Global Ship Hull Description Using Single RBF

D. Ban & B. Ljubenkov
University of Split, Faculty of Electrical Engineering, Mechanical Engineering and Naval Architecture, (FESB) Ruđera Boškovića 32, 21000 Split, Croatia

ABSTRACT: The advances in ship hull geometry modeling, based on computers development in recent decades, led to parametric hull form description based on Basis spline methods. After it has been shown that global ship's hull form can be described using a single NURB spline description, it will be shown that it is possible to describe hull form using single, global, explicit RBF description, too. Interpolation and approximation methods based on radial basis functions will be investigated for that purposes, using L_1 norm. The calculation procedure will be tested for theoretical and calculated offset data, simulating ship's hull form described by cloud of points.

1 INTRODUCTION

Methods based on parametric curves and surfaces, such as B-spline and NURB-spline, are common methods used for ship geometry description in computer graphics today. Their basic advantage over other description methods in shipbuilding is possibility of free-form modeling of various complex ship hull shape geometries with knuckles and discontinuities, as well as description of non-bijective hulls. Several authors have been trying to describe ship hull form globally, like Norskov-Lauritsen (1985), Standerski (1988), using various methods, but recently Lu et al. (2005, 2007) have shown that ship geometry can be described using single NURB-spline. Nevertheless, although widely used, NURB-splines do not enable direct computation of geometric and hydrostatic properties of the ship hull necessary for further calculation of ship's overall properties, what is desirable property of some description method.

On the other hand, meshless radial basis functions (RBF) are relatively new geometry description method that is not proved in the description of highly complex geometries such is ship hull geometry, yet. Radial basis functions represent analytical description method that potentially enables direct calculation of ship's hydrostatic particulars. It is shown in Ban (2012) and Ban et al. (2014), that it is possible to describe and calculate ship's hydrostatic particulars directly using 2D composition of cubic and linear polynomial RBFs, and it assumed it is possible to do it using 3D methods, too.

This paper will investigate global description of ship geometry using single RBF, with focus on the choice of norm for achieving that task. It will be shown that arbitrary ship geometry can be described using single RBF is possible with suitable choice of radial basis function and norm argument.

Calculation results and quality of description will be examined on a description of complex test-hull form of typical car-truck carrier.

2 HULL SURFACE RECONSTRUCTION PROBLEMS

One of the main goals of any hull geometry description is extraction of their specific properties like form discontinuities, i.e. knuckles and form breaks, simultaneously keeping smoothness of description where necessary. When radial basis functions are used for solving hull surface reconstruction problem, they belong to positive definite reproducing kernel Hilbert spaces (RKHS), that ensure point-wise convergence and have ortho-normal bases, Fasshauer (2007).

Standard RBFs have L_2 norm between input points, as function argument, that is additionally squared to ensure invertibility of scattered data interpolation matrix, i.e. existence of positive definite functions. They are usually defined as linear combination of certain basis functions

$$f(\mathbf{x}) = \sum_{i=1}^{O} w_i B_i = \sum_{i=1}^{O} w_i \Phi_i(\mathbf{x}) = \sum_{i=1}^{O} w_i \varphi\left(\|\mathbf{x}, \mathbf{t}_i\|_2\right) \quad (1)$$

where $\mathbf{x}_j, j = 1, \ldots, N$; $\mathbf{x} \in IR^s$ is input data set, B_i are basis functions, Φ_i are radial basis functions, $\mathbf{t}_i$ are the development centers of RBF with $i = 1, \ldots, O$; where O is the number of centers, w_i are RBF network weight coefficients, φ is radial basis function based on Euclidian L_2 norm between input data and centers, and $f(x)$ is the generalized interpolation/ approximation function.

It should be noticed that in the case of ship hull surface reconstruction, input data set $\mathbf{x}$ consists of longitudinal and vertical coordinates with $\mathbf{x} \equiv \{x, z\}$, and output data set $\mathbf{y}$ equals half-breadths of observed input data set points.

