# Hölder continuity of Oseledets splitting for semi-invertible operator cocycles

Davor Dragičević

University of New South Wales

(joint work with Gary Froyland)

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# Preliminaries

Let $(X, \mathcal{B}, \mu)$ be a probability space and let $f\colon X \to X$ be an invertible transformation that preserves measure $\mu$. We assume that $\mu$ be ergodic. Furthermore, let $M_d$ denote the space of all real matrices of order $d$.

## Definition

A measurable map $\mathcal{A}\colon X \times \mathbb{Z} \to M_d$ is a *cocycle* over $f$ if:

1. $\mathcal{A}(x, 0) = \mathrm{Id}$ for every $x \in X$;
2. $\mathcal{A}(x, n+m) = \mathcal{A}(f^m(x), n)\mathcal{A}(x, m)$ for every $x \in X$ and $m, n \in \mathbb{Z}$.

A map $A\colon X \to M_d$ given by $A(x) = \mathcal{A}(x, 1)$, $x \in X$ is called a *generator* of a cocycle $\mathcal{A}$.

# Metric on Grassmanian

For a subspace $W \subset \mathbb{R}^d$ and $v \in \mathbb{R}^d$ we define

$$d(v, W) = \inf\{\|v - w\| : w \in W\}.$$

For two subspaces $V$ and $W$ of $\mathbb{R}^d$ we define the distance between them by

$$d(V, W) = \max\left\{ \sup_{v \in V, \|v\|=1} d(v, W), \sup_{w \in W, \|w\|=1} d(w, V) \right\}.$$

**Proposition**

*We have $d(V, W) = \|P_V - P_W\|$, where $P_V$ and $P_W$ denote orthogonal projections onto $V$ and $W$ respectively.*

# Metric on Grassmanian

Assume now that $X$ is a metric space with a distance $\rho$ and let $\Lambda \subset X$. We say that a family $E(x)$, $x \in \Lambda$ of subspaces of $\mathbb{R}^d$ is *Hölder continuous* on $\Lambda$ if there exist $L, \varepsilon > 0$ and $\beta \in (0, 1]$ such that

$$d(E(x), E(y)) \leq L\rho(x, y)^{\beta},$$

for $x, y \in \Lambda$ such that $\rho(x, y) \leq \varepsilon$.

# Metric on Grassmanian

**Lemma**

*Assume that $E(x)$ and $F(x)$ are orthogonal subspaces for each $x \in \Lambda$. We have*

1. *if $E(x)$, $x \in \Lambda$ is a Hölder continuous family then $E(x)^{\perp}$, $x \in \Lambda$ is also a Hölder continuous family;*
2. *if $E(x)$, $x \in \Lambda$ and $E(x) \oplus F(x)$, $x \in \Lambda$ are Hölder continuous families then $F(x)$, $x \in \Lambda$ is a Hölder continuous family.*

# Oseledets theorem

## Theorem

*Assume that $\mathcal{A}$ is a cocycle over $f$ with values in $GL_d$ and that*

$$\log^+\|A\|, \ \log^+\|A^{-1}\| \in L^1(\mu).$$

*Then, there exist numbers (Lyapunov exponents of $\mathcal{A}$ w.r.t. $\mu$) $\lambda_k > \ldots > \lambda_1$ and for $\mu$-a.e. $x \in X$ an decomposition*

$$\mathbb{R}^d = E_1(x) \oplus \ldots \oplus E_k(x)$$

*such that $A(x)E_i(x) = E_i(f(x))$ and*

$$\lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(x,n)v\| = \lambda_i, \quad \textit{for } v \in E_i(x) \setminus \{0\} \textit{ and } i = 1, \ldots, k.$$

# Result of Araujo, Bufetov and Filip, JLMS 2016

We say that a cocycle $\mathcal{A}$ is Hölder continuous if there exist $C, \nu > 0$ such that

$$\|A(x) - A(y)\| \leq C\rho(x,y)^\nu, \quad \text{for } x, y \in X.$$

**Theorem**

*Assume that $X$ is compact and that $f$ if bi-Lipschitz. Moreover, let $\mathcal{A}$ be a Hölder continuous cocycle over $f$ with values in $GL_d$. Then, for each $\varepsilon > 0$ there exists a compact set $\Lambda \subset X$, $\mu(\Lambda) > 1 - \varepsilon$ such that $x \mapsto E_i(x)$ is Hölder continuous on $\Lambda$ for each $i \in \{1, \ldots, k\}$.*

