Y, Z is rotated around its origin so that axis Z moves to a point with geographic coordinates φ_P, λ_P. First, the system X, Y, Z is rotated around axis Z by angle λ_P, then around the new axis Y by $\frac{\pi}{2} - \varphi_P$, and finally around the third new axis Z by θ_P. This can be expressed formally as:

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} \cos\theta_P & \sin\theta_P & 0 \\ -\sin\theta_P & \cos\theta_P & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\varphi_P & 0 & -\cos\varphi_P \\ 0 & 1 & 0 \\ \cos\varphi_P & 0 & \sin\varphi_P \end{bmatrix} \begin{bmatrix} \cos\lambda_P & \sin\lambda_P & 0 \\ -\sin\lambda_P & \cos\lambda_P & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{7}$$

After the matrixes on the right side are multiplied, we get

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = R \begin{bmatrix} \sin\varphi_P \cos\varphi \cos(\lambda-\lambda_P)\cos\theta_P - \cos\varphi_P \sin\varphi \cos\theta_P + \cos\varphi \sin(\lambda-\lambda_P)\sin\theta_P \\ -\sin\varphi_P \cos\varphi \cos(\lambda-\lambda_P)\sin\theta_P + \cos\varphi_P \sin\varphi \sin\theta_P + \cos\varphi \sin(\lambda-\lambda_P)\cos\theta_P \\ \cos\varphi_P \cos\varphi \cos(\lambda-\lambda_P) + \sin\varphi_P \sin\varphi \end{bmatrix} \tag{8}$$

Taking (2) into account, we get

$$\begin{aligned} \cos\varphi'\cos\lambda' &= \sin\varphi_P \cos\varphi \cos(\lambda-\lambda_P)\cos\theta_P - \cos\varphi_P \sin\varphi \cos\theta_P + \cos\varphi \sin(\lambda-\lambda_P)\sin\theta_P \\ \cos\varphi'\sin\lambda' &= -\sin\varphi_P \cos\varphi \cos(\lambda-\lambda_P)\sin\theta_P + \cos\varphi_P \sin\varphi \sin\theta_P + \cos\varphi \sin(\lambda-\lambda_P)\cos\theta_P \\ \sin\varphi' &= \cos\varphi_P \cos\varphi \cos(\lambda-\lambda_P) + \sin\varphi_P \sin\varphi \end{aligned} \tag{9}$$

Figure 3. Effect of angle θ_P ($0°, 30°, 60°$) on the appearance of a world map in the oblique aspect of a Winkel Tripel Projection, with constants $\varphi_P = 45°$ and $\lambda_P = 30°$.

The effect of angle θ_P is represented in Fig. 3. If special case that $\theta_P = 0$ applies, the formulae (9) are simpler:

$$\begin{aligned}
&\cos\varphi'\cos\lambda' = \sin\varphi_P \cos\varphi\cos(\lambda-\lambda_P) - \cos\varphi_P \sin\varphi \\
&\cos\varphi'\sin\lambda' = \cos\varphi\sin(\lambda-\lambda_P) \\
&\sin\varphi' = \cos\varphi_P \cos\varphi\cos(\lambda-\lambda_P) + \sin\varphi_P \sin\varphi\,.
\end{aligned} \tag{10}$$

The third equation in (10) is identical to formula (5), and if we divide the second equation by the first from (10), we get (6). From there, we can conclude that expressions (5) and (6) are a special case of a relation between the coordinates φ, λ and φ', λ'. In this special case, the circle for which $\theta_P = 0$ was taken implicitly as the initial main circle. This is similar to the implicit assumption in (1) that the initial meridian is the one for which $\lambda = 0$. However, that meridian does not need to be the central meridian of the mapped area, so map projection formulae often include $\lambda - \lambda_P$ rather than λ, which is equivalent to the rotation of a coordinate system around axis Z by angle λ_P.

Beineke (1983) also noted that this was generally a composition of three rotations, i.e. a rotation defined with three angles/parameters (1983). He defined the *projection origin* (*Entwurfsnullpunkt*) as point $P_0(\varphi_0, \lambda_0)$ with properties $x = x(\varphi_0 = 0, \lambda_0 = 0) = 0$ and $y = y(\varphi_0 = 0, \lambda_0 = 0) = 0$ and differentiated between rotation matrixes for:

- cylindrical, indirect (*vermittelnden*), prenumerated (*umbezifferten*) and polyconical projections
- azimuthal projections
- conical projections.