For basis functions B_j to be invertible, their interpolation matrix must be in Haar space, i.e. satisfy condition

$$\det(B_j(x_i)) \neq 0 \tag{2}$$

The solution of scattered data interpolation problem can be obtained by calculating weight coefficients matrix $\mathbf{w}$ as

$$\mathbf{w} = \mathbf{H}^{-1} \cdot \mathbf{y} \tag{3}$$

where $\mathbf{y}$ is target vector (output data set), and H is interpolation matrix, with elements $H_{ji} = \|\mathbf{x}_j - \mathbf{x}_i\|$, $j, i = 1, \ldots, N$.

Main theory, regarding RBFs with L_2 norm, is focused on ensuring the existence of positive definite functions, Wendland (2005), with basis set on ball-in-a-cone condition as shown on Figure 1, below

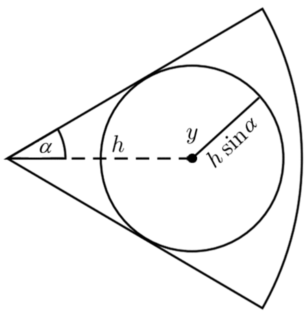

Figure 1: Ball-in-a-cone condition

Nevertheless, this condition limits the applicability of those functions regarding their belonging tangent values for higher α angle values between curve/surface and their tangents near input points. Additionally, one of the main tasks of RBF researchers is founding and improving interpolation error bounds, therefore forcing basis functions like multiquadric RBFs, Gaussian RBFs and others that posses error bounds. Because of above, there exist minimal fill distance value $h_{X,\Omega}$ between input points, producing convergence and computation stability problems, connected with inversion of scattered data interpolating matrix, with

$$h_{X,\Omega} = \sup_{x \in \Omega} \min_{x_j \in X} \|x - x_j\|_2 \tag{4}$$

Therefore, it is not possible to set as many points as needed in order to describe some hull geometry property using L_2 norm in reproducing functions using multiquadrics and Gaussian RBFs.

It will be shown in this paper that RBFs with L_1 norm can generally solve description problem, together with radial powers RBFs that do not posses error bounds and does not have infinite smoothness.

3 THE CHOICE OF RBF NORM

3.1 *L_1 norm*

One of the solutions of RBF reproduction problem for some arbitrary ship geometry is in using argument based on L_1 norm instead of L_2 norm, where no ball-in-cone condition exist, as well as no limitation of fill distance between points $h_{X,\Omega}$, as shown in Dyn et al. (1989) and Fasshauer (2007). But L_1 norm is not used in theory for surface description using RBFs due to well-known singularity problem in the description of simple input set consisting of $\mathbf{x} = \{(0,0), (1,0), (1,1), (0,1)\}$. Nevertheless, this problem is easily solvable by adding new points using van der Corput sequences, or setting points non-symmetrically using rotation.

Additionally, the problem can be solved numerically without adding new points, using non-existence of $h_{X,\Omega}$ for L_1 norm, and "spoiling" one of the points in input data set by adding small imperfection of order of variable last decimal, with $\mathbf{x}' = \{(0,0), (1,0), (1,1), (0,1.0001)\}$. Its L_1 norm distance matrix D_1 is than

$$D_1 = \begin{bmatrix} 0 & 1 & 2 & 1.0001 \\ 1 & 0 & 1 & 2.0001 \\ 2 & 1 & 0 & 1.0001 \\ 1.0001 & 2.0001 & 1.0001 & 0 \end{bmatrix}$$

Using that non-limiting characteristic of L_1 norm, D_1 matrix becomes non-singular and its inversion can be easily derived as

$$\begin{bmatrix} -5.000499999994998 & 5.000499999994998 & -4.999999999994999 & 4.999999999995000 \\ 5.000499999995000 & -5.000750012494375 & 5.000499999994999 & -4.999750012494375 \\ -4.999999999995000 & 5.000499999994999 & -5.000499999994999 & 4.999999999995000 \\ 4.999999999995000 & -4.999750012494375 & 4.999999999995000 & -4.999750012494375 \end{bmatrix}$$

Also, the interpolating matrix is always invertible if input points are all different. Input points can be arbitrarily set wherever needed, especially at form breaks and near them, with number suitable for precise description. Thus, both Runge and Gibbs phenomena can be solved for 2D problems, as shown in

Ban (2012), Ban et al. (2014). Moreover, this corresponds to real situation with actual ship hull geometries, where no ideal mathematical surfaces exist.