# Work of Brin

**Lemma**

*Let $(A_n)_{n\geq 1}$, $(B_n)_{n\geq 1}$ be two sequences of real matrices satisfying $\|A_n - B_n\| \leq \delta a^n$ and suppose there exist subspaces $E, E', F, F'$ of $\mathbb{R}^d$ satisfying $\mathbb{R}^d = E \oplus F = E' \oplus F'$ such that:*

1. *Upper bound for the growth of $A_n/B_n$ on $E/E'$.*
2. *Lower bound for the growth of $A_n/B_n$ on $F/F'$.*
3. *Bound for the angle between $E/F$ and $E'/F'$.*

*Then $d(E, E') \leq C\delta^\gamma$.*

# Work of Brin

**Lemma**

*Assume that $\mathcal{A}$ is a Hölder continuous cocycle and that there exists $L > 0$ such that $f$ is Lipschitz with constant $L$ and such that $\|\mathcal{A}(x,n)\| \le L^n$ for $n \ge 0$ and $x$ in some fixed compact set $\Lambda \subset X$. Then, there exist $a, \nu > 0$ such that $\|\mathcal{A}(x,n) - \mathcal{A}(y,n)\| \le a^n d(x,y)^\nu$ for $x, y \in \Lambda$ and $n \ge 0$.*

Applications:

1. stable/unstable subspaces of an Anosov diffeomorphism of a comp. Riem. manifold $M$ are Hölder continuous on $M$;
2. the same holds for nonuniformly hyperbolic $f$ but on compact sets of arbitrary large measure.

# Semi-invertible Oseledets theorem

**Theorem (Froyland, Lloyd, Quas, 2010)**

*Assume that $\mathcal{A}$ is a cocycle over $f$ with values in $M_d$ and such that*

$$\log^+\|A\| \in L^1(\mu).$$

*Then, there exists numbers $\infty > \lambda_k > \ldots > \lambda_1 \geq -\infty$ and for $\mu$-a.e. $x \in X$ an decomposition*

$$\mathbb{R}^d = E_1(x) \oplus \ldots \oplus E_k(x)$$

*such that $A(x)E_i(x) \subset E_i(f(x))$ and*

$$\lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(x,n)v\| = \lambda_i, \quad \text{for } v \in E_i(x) \setminus \{0\} \text{ and } i = 1, \ldots, k.$$

# Result of D. and Froyland, to appear in ETDS

**Theorem**

*Assume that $X$ is compact and that $f$ if bi-Lipschitz. Moreover, let $\mathcal{A}$ be a Hölder continuous cocycle over $f$ with values in $M_d$. Then, for each $\varepsilon > 0$ there exists a compact set $\Lambda \subset X$, $\mu(\Lambda) > 1 - \varepsilon$ such that $x \mapsto E_i(x)$ is Hölder continuous on $\Lambda$ for each $i \in \{1, \ldots, k\}$.*

We present the sketch of the proof.

## Theorem

*Let $\mathcal{A}$ be a cocycle over $f$ with Lyapunov exponents $-\infty \leq \lambda_1 < \cdots < \lambda_k$ and take $i \in \{1, \ldots, k\}$. Let $E(x) = \bigoplus_{j=1}^{i} E_j(x)$ and $F(x) = \bigoplus_{j=i+1}^{k} E_j(x)$.*

*Then, there exists a Borel set $\Lambda \subset X$ such that $\mu(\Lambda) = 1$ and for each $\varepsilon > 0$ there are measurable tempered functions $C, K \colon \Lambda \to (0, \infty)$ with the property that for every $x \in \Lambda$ we have that:*

1. *for each $v \in F(x)$ and $n \geq 0$, $\|\mathcal{A}(x, n)v\| \geq \frac{1}{C(x)} e^{(\lambda_{i+1}-\varepsilon)n}\|v\|$;*
2. *for each $v \in E(x)$ and $n \geq 0$, $\|\mathcal{A}(x, n)v\| \leq C(x)e^{(\lambda_i+\varepsilon)n}\|v\|$, (for $i = 1$, if $\lambda_1 = -\infty$, replace $\lambda_1$ with any number in $(-\infty, \lambda_2)$);*
3. *(Tempered angles): for each $u \in E(x)$ and $v \in F(x)$,*
   $$\|u\| \leq K(x)\|u + v\| \quad \text{and} \quad \|v\| \leq K(x)\|u + v\|.$$

Let $E(x) = \bigoplus_{j=1}^{i} E_j(x)$ and $F(x) = \bigoplus_{j=i+1}^{k} E_j(x)$.

