Two divisors of $(n^2+1)/2$ summing up to $\delta n + \delta \pm 2$, δ even

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Previous results

Theorem (Ayad, Luca ([1])). *There do not exist an odd integer $n > 1$ and two positive divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = n + 1$.*

Dujella and **Luca** ([3]) deal with more general issue where $n+1$ is replaced by an arbitrary linear polynomial $\delta n + \varepsilon$, $\delta, \varepsilon \in \mathbb{Z}$, $\delta > 0$.

Since d_1, d_2 are divisors of a sum of two coprime squares, then $d_1 \equiv d_2 \equiv 1 \pmod 4$ and because of $d_1 + d_2 = \delta n + \varepsilon$, there are two possible cases

$$\delta \equiv \varepsilon \equiv 1 \pmod 2 \text{ or } \delta \equiv \varepsilon + 2 \equiv 0 \text{ or } 2 \pmod 4.$$

In [3], authors deal with the first case, we deal with the second case. More precisely, we deal with one parametric families of coefficients of the linear polynomial $\delta n + \varepsilon$, namely we deal with $(\delta, \varepsilon) = (\delta, \delta \pm 2)$.

Case $d_1 + d_2 = \delta n + \delta + 2$

Theorem 1. *There exist infinitely many odd positive integers n for which there exist divisors $d_1, d_2 > 1$ of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = \delta n + \delta + 2$.*

Proof. There exists $d \in \mathbb{N}$ such that

$$d_1 d_2 = \frac{g(n^2+1)}{2d}, \quad g = \gcd(d_1, d_2).$$

From the identity

$$(d_2 - d_1)^2 = (d_1 + d_2)^2 - 4d_1 d_2, \tag{1}$$

we get

$$X^2 - d(\delta^2 d - 2g)Y^2 = 4\delta^2 dg + 8d\delta g + 8dg - 4g^2, \tag{2}$$

for $X = (\delta^2 d - 2g)n + d\delta(\delta+2)$, $Y = d_2 - d_1$. For $d = g$ and $X = dX'$ the equation (2) becomes pellian equation

$$X'^2 - (\delta^2 - 2)Y^2 = 4(\delta+1)^2. \tag{3}$$

After introducing supstitutions $X' = 2(\delta+1)U$, $Y = 2(\delta+1)V$ and dividing (3) by $(2(\delta+1))^2$, we get a Pell's equation

$$U^2 - (\delta^2 - 2)V^2 = 1, \tag{4}$$

that has infinitely many solutions (U, V) generated by the recursive formulas. The fundamental solution of (4) is $(U_1, V_1) = (\delta^2 - 1, \delta)$. Consequently, (3) has infinitely many solutions (X', Y).
For $X = 2d(\delta+1)U$ and $X = (\delta^2 d - 2d)n + d\delta(\delta+2)$, we obtain

$$n = \frac{2(\delta+1)U - \delta(\delta+2)}{\delta^2 - 2}. \tag{5}$$

Because $U \equiv 1 \pmod{(\delta^2 - 1)}$ is satisfied for every solution of (4), we conclude $(\delta^2 - 2) \mid (2(\delta+1)U - \delta(\delta+2))$, so n are positive and odd integers.

Example

$$\begin{cases} n = 2\delta + 1, \\ d_1 = 1, \\ d_2 = 2\delta^2 + 2\delta + 1. \end{cases}$$

$$\begin{cases} n = 4\delta^3 + 4\delta^2 - 1, \\ d_1 = 2\delta^2 + 2\delta + 1, \\ d_2 = 4\delta^4 + 4\delta^3 - 2\delta^2 - 2\delta + 1. \end{cases}$$

$$\begin{cases} n = 8\delta^5 + 8\delta^4 - 8\delta^3 - 8\delta^2 + \delta + 1, \\ d_1 = 4\delta^4 + 4\delta^3 - 2\delta^2 - 2\delta + 1, \\ d_2 = 8\delta^6 - 12\delta^4 - 12\delta^3 + 4\delta^2 + 4\delta + 1. \end{cases}$$

Case $d_1 + d_2 = \delta n + \delta - 2$

Proposition 1. *If Schinzel's hypothesis H is true, then for all $\delta \equiv 4, 6 \pmod 8$ there exist infinitely many odd integers n for which there exist divisors d_1, d_2 of $\frac{n^2+1}{2}$ such that $d_1 + d_2 = \delta n + \delta - 2$.*

Proof. After applying similar methods described in Theorem 1, we get

$$X^2 - 2(2k^2 - 2k + 1)(\delta k - 1)(\delta k - \delta + 1)Y^2 = (2g(2k-1))^2. \tag{6}$$