3.2 L_2 norm

Radial basis functions are usually defined for L_2 norm as basis function argument. As mentioned before, L_2 norm is usually squared to ensure existence of positive definite functions, and error bounds of chosen functions, and denoted

$$\left\|\mathbf{x}-\mathbf{x}_i\right\|_2 \text{ or } L_2^2$$

Because of that, mostly used RBFs, multiquadric (MQ) and Gaussian RBFs, usually suffer from fill distance limitations, Wendland (2005), Fasshauer (2007).

Anyway, they are usually used because of basic RKHS property connected with Hilbert's scalar product of vectors, as shown in Wendland (2005), as

$$\Phi(x, y)=\Phi(\,\cdot\,, x)\cdot\Phi(\,\cdot\,, y)$$

that enables smooth representation of the description when RBFs with L_2 norm are used.

In order to investigate other L_2 norm exponents, additional norm exponent γ will be introduced, equaling two in standard norm RBF definitions, thus ensuring existence of positive definite functions.

Anyway, if radial powers RBFs are used, they do not suffer from fill distance limitation, as will be shown in the paper below. Therefore, it is necessary to choose suitable radial basis function for hull surface description depending on the norm of basis function argument.

4 SELECTION OF RADIAL BASIS FUNCTION

There are several basis function usually used for surface description in scattered data interpolation theory. Among them, multiquadrics and Gaussian radial basis function are commonly used when global surface reconstruction is done, ensuring required smoothness and precision, together with required positive definiteness. Their respective definitions with $r = \|.\|_2$, Ban et al. (2009, 2010), are

$$\left(r^2+c^2\right)^\beta, \beta>0, \beta \notin 2IN, x \in IR^s \tag{5}$$

$$e^{-cr^2}, c>0 \tag{6}$$

where c is shape parameter of basis functions.

They usually have L_2 norm as argument and therefore suffer from the distance between input points $h_{X,\Omega}$ limitations. Because of that, required precision of arbitrary hull surface description cannot be achieved, neither hull geometry features extracted. Moreover, because of basis functions complexity, the calculation time for matrix inversion is very long compared to free-form parametric methods. Therefore, L_1 norm will be used instead of L_2 norm, for those functions.

Additionally, radial powers basis functions (RP) without spectral convergence, Fasshauer (2007), i.e. with finite smoothness, will be investigated here, with definition

$$(-1)^{\beta} r^{\beta}, \beta>0, \beta \notin 2IN, x \in IR^s \tag{7}$$

Above multiquadric and Gaussian functions in (5) and (6) can be rewritten using additional exponent γ with

$$\left(r^\gamma+c^\gamma\right)^\beta, \beta>0, \beta \notin 2IN, \gamma \in IN, x \in IR^s \tag{8}$$

$$e^{-cr^\gamma}, c>0, \gamma \in IN \tag{9}$$

It can be seen from above radial powers RBF definition that this type of radial basis differs from multiquadrics for they do not have shape parameter c, therefore having much simpler form. Except multiquadrics, radial powers RBFs are also similar to polynomial RBFs for integer exponent values. Therefore, polynomial RBFs will be investigated here, also, together with functions with exponent γ equal one, and radial powers will be observed as polynomial RBFs with L_2 norm.

In order to lower computational time, simple basis functions are required, with polynomial RBFs being the most promising candidate. When L_1 norm is introduced, polynomial RBFs can be obtained from multiquadrics or radial powers and they will be investigated further. Although having the simplest basis, polynomial RBFs are not widely used because of above mentioned singularity in the description of the symmetric square problem or equally distanced points as described in Mairhuber-Curtis theorem. But, it is also shown above, that this problem can be easily solved using L_1 norm and adding theoretically infinitesimal point imperfections.

