A NOTE ON HARMONIC QUADRANGLE IN ISOTROPIC PLANE

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ABSTRACT: The *harmonic quadrangle* is a cyclic quadrangle $ABCD$ with the following property: the point of intersection of the tangents at the vertices A and C lies on the line BD, and the intersection point of the tangents at the vertices B and D lies on the line AC. If one of the requests is fulfilled, the other one automatically follows. In this paper we give some characterizations of harmonic quadrangles among the cyclic ones. We also study a new harmonic quadrangle associated to the given harmonic quadrangle $ABCD$.

Keywords: isotropic plane, cyclic quadrangle, harmonic quadrangle

1. INTRODUCTION

An isotropic plane is a real projective plane where the metric is induced by a real line f and a real point F, incident with it. The ordered pair (f,F) is called the absolute figure of the isotropic plane. All straight lines through the absolute point F are called isotropic lines, and all points incident with f are called isotropic points. Two lines are parallel if they intersect in an isotropic point, and two points are parallel if they lie on the same isotropic line. In the affine model of the isotropic plane where the coordinates of the points are defined by

$$x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}$$

the absolute line f has the equation $x_0 = 0$ and the absolute point F has the coordinates $(0,0,1)$. For two non-parallel points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ a distance is defined by $d(A,B) = x_B - x_A$, and for two parallel points $A = (x, y_A)$ and $B = (x, y_B)$ a span is defined by $s(A,B) = y_B - y_A$. Two non-parallel lines p and q given by the equations $y = k_p x + l_p$ and $y = k_q x + l_q$ form an angle defined by $\angle(p,q) = k_q - k_p$, [5].

2. ON HARMONIC QUADRANGLE IN ISOTROPIC PLANE

In [6] it is shown that in order to prove geometric facts for each cyclic quadrangle, it is sufficient to give a proof for the standard cyclic quadrangle. We consider the standard cyclic quadrangle $ABCD$ with the circumscribed circle given by

$$y = x^2 \tag{1}$$

and vertices are chosen to be

$$A = (a,a^2), B = (b,b^2), C = (c,c^2), D = (d,d^2), \tag{2}$$

with a,b,c,d being mutually different real numbers, where $a < b < c < d$. The following two lemmas characterize such a quadrangle.

Lemma 1 *([6], p. 267) For any cyclic quadrangle ABCD there exist four distinct real numbers a,b,c,d such that, in the defined canonical affine coordinate system, the vertices have the form (2), the circumscribed circle has the equation (1) and the sides are given by*

$$\begin{aligned} &AB \ldots y = (a+b)x - ab, \quad &DA \ldots y = (a+d)x - ad,\\ &BC \ldots y = (b+c)x - bc, \quad &AC \ldots y = (a+c)x - ac,\\ &CD \ldots y = (c+d)x - cd, \quad &BD \ldots y = (b+d)x - bd. \end{aligned} \tag{3}$$

Tangents $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}$ of the circle (1) at the points (2) are of the form

$$\begin{aligned} &\mathscr{A} \ldots y = 2ax - a^2, \quad &\mathscr{B} \ldots y = 2bx - b^2,\\ &\mathscr{C} \ldots y = 2cx - c^2, \quad &\mathscr{D} \ldots y = 2dx - d^2. \end{aligned} \tag{4}$$

The points of intersection of the tangents in (4) are

$$\begin{array}{l} T_{AB} = \mathscr{A} \cap \mathscr{B} = (\frac{a+b}{2}, ab), \\ T_{AC} = \mathscr{A} \cap \mathscr{C} = (\frac{a+c}{2}, ac), \\ T_{AD} = \mathscr{A} \cap \mathscr{D} = (\frac{a+d}{2}, ad), \\ T_{BC} = \mathscr{B} \cap \mathscr{C} = (\frac{b+c}{2}, bc), \\ T_{BD} = \mathscr{B} \cap \mathscr{D} = (\frac{b+d}{2}, bd), \\ T_{CD} = \mathscr{C} \cap \mathscr{D} = (\frac{c+d}{2}, cd). \end{array} \tag{5}$$

An allowable triangle is a triangle whose sides are non isotropic lines, see [2]. According to [6] the *allowable cyclic quadrangle* is the cyclic quadrangle having the allowable diagonal triangle.