We deal with the associated Pell's equation

$$U^2 - 2(2k^2 - 2k + 1)(\delta k - 1)(\delta k - \delta + 1)V^2 = 1. \tag{7}$$

Solutions of (6) are $(X, Y) = (2g(2k-1)U,\ 2g(2k-1)V)$, where (U, V) are solutions of (7). Additionally, solutions of (6) have to satisfy the congruence

$$X \equiv d\delta\varepsilon \equiv d\delta(\delta - 2) \pmod{2(\delta k - 1)(\delta k - \delta + 1)}, \tag{8}$$

if we require $n \in \mathbb{N}$. Let $a = 2k^2 - 2k + 1$, $b = \delta k - 1$, $c = \delta k - \delta + 1$. Now, (7) becomes $U^2 - 2abcV^2 = 1$. Let (U_0, V_0) be its fundamental solution. We have $(U_0 - 1)(U_0 + 1) = 2abcV_0^2$. Obviously, a, b, c are odd, which implies V_0 is even. Let $V_0 = 2st$, $s, t \in \mathbb{N}$. We deal with the equation

$$(U_0 - 1)(U_0 + 1) = 8abcs^2t^2.$$

For a, b, c prime prime numbers, the following factorizations are possible:

$$\begin{array}{llll}
1^{\pm}) & U_0 \pm 1 = 2abcs^2,\ U_0 \mp 1 = 2^2t^2, & 5^{\pm}) & U_0 \pm 1 = 2bcs^2,\ U_0 \mp 1 = 2^2at^2, \\
2^{\pm}) & U_0 \pm 1 = 2^2abcs^2,\ U_0 \mp 1 = 2t^2, & 6^{\pm}) & U_0 \pm 1 = 2as^2,\ U_0 \mp 1 = 2^2bct^2, \\
3^{\pm}) & U_0 \pm 1 = 2abs^2,\ U_0 \mp 1 = 2^2ct^2, & 7^{\pm}) & U_0 \pm 1 = 2bs^2,\ U_0 \mp 1 = 2^2act^2, \\
4^{\pm}) & U_0 \pm 1 = 2acs^2,\ U_0 \mp 1 = 2^2bt^2, & 8^{\pm}) & U_0 \pm 1 = 2cs^2,\ U_0 \mp 1 = 2^2abt^2.
\end{array}$$

From $U_0^2 \equiv 1 \pmod{(\delta k - 1)}$ and $U_0^2 \equiv 1 \pmod{(\delta k - \delta + 1)}$, we may assume

$$U_0 \equiv -1 \pmod{(\delta k - 1)}, \quad U_0 \equiv 1 \pmod{(\delta k - \delta + 1)}. \tag{9}$$

Let $k \equiv 3 \pmod 8$. In this case, $a \equiv 5 \pmod 8$, $b \equiv 3 \pmod 8$, $c \equiv 1 \pmod 8$. Our goal is to prove that it is always possible to find infinitely many integers k such that only cases 4^-) and 7^+) of the above factorizations are satisfied. This implies that the congruence (8) is satisfied which, again, implies that (X, Y) are integer solutions. We deal with each of the cases $1^{\pm}) - 8^{\pm})$ separately.
If we require that conditions

$$\left(\frac{a}{c}\right) = \left(\frac{c}{a}\right) = -1 \quad \text{and} \quad \left(\frac{c}{b}\right) = \left(\frac{b}{c}\right) = 1, \tag{10}$$

are satisfied, then only $4^-), 7^+)$ are possible.

In the end, we show that we can find infinitely many integers k such that $k \equiv 3 \pmod 8$, that are satisfied conditions (10) and that the integers a, b, c are simultaneously prime numbers.

Analogous claims for $\delta \equiv 0, 2 \pmod 8$ require different approach and methods and are still open problems.

References

[1] M. Ayad and F. Luca, *Two divisors of $(n^2+1)/2$ summing up to $n+1$*, J. Théor. Nombres Bordeaux **19** (2007), 561–566.

[2] S. Bujačić, *Two divisors of $(n^2+1)/2$ summing up to $\delta n + \varepsilon$, for δ and ε even*, Miskolc Math. Notes, 15 (2) (2014), 333-344.

[3] A. Dujella and F. Luca, *On the sum of two divisors of $(n^2+1)/2$*, Period. Math. Hungar. **65** (2012), 83–96.

[4] A. Schinzel and W. Sierpiński, *Sur certaines hypothèses concernant les nombres premiers*, Acta Arith. **4** (1958), 185–208, Corrigendum, 5 (1959), 259.