It can be noticed that shape parameter c can be left out from brackets or completely omitted, what simplifies the interpolation matrix and makes it easier for calculation, as will be shown further in the paper.

In order to find suitable polynomial RBF's main function exponents β for surface description, corresponding β - *RMSE* sensitivity diagram for Franke's 2D function description with randomly distanced input points is done, as shown on Figure 2, below. It can be observed that odd integer values are producing jumps in *RMSE* values, and therefore are not suitable for calculation. The same is with equally distanced points.

(It should be noticed that mathematicians usually call surface reconstruction as 2D description, and not 3D, as in Franke's 2D function name.)

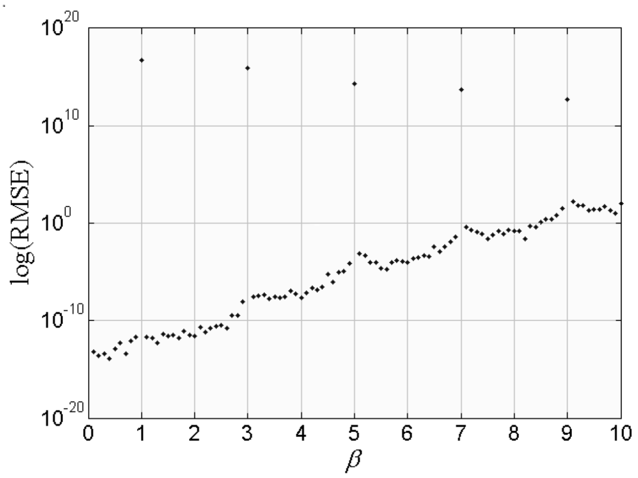

Figure 2: β - *RMSE* sensitivity diagram for Polynomial RBF description of Franke's 2D function

Polynomial RBFs definition for 3D description therefore can be written as

$$f(x) = \sum_{j=1}^{N} w_i \left| \mathbf{x} - \mathbf{t}_i \right|^{\beta} + c \,, \beta \in IR^{+} \setminus (2 \cdot IN - 1) \quad (10)$$

where main function exponent β values must avoid odd integer values.

This fact confirms why no linear polynomial RBFs are used in surface description, in theory. According to above definition, only even integer values are allowable or rational exponent values. Therefore, analytical RBF solution, for curves with discontinuities description in 2D problem, using composition of polynomial RBFs, as described in Ban et al. (2014), cannot be applied for 3D description problems.

In order to investigate possibility of RBF description of ship hull surfaces, the description of test hull form of a car-carrier will be shown in next chapter of this paper. Scattered data interpolation and approximation methods will be used for that purpose, for above mentioned radial basis functions, with L_1 and L_2 norm.

5 DESCRIPTION OF ARBITRARY HULL FORM USING SINGLE RBF

The most important characteristic of some arbitrary geometry description method is the possibility of all geometry characteristics reconstruction. In the case of global description of ship hull geometry, it is therefore necessary to enable the description of knuckles, flat parts and hull boundaries, together with smooth description of curved hull parts.

When analytical, direct computational methods, based on set of points are used, it means that input points must be distributed in such way that all hull features can be extracted. This cannot be achieved using traditional description based on mesh-based wire-frame hull geometry description with equally distributed frame sections, where description is based on curves.

In order to enable geometry features extraction it is necessary to have additional points near form knuckles, breaks and boundaries, which can be obtained twofold. One way is by computing additional points from wire-frame 2D description of ship geometry, and the other by geometry scanning using some of modern scanning techniques and obtaining cloud of points from which necessary points can be obtained.

Figure 3, below, shows total input data set for the description of test-hull, the hull form of a car-truck carrier with fore and aft bulbs, flat of a side, bottom deadrise, rounded bow and transom.