Barton (1976) cited Cayley (1843, 1846, 1849) and proposed expressing spatial rotation using three parameters, *p*, *q* and *r*, with formulae containing only basic calculations, including squaring, without trigonometric or other transcendent functions. He demonstrated that various projection aspects could be obtained so as to enable selected areas of Earth to be represented with less distortion, rather than resorting to another projection.

Rotation in a plane is an isometry with one fixed point. Rotation in three-dimensional space is an isometry with one fixed straight line. Therefore, each rotation in space is rotation around a straight line. Rotation composition is a rotation. For example, if we have three consecutive rotations around three different straight lines, the result can also be interpreted as one rotation around one fixed straight line. This is an interesting property known in mathematics as Euler's rotation theorem, but we are not going to use it further in this paper.

Finally, in his formulae for transforming cartographic networks, Snyder ((5-7)–(5-10) in Snyder 1987) used three parameters, named α, λ_0 and β, but which he introduced in a different way.

4. Map projection aspect

If we want to understand the concept of aspects in map projection theory, we first have to define the standard geographic coordinate system using geographic parameterization of a sphere. Analogously, we have to define the generalized or pseudogeographic coordinate system using pseudogeographic parameterization of a sphere. Then, we can establish the relation between the two spherical coordinate systems. It is possible to obtain each from the other by rotation around the centre of a sphere, and each rotation in space is defined by three parameters.

Subsequently, the basic equations of map projection must be defined, or an equivalent representation of map projection, which is a question of agreement and/or custom. The map projection aspect can now be defined as the position of the projection axis in relation to the axis of geographic sphere parameterization.

At first, this may seem to provide a solution to the aspect problem, but this is not the case. The equations of a certain projection in a pseudogeographic system must still be formulated. This is equivalent to Snyder's consideration of basic projection, a term he left undefined (Snyder 1987). Therefore, we will attempt to resolve the issue. Each definition is a question of agreement and/or custom. We propose the following definitions, inspired by Lee (1944):

- *Cylindrical projections* are projections in which pseudomeridians are represented as parallel straight lines and pseudoparallels are represented as parallel straight lines perpendicular to meridian images. Cylindrical projection equations in a pseudogeographic system are as follows:

$$x = x(\varphi'),\ y = k\lambda',\ 0 < k < 1.$$

- *Conic projections* in the narrower sense are projections in which pseudomeridians are represented as straight lines intersecting at a certain point, while pseudoparallels are represented as concentric circle arcs, with the angle between any two pseudomeridians being lesser than the corresponding difference of the corresponding pseudogeographic latitudes. Conical projection equations in a pseudogeographic system are as follows:

$$\rho = \rho(\varphi'),\ \delta = k\lambda',\ 0 < k < 1.$$

- *Azimuthal projections* are projections in which pseudomeridians are represented as straight lines intersecting at a certain point and forming an angle equal to the difference of corresponding pseudogeographic latitudes, while parallels are concentric circles with a common centre at the point of intersecting pseudomeridians. Azimuthal projection equations in the pseudogeographic system are as follows:

$$\rho = \rho(\varphi'),\ \delta = \lambda'.$$

- *Pseudocylindrical projections* are projections in which pseudomeridians are represented as curves symmetrical to the central pseudomeridian, which is mapped as a straight line, while pseudoparallels are represented as parallel straight lines perpendicular to the central meridian image. Pseudocylindrical projection equations in the pseudogeographic system are as follows:

$$x = x(\varphi'),\ y = y(\varphi', \lambda').$$

- *Pseudoconic projections* are projections in which pseudomeridians are represented as curves symmetrical to the central pseudomeridian, which is mapped as a straight line, while pseudoparallels are mapped as arcs of concentric circles. Pseudoconic projection equations in the pseudogeographic system are as follows:

$$\rho = \rho(\varphi'),\ \delta = \delta(\varphi', \lambda').$$

- *Winkel Tripel projection* is an example of a mixed projection. Winkel Tripel projection equations in the pseudogeographic system are as follows:

$$x = x(\varphi', \lambda') = \frac{R}{2}\left(\varphi' + \frac{\sin\varphi'}{\operatorname{sinc} z}\right),\ y = y(\varphi', \lambda') = \frac{R}{2}\left(\lambda'\cos\varphi'_0 + \frac{2\cos\varphi'\sin\frac{\lambda'}{2}}{\operatorname{sinc} z}\right),$$

where

$$\cos z = \cos\varphi'\cos\frac{\lambda'}{2},\ \operatorname{sinc} z = \begin{cases} 1 & \text{za } z = 0 \\ \dfrac{\sin z}{z} & \text{za } z \neq 0 \end{cases},\ \varphi'_0 = const.$$

If we want to refer to the aspect of any other map projection, we must provide a definition in a similar way in a pseudogeographic system. Likewise, according to Snyder (1987), it is necessary to agree on the basic projections. In fact, projection categorization according to the shape of the normal cartographic network (cylindrical, conic, azimuthal, pseudocylindrical, etc.) is not unique, because, for example, orthographic projection is both azimuthal and pseudocylindrical. Therefore, additional sorting criteria are required. We can agree on the following additional criteria:

- the author's definition of basic equations of map projection or an equivalent representation of map projection
- the compatibility of the graticule's appearance with projection distortion distribution
- the simpler appearance of the graticule
- simpler mathematical expressions

In orthographic projection, the author is unknown, while the graticule, in the form of a pattern of straight lines and concentric circles, is somewhat simpler than ellipses and parallel straight lines, although this is debatable. However, the deciding criterion should be the distortion distribution, according to which orthographic projection should be considered azimuthal. Taking this into account, orthographic projection equations in the pseudogeographic system are as follows:

$$\rho = R\cos\varphi',\ \delta = \lambda'.$$

Even though it is theoretically possible to consider each projection in various aspects, does this always make sense? Examples of projections for which it is probably not justified include the bipolar oblique conic conformal, modified polyconic (world map projection), two-point equidistant projection, Wiechel's projection, Berghaus star projection, Briesemeister's projection, armadillo projection and Robinson's projection. Descriptions of all these projections can be found in the album by Snyder and Voxland (1989).

We propose the following definitions.

Basic equations of map projection are map projection equations which define a map projection in a pseudogeographic system.
Note: The selection of basic equations for a map projection is a question of agreement and/or custom. By selecting the basic equations of a map projection, one of its aspects is implicitly or explicitly defined. For cylindrical projections, meridians are represented as parallel straight lines, while parallels are represented as parallel straight lines perpendicular to meridian images. For Robinson's or Winkel Tripel projections, they take the form conceived by their authors. In fact, if we have projection equations in a geographic coordinate system, then we obtain equations in the pseudogeographic system by formally replacing geographic coordinates with pseudogeographic coordinates. However, this still does not guarantee the basic equations of a map projection, because the equations of any projection in any aspect can be written in the geographic coordinate system.

4.1. Aspect according to projection axis position

The *projection axis* is the axis of pseudogeographic parameterization of a sphere, based on which the basic equations of map projection are defined.
Coordinate axis Z' is the projection axis in this paper. If the basic equations of a map projection are given using geographic coordinates, then the projection axis is identical to the axis of geographic sphere parameterization. The axis of geographic sphere parameterization in this paper is coordinate axis Z.

The *projection aspect* is the position of the projection axis in relation to the geographic sphere parameterization axis.
The aspect can be normal, transverse or oblique.

The *normal aspect* is the aspect in which the projection axis coincides with the geographic sphere parameterization axis.
A Winkel Tripel projection in the normal aspect is shown in Fig. 4.

The *transverse aspect* is the aspect in which the projection axis is perpendicular to the geographic sphere parameterization axis.
A Winkel Tripel projection in the transverse aspect is shown in Fig. 5.

The *oblique aspect* is neither normal nor transverse.
A Winkel Tripel projection in the oblique aspect is shown in Fig. 3.