### Lemma

$\limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{A}(x,n)|E(x)\| \le \lambda_i$ *for* $\mu$*-a.e.* $x \in X$.

### Lemma

$\lim_{m\to\pm\infty} \frac{1}{m} \log \sup_{n\ge 0}\{\|\mathcal{A}(f^m(x),n)|E(f^m(x))\| \cdot e^{-(\lambda_i+\varepsilon)n}\} = 0$ *for* $\mu$*-a.e.* $x \in X$.

### Lemma

$\int_X \log^+ \|(A(x)|F(x))^{-1}\| \; d\mu(x) < \infty$

How to bound the angle between $E(x)$ and $F(x)$?

## Lemma

*Assume that* $\Lambda$ *is an* $f$*-invariant set and let* $E(x) \subset \mathbb{R}^d$ *and* $F(x) \subset \mathbb{R}^d$, $x \in \Lambda$ *be* $\mathcal{A}$*-invariant families of complementary subspaces with the property that there exist* $\lambda_1 < \lambda_2$, $\varepsilon > 0$ *and a measurable tempered function* $C \colon \Lambda \to (0, \infty)$ *such that*

1. *for* $x \in \Lambda$, $v \in E(x) \oplus F(x)$ *and* $n \geq 0$, $\|\mathcal{A}(x,n)v\| \leq C(x)e^{(\lambda_2+\varepsilon)n}\|v\|$; *(global upper bound on growth)*
2. *for* $x \in \Lambda$, $v \in F(x)$ *and* $n \geq 0$, $\|\mathcal{A}(x,n)v\| \geq \frac{1}{C(x)}e^{(\lambda_2-\varepsilon)n}\|v\|$; *(lower bound for the growth on* $F(x)$*)*
3. *for* $x \in \Lambda$, $v \in E(x)$ *and* $n \geq 0$, $\|\mathcal{A}(x,n)v\| \leq C(x)e^{(\lambda_1+\varepsilon)n}\|v\|$; *(upper bound for the growth on* $E(x)$*)*

*Then, there exists a measurable tempered function* $K \colon \Lambda \to (0, \infty)$ *such that* $\|v_1\| \leq K(x)\|v_1 + v_2\|$ *and* $\|v_2\| \leq K(x)\|v_1 + v_2\|$, *for* $v_1 \in E(x)$ *and* $v_2 \in F(x)$.

Fix $i \in \{1, \dots, k\}$ and set

$$\Lambda_l = \{x \in \Lambda : C(x) \leq l \quad \text{and} \quad K(x) \leq l\}.$$

Then, $\Lambda = \cup_{l=1}^{\infty} \Lambda_l$, $\Lambda_l \subset \Lambda_{l+1}$ and $\Lambda_l$ is compact. Application of Brin's lemmas gives the Hölder continuity of $x \mapsto \oplus_{j=1}^{i} E_j(x)$ on each $\Lambda_l$ for each $l \in \mathbb{N}$. Thus, we have the Hölder continuity of $x \mapsto \oplus_{j=1}^{i} E_j(x)$ on compact sets of arbitrarily large measure. How to prove the same for $x \mapsto \oplus_{j=i+1}^{k} E_j(x)$?

Denote by $\mathcal{A}^*$ the cocycle over $f^{-1}$ with generator $A^* \circ f^{-1}$.

**Theorem**

*The Lyapunov exponents of the semi-invertible cocycle $\mathcal{A}^*$ are the same as those of the semi-invertible cocycle $\mathcal{A}$. Furthermore, the Oseledets subspace that corresponds to $\lambda_i$ is given by*

$$\left(\bigoplus_{j\neq i} E_j(x)\right)^{\perp}.$$

By applying what is proved for adjoint cocycle, we obtain the Hölder continuity (on compact sets of a.l.m) of

$$x \mapsto \left(\bigoplus_{j\neq 1} E_j(x)\right)^{\perp} \oplus \ldots \oplus \left(\bigoplus_{j\neq i} E_j(x)\right)^{\perp} = \left(E_{i+1} \oplus \ldots \oplus E_k(x)\right)^{\perp}.$$