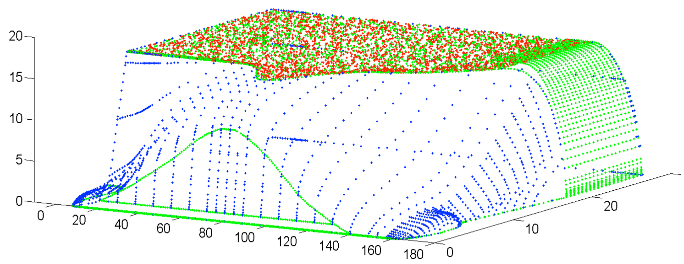

Figure 3: Total input set of points for test-hull form

The points marked blue on the Figure 3 are original points obtained from Table of offsets of test ship, with N_0 = 1,372, while other points are calculated randomly using 2D and 3D curve calculations from its wire-frame representation, as shown on Figure 4, below. Minimal distance between points in this case is 10^{-4} (m). In order to avoid the description of flat deck, input data set $\mathbf{x} \equiv \{x, z\}$ is bound for z values to 26.5 (m), and then scaled and normalized to range [0, 1]. Total number of points obtained for calculation in this way is $N \cong 10{,}200$.

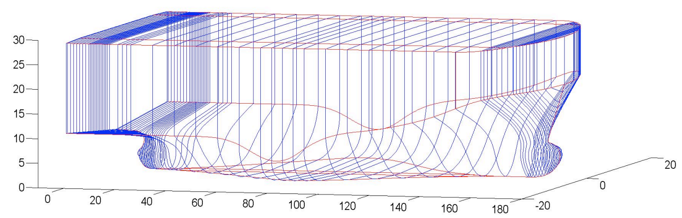

Figure 4: Wire-frame model of test-hull form

The calculation methods used in this paper are interpolation and approximation of scattered hull form data, using polynomial RBFs with L_1 and L_2 norm.

5.1 *3D RBF Interpolation Results*

Tables 1 and 2, below, contain RBF description results for chosen radial basis functions for L_1 and L_2 norm, respectively. The columns 4, 5, 6 and 7 denote Root-Mean-Squared-Error (*RMSE*), maximum absolute error of description (*Err*), in meters, and interpolation and generalization times, in seconds. All calculations are performed on laptop, with Intel Core i3 processor, with generalization set consisting of $N_G \cong 170{,}000$ points.

Table 1: The results of RBF interpolation of test-hull form using L_2 norm

Function	β	γ	*RMSE*	*Err*	*Int*	*Gen*
MQ (c=0.001)	0.5	2	$8.925\cdot10^{18}$	-	225	112
MQ (c=0.001)	0.5	1	$6.263\cdot10^{-4}$	0.229	191	119
Gaussian (c=1)	1		$3.613\cdot10^{18}$	-	135	120
RP	1		$6.710\cdot10^{-4}$	0.068	115	89
RP	3		$4.441\cdot10^{15}$	-	105	239
RP	0.5		$1.079\cdot10^{-8}$	0.056	93	246

Table 2: The results of RBF interpolation of test-hull form using L_1 norm

Function	β	γ	*RMSE*	*Err*	*Int*	*Gen*
MQ (c=0.001)	0.5	2	$1.227\cdot10^{11}$	-	110	119
MQ (c=0.001)	0.5	1	$5.600\cdot10^{-5}$	0.199	140	115
Gaussian (c=1)	1		$4.248\cdot10^{-3}$	0.230	89	148
PRBF (RP)	0.5		$4.170\cdot10^{-8}$	0.068	115	89
PRBF (RP)	1.5		$2.670\cdot10^{2}$	-	280	256
PRBF (RP)	0.25		$4.181\cdot10^{-10}$	0.095	88	254

The result of test-hull form description using polynomial RBFs interpolation with L_1 norm is shown on Figure 5, below. It can be seen that all form features are described, but no required smoothness is achieved.