Lemma 2 *The diagonal points U, V, W of the allowable cyclic quadrangle ABCD are of the form*

$$\begin{aligned} U &= (\frac{ac-bd}{a+c-b-d}, \frac{ac(b+d)-bd(a+c)}{a+c-b-d}), \\ V &= (\frac{ab-cd}{a+b-c-d}, \frac{ab(c+d)-cd(a+b)}{a+b-c-d}), \\ W &= (\frac{ad-bc}{a+d-b-c}, \frac{ad(b+c)-bc(a+d)}{a+d-b-c}), \end{aligned} \tag{6}$$

and the sides of the diagonal triangle are given with

$$\begin{aligned} UV \ldots y &= \frac{2(ad-bc)}{a+d-b-c}x - \frac{ad(b+c)-bc(a+d)}{a+d-b-c}, \\ UW \ldots y &= \frac{2(ab-cd)}{a+b-c-d}x - \frac{ab(c+d)-cd(a+b)}{a+b-c-d}, \\ VW \ldots y &= \frac{2(ac-bd)}{a+c-b-d}x - \frac{ac(b+d)-bd(a+c)}{a+c-b-d}, \end{aligned} \tag{7}$$

where $a+c-b-d \neq 0$, $a+b-c-d \neq 0$, $a+d-b-c \neq 0$.

Apparently, conditions $a+c-b-d \neq 0$, $a+b-c-d \neq 0$, and $a+d-b-c \neq 0$ are the conditions for the cyclic quadrangle $ABCD$ to be allowable.

The geometry of harmonic quadrangle firstly has been discussed in [1]. This paper presents the sequel to such an investigation. The *harmonic quadrangle* is a cyclic quadrangle $ABCD$ with the following property: the point of intersection of the tangents at the vertices A and C lies on the line BD, and the intersection point of the tangents at the vertices B and D lies on the line AC. If one of the requests is fulfilled, the other one automatically follows. Due to Theorem 3 there are many other characterizations of harmonic quadrangles among the cyclic ones.

Lemma 3 *[1] Let ABCD be an allowable cyclic quadrangle with vertices given by (2), sides by (3) and tangents of its circumscribed circle (1) at its vertices given by (4). These are the equivalent statements:*

1. *the point $T_{AC} = \mathscr{A} \cap \mathscr{C}$ is incident with the diagonal BD;*
2. *the point $T_{BD} = \mathscr{B} \cap \mathscr{D}$ is incident with the diagonal AC;*
3. *the equality*

   $$d(A,B) \cdot d(C,D) = -d(B,C) \cdot d(D,A) \tag{8}$$

   holds;
4. *the equality*

   $$2(ac+bd) = (a+c)(b+d) \tag{9}$$

   holds.

Since the properties 1-3 have completely geometrical sense, the property 4 does not depend on the choice of the affine coordinate system. Choosing the y-axis to be incident with the diagonal point U, because of (6) $ac = bd$ follows. Since $ac < 0$ and $bd < 0$, we can use the notation