Now we wish to establish the Hölder continuity of $x \mapsto E_i(x)$. Set $F(x) = E_i(x) \oplus \ldots \oplus E_k(x)$ which is Hölder continuous on $\Lambda_l$. Let $P(x)$ be an orthogonal projection onto $F(x)$. Choose $x, y \in \Lambda_l$ and set $A_n = \mathcal{A}(x,n)P(x)$ and $B_n = \mathcal{A}(y,n)P(y)$. Then,

$$\|A_n v\| \le l e^{(\lambda_i+\varepsilon)n}\|v\|, \quad \text{for } v \in F(x)^\perp \oplus E_i(x) \text{ and } n \ge 0$$

as well as

$$\frac{1}{l} e^{(\lambda_{i+1}-\varepsilon)n}\|v\| \le \|A_n v\|, \quad \text{for } v \in \oplus_{j=i+1}^k E_j(x) \text{ and } n \ge 0.$$

Same estimates hold for $B_n$. Also, it possible to bound $\|A_n - B_n\|$. Then, second Brin's lemma will give Hölder continuity of $x \mapsto F(x)^\perp \oplus E_i(x)$ which implies the Hölder continuity of $x \mapsto E_i(x)$.

## Theorem

*Let $X$ be a compact metric space, $f : X \circlearrowleft$ be a bi-Lipschitz ergodic transformation, $\mathcal{H}$ a Hilbert space, and $A : X \to \mathcal{B}(\mathcal{H})$ take values in the space of all compact operators. Furthermore, assume that $x \mapsto A(x)$ is Hölder continuous in the operator norm topology. Then either:*

1. *There is a* **finite** *sequence of numbers $\lambda_1 > \lambda_2 > \cdots > \lambda_k > \lambda_\infty = -\infty$ and a $\mu$-continuous decomposition $\mathcal{H} = E_1(x) \oplus \cdots \oplus E_k(x) \oplus E_\infty(x)$ such that $A(x)E_i(x) = E_i(f(x)), i = 1, \ldots, k$ and $A(x)E_\infty(x) \subset E_\infty(f(x))$ and $\lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(x,n)v\| = \lambda_i$, for $v \in E_i(x) \setminus \{0\}$, $i \in \{1, \ldots, k\} \cup \{\infty\}$. Moreover, each $E_i(x), i = 1, \ldots, k$, is a finite-dimensional subspace of $\mathcal{H}$. [F/Lloyd/Quas'13]*

   *The maps $x \mapsto E_i(x)$, $i = 1, \ldots, k, \infty$ are Hölder continuous on a compact set of arbitrarily large measure.*

## Theorem (cont...)

2 *There exists an* **infinite** *sequence of numbers* $\lambda_1 > \lambda_2 > \cdots > \lambda_k > \ldots > \lambda_\infty = -\infty$ *and a* $\mu$*-continuous decomposition* $\mathcal{H} = E_1(x) \oplus \cdots \oplus E_k(x) \oplus \cdots \oplus E_\infty(x)$ *such that* $A(x)E_i(x) \subset E_i(f(x))$ *(with equality if for* $i \in \mathbb{N}$*) and* $\lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(x,n)v\| = \lambda_i$, *for* $v \in E_i(x) \setminus \{0\}$, $i \in \mathbb{N} \cup \{\infty\}$. *Moreover, each* $E_i(x), i \neq \infty$ *is a finite-dimensional subspace of* $\mathcal{H}$.

*The maps* $x \mapsto E_i(x)$, $i \neq \infty$ *are Hölder continuous on a compact set of arbitrarily large measure.*

# Example

Let $M \subset \mathbb{R}^d$ be compact and $T_x \colon M \to M$, $x \in X$ a measurable family of maps on $M$ such that $x \mapsto T_x(y)$ is Hölder for each $y$. Define $\mathcal{L}_x \colon L^2(M) \to L^2(M)$ by

$$\mathcal{L}_x f(y) = \int_M \lambda_{B_\varepsilon(y)}(z) T_x(z) f(z) \, dz.$$

Then, the cocycle with generator $x \to \mathcal{L}_x$ is compact and Hölder continuous. So, the results of our paper are applicable.

# Applications

1. *Random Markov chains:* cocycles of stochastic matrices. Under some conditions the top Oseledets subspace is one-dimensional and the positive and normalized vectors $v(x)$ that belong to it satisfy $v(f(x)) = v(x)A(x)$.
2. *Lagrangian coherent structures:* study of the fluid flow (for example ocean), interested in detecting parts of the fluid that decay to equilibrium slowly (for example eddies). Main tool: transfer operators which can be numerically approximated by matrices (Ulam scheme). Second Oseledets subspace corresponds to elements which decay at the slowest rate.