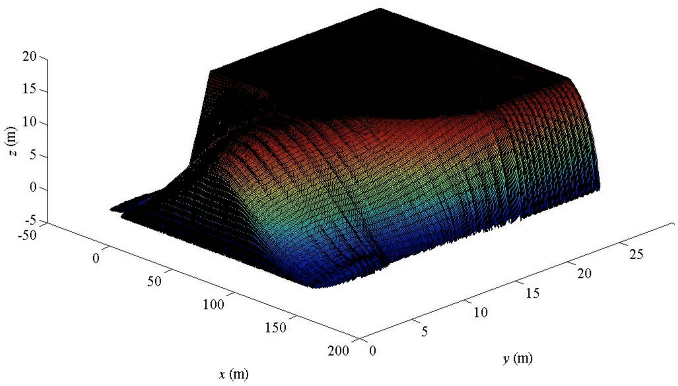

Figure 5: The description of test-hull form using polynomial RBFs with L_1 norm

Therefore, the description of test-hull form using polynomial RBFs with L_2 norm will be observed also, with improved smoothness, as one of the main goals of hull surface reconstruction.

Figure 6, below, shows description of test-hull form using PRBF with $\beta = 0.5$ and L_2 norm.

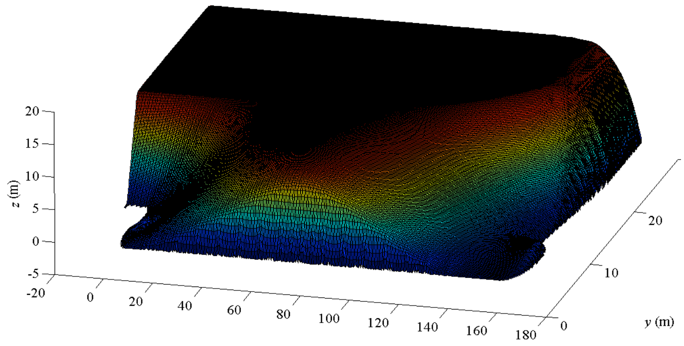

Figure 6: The description of test-hull form using polynomial RBFs with L_2 norm

It can be seen that smooth representation is obtained using polynomial RBFs with L_2 norm that describes all test hull form geometrical features, while flat of the side is not ideally flat and has some jumps. The largest error of the description is on the side of the ship with 56 (mm) error, not satisfying the requirement of 10^{-3} (m) accuracy.

5.2 *3D RBF Approximation Results*

Approximation based RBF methods in 3D description of ship hull form are checked also in this paper, with results shown in Table 3, below.

Table 3: The results of RBF approximation of test-hull form using L_2 norm

Function	β	γ	*RMSE*	*Err*	*App*	*Gen*
MQ (c=0.001)	0.5	2	$1.429\cdot10^{2}$	-	256	63
MQ (c=0.001)	0.5	1	$9.072\cdot10^{-4}$	0.258	344	115
Gaussian (c=1)	1		$5.439\cdot10^{-7}$	0.360	470	121
RP	1		$2.756\cdot10^{-6}$	0.091	391	81
RP	3		$2.670\cdot10^{2}$	-	280	256
RP	0.5		$1.762\cdot10^{-4}$	0.072	376	172

All results in Table 3, above, are obtained for Leave-One-Out (LOO) method, where only one point is allowed to be left-out of the input data set, to achieve acceptable description result. Otherwise, no acceptable result is obtained.

This means, it is necessary to use large number of points, and in that case computational time is very large. In general, computational time for approximation is about ten times larger than for interpolation when radial basis functions with L_1 norm or RP with L_2 norm are chosen. In order to obtain required results, total number of points is lowered to $N \cong 6900$, in this case.

It can be concluded that RBF interpolation methods are favorable comparing to RBF approximation methods regarding their computational time, where no gain is obtained regarding accuracy and smoothness, when approximation is used.

6 DESCRIPTION USING DATA CLOUD

It can be observed in previous chapter that RBF descriptions of arbitrary surface depend on the position of input data set of points.

Regardless basis function chosen and its belonging parameters, the quality of surface reconstruction depends on the proper setup of points. Above description error can be lowered with suitable distribution of points near hull boundaries, knuckles and bottom. This can be achieved either by measuring actual ship hull or by calculation of points on required positions using previous wire-frame PRBF curve description of test-ship as shown on Figure 4, using polynomial RBFs description of curves as described in Ban et al. (2014).