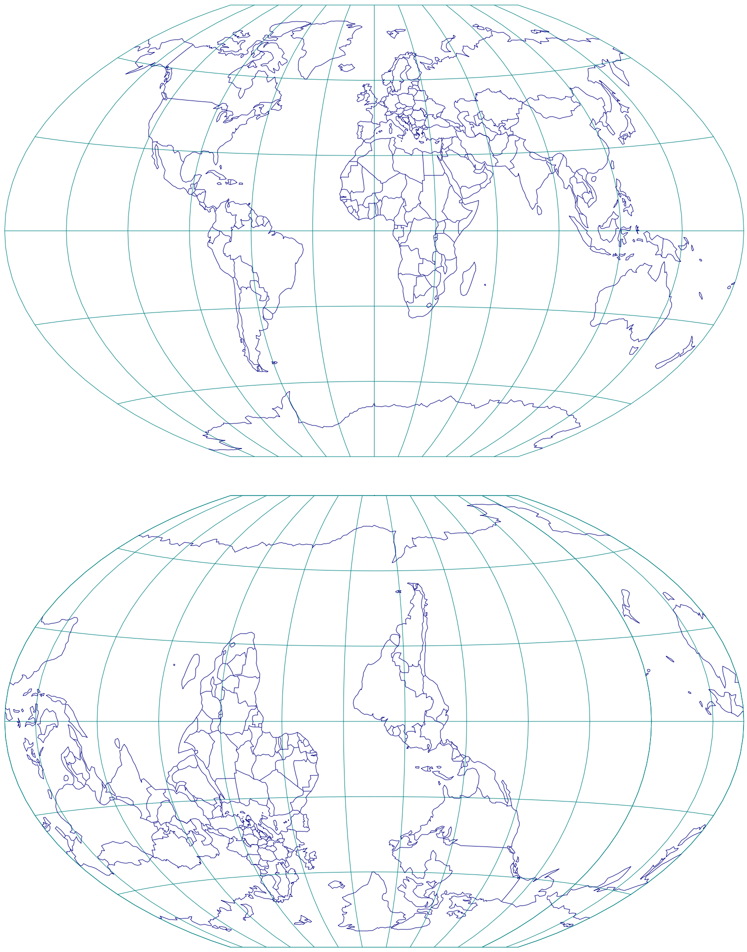


Figure 4. Two world maps in the normal aspect of a Winkel Tripel projection: for $\varphi_P = 90°$ and $\varphi_P = -90°$ respectively.

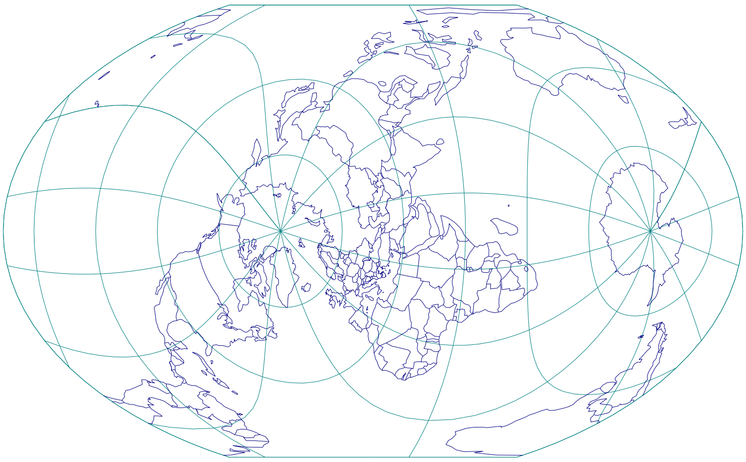
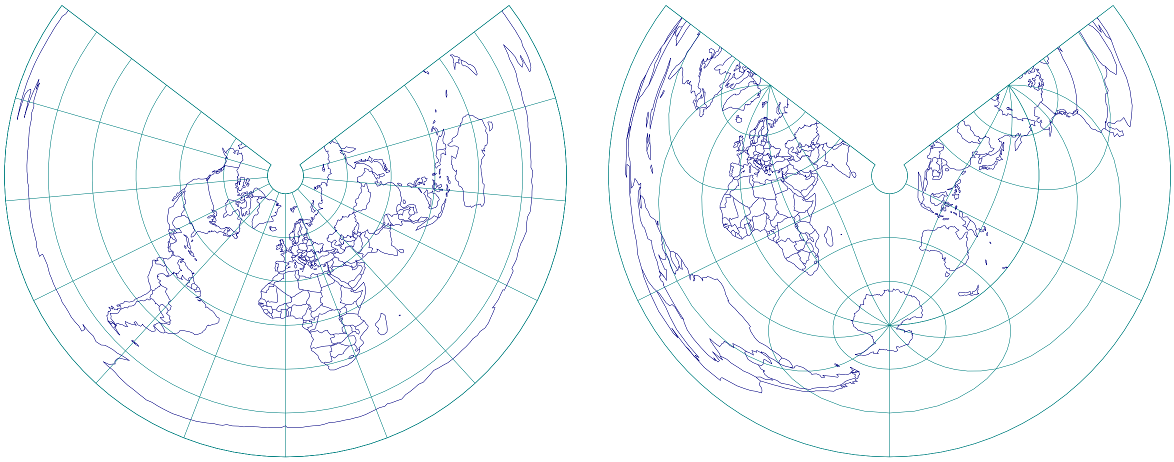


Figure 5. Three world maps in the transverse aspect of a Winkel Tripel projection, $\varphi_P = 0°$.

4.2. Aspect according to representing the area in the map centre

In references on map projections, the aspect is sometimes defined according to the area represented in the central part of a map (Snyder 1987, Grafarend and Krumm 2006, Fenna 2007). According to this definition, the aspect can be polar, equatorial or oblique.

If, by polar aspect, we mean the representation of a pole in the map centre, then the polar aspect is not defined by the position of the projection axis in relation to the geographic sphere parameterization axis. So normal azimuthal and transverse cylindrical projections can be in the polar aspect. Likewise, if by the equatorial aspect we mean the representation of the equator in the map centre, then the equatorial aspect is not defined by the position of the projection axis in relation to the geographic sphere parameterization axis. For example, transverse azimuthal and normal cylindrical projections can be in the equatorial aspect. The projections in Figures 6a and 6b might be considered polar projections by one person, but equatorial projections by another. This example shows that the definitions of the polar and equatorial aspects are imprecise and ambiguous, and should be avoided.



(a) (b)

Figure 6. Normal (a) and transverse (b) conic projection (equidistant along meridians). Illustration of problematic choices of polar or equatorial aspects.

5. Conclusion

Since changing the projection aspect does not affect distortion distribution, it is clear that the choice of an adequate aspect depends on the form and extent of the area we wish to represent on a map. In other words, the choice of a suitable aspect enables an arbitrary area on the globe to be located with smaller distortions. Thus, one requirement for the successful application of various aspects of a projection is knowledge of the distortion distribution of the projection.

After studying the relevant literature, we conclude that there is no generally accepted definition of a projection aspect, even though the term itself is widely used. Furthermore, we have shown that a projection aspect can be defined without any auxiliary surfaces, which often do not even exist in map projection theory. The authors of this paper propose a rigorous, mathematically based definition of aspects. They note that the transition from one aspect to another is equivalent to the rotation of a spatial coordinate system unambiguously defined by three parameters rather than two, as is frequently found in references.

In reality, aspects can be defined in two ways which are not mutually exclusive. According to the first definition, the aspect is the position of the projection axis in relation to the geographic sphere parameterization axis. The projection axis is the axis of the pseudogeographic sphere parameterization, based on which the basic equations for the map projection are defined, and the basic equations for the map projection are the map projection equations based on which the map projection in the pseudogeographic system is defined. This choice is an issue of agreement and/or custom. According to this definition, map projection aspects can be normal, transverse or oblique.

According to the second definition, the aspect is the representation of an area in the central part of the map. Aspects can be polar, equatorial or oblique. For example, it is possible for a projection to be normal and polar, but a projection can also be normal and equatorial. The authors of the present paper do not recommend this definition, due to its ambiguity.

The issue of ellipsoid projection aspect will be left for another occasion. According to Kavrayskiy (1958), it is possible to consider generalized transverse and oblique projections onto an ellipsoid, i.e. ellipsoid projections which would be transformed into transverse or oblique projections if they were applied on a sphere. In addition, an ellipsoid can be mapped onto a sphere and that image can be mapped onto a plane in some projection applied in transverse or oblique aspects. This is known as the double mapping of an ellipsoid.

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