In addition to reconstruction theory, the description theory in computer graphics usually observes some geometry regarding its energy of description. Total energy is divided it into energy of smooth part of the description and energy of discontinuities, in it, where energy of discontinuities can be divided into energy of boundaries and energy of inner discontinuities of description.

Total energy of the description E can be than written as

$$E = E_s + E_b + E_d \tag{11}$$

where E_s is energy of the smooth part of the description, E_b is energy of boundary and E_d is energy of inner discontinuities.

Belonging input points then correspond to the energies they describe, with denser description of boundaries and inner discontinuities. Therefore it is necessary to describe ship hull using B-rep description, as shown on the Figure 7, below. B-rep description of arbitrary geometry consists of curve based boundaries and discontinuities and smooth parts of the geometry bounded by that 3D curves.

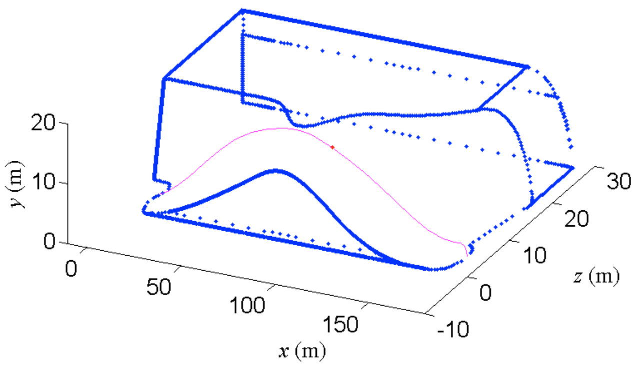

Figure 7: B-rep description of test hull, with generation of a point using waterline PRBF description

Figure 7 is also showing the example of automatic longitudinal generation of data cloud points on arbitrary waterlines using polynomial RBFs, based on previous wire-frame description of ship hull shown on Figure 4, with curves described using their corresponding polynomial RBFs, too, as described in Ban et al. (2014).

In order to obtain dense description of discontinuities, Chebyshev distribution of points is used, for ship hull divided into parts bounded by discontinuity curves. Five separated parts of test hull form can be observed on Figure 7: flat of a side, two parts of rounded freeboard on a bow, flat deadrise and main smooth part of a test hull. Figure 8, below, shows input data set with randomly distributed points for smooth part of test hull form description and Chebyshev based distribution of points for discontinuities.

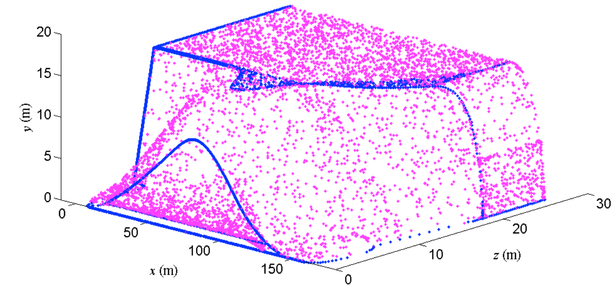

Figure 8: Generated data cloud for test-hull form using 2D wire-frame PRBF description

This distribution of points, as shown on Figure 8, above, corresponds to the reconstruction of geometry from data set cloud that can be obtained by actual hull form measuring. Similar to calculating input points, the points are chosen from data clouds randomly with care for description of discontinuities. Supervised or non-supervised learning methods can be used, or their combination, in order to define input data set adequately.

After generating input data set, the description of test hull form can be performed using polynomial RBFs, i.e. radial powers, with L_2 norm and function exponent $\beta = 0.5$, as shown before. The figure showing this description is similar to previous description on Figure 6, and is therefore omitted here.

The results of test hull form description are then tested for smoothness, using waterlines and sections plans as shown on Figures 9 and 10, below.

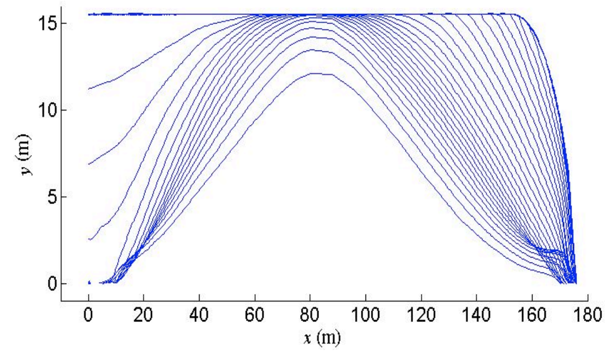

Figure 9: Test hull form waterlines plan described using data cloud of points and Polynomial RBF with L_2 norm $\beta = 0.5$

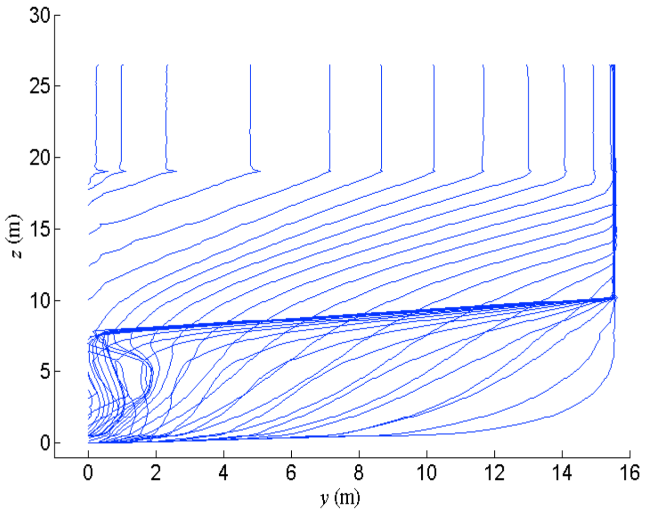

Figure 10: Test hull form sections plan described using data cloud of points and Polynomial RBF with L_2 norm $\beta = 0.5$

It can be noticed from Figures 9 and 10 that description of test hull form using polynomial RBFs gives relatively acceptable results regarding smoothness in both longitudinal and transversal directions. Nevertheless, there are areas of the description near discontinuities and boundaries where oscillations occur, where denser description of points is necessary to obtain required accuracy. It can be concluded that combination of wire-frame point data and cloud data should be used to obtain acceptable results, i.e. ship surface reconstruction procedures should combine supervised and unsupervised learning methods.

Except above mentioned, minimal number of points and their position in reconstruction theory of surfaces is not known in theory yet, and additional work should be done to solve this problem.

7 CONCLUSION

Analytical description methods, based on scattered data interpolation, are enabling a single radial basis function 3D description of arbitrary hull form, using radial functions with L_1 norm as argument. Among basis functions chosen, polynomial RBFs, i.e. radial powers, produce acceptable results. Moreover, the best results over all observed basis functions are obtained using radial powers RBFs with L_2 norm, also.

Achieved smoothness and accuracy of description of order of $5 \cdot 10^{-2}$ (m) is still not below description error of 10^{-3} (m) required in shipbuilding, but the results are promising. The smoothness of the description, i.e. proper description of all hull form features is still not matching NURB-splines, but the improvement is significant.

The calculation times of interpolation methods, for large, full matrices with around 10^8 elements, are around 10^2 seconds using older Intel Core i3 processor, and it is expected that that time can be further lowered on faster hardware configuration, with aim of being closer to free-form methods.

Furthermore, the form of polynomial, i.e. radial powers RBFs, is very promising regarding direct solvability of double integrals of ship's hydrostatics, i.e. calculation of volume displacement and centre of buoyancy force, since its form is the simplest possible regarding integrability and possibility of solving intersection problem. This will be investigated in further work of the authors.

Except investigated methods, partition of unity methods could be promising in 3D global RBF geometry description, potentially enabling local free-form deformation of original hull.

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