$$ac = bd = -k^2. \tag{10}$$

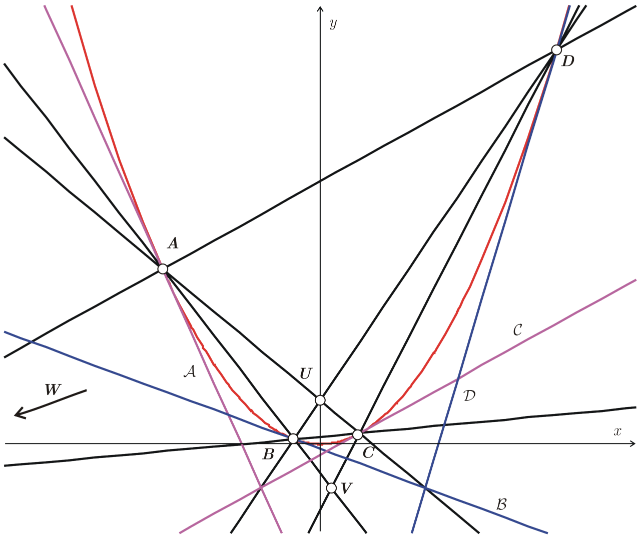

Figure 1: The harmonic quadrangle

Thus the diagonal point U, V, W turns into

$$U = (0, k^2), V = (\tfrac{ab-cd}{a+b-c-d}, -k^2), \\ W = (\tfrac{ad-bc}{a+d-b-c}, -k^2). \tag{11}$$

Such a harmonic quadrangle is said to be a *standard harmonic quadrangle*. As every harmonic quadrangle can be represented in the standard position, in order to prove geometric facts for each harmonic quadrangle, it is sufficient to give a proof for the standard harmonic quadrangle. Figure 1 presents one standard harmonic quadrangle.

3. FEW NEW PROPERTIES OF HARMONIC QUADRANGLE IN ISOTROPIC PLANE

Hereby, we present some interesting results that characterize the harmonic quadrangle among cyclic ones. For the Euclidean counterparts see [4].

Theorem 1 *Let $ABCD$ be an allowable cyclic quadrangle. $ABCD$ is the harmonic quadrangle if and only if the isotropic line through the point D intersects sides AB, BC, CA in the points P, Q, R, respectively, such that $s(Q,R) = s(R,P)$. The analogous statements for the other three vertices are valid as well.*

Proof: The isotropic line through D has the equation $x = d$. From (3) we get

$$P(d, d(a+b) - ab),$$
$$Q(d, d(b+c) - bc),$$
$$R(d, d(a+c) - ac).$$

Now we have

$$s(Q,R) = d(a+c) - ac - [d(b+c) - bc] = \\ = (a-b)(d-c),$$

$$s(R,P) = d(a+b) - ab - [d(a+c) - ac] = \\ = (b-c)(d-a).$$

Obviously, $s(Q,R) = s(R,P)$ precisely when $ab + ad + bc + cd = 2ac + 2bd$, i.e. when $ABCD$ is a harmonic quadrangle.

Theorem 2 *Let $ABCD$ be an allowable cyclic quadrangle. If $C_{AB} = AB \cap \mathscr{C}$, $D_{AB} = AB \cap \mathscr{D}$, $A_{BC} = BC \cap \mathscr{A}$, $D_{BC} = BC \cap \mathscr{D}$, $A_{CD} = CD \cap \mathscr{A}$, $B_{CD} = CD \cap \mathscr{B}$, $B_{AD} = AD \cap \mathscr{B}$, $C_{AD} = AD \cap \mathscr{C}$ are considered, then the following statements are equivalent:*

1. *$ABCD$ is a harmonic quadrangle.*
2. *The lines BA_{CD}, DA_{BC}, AC are concurrent.*
3. *The lines AB_{CD}, CB_{AD}, BD are concurrent.*
4. *The lines BC_{AD}, DC_{AB}, AC are concurrent.*
5. *The lines AD_{BC}, CD_{AB}, BD are concurrent.*

Proof: We will prove that statement 1 is equivalent to statement 5. Used calculations are simple, but long, so hereby we present only the final results of them.

The points D_{AB} and D_{BC} have the following coordinates

$$D_{AB} = \left(\frac{ab - d^2}{a+b-2d}, \frac{2abd - d^2(a+b)}{a+b-2d} \right),$$

$$D_{BC} = \left(\frac{bc - d^2}{b+c-2d}, \frac{2bcd - d^2(b+c)}{b+c-2d} \right).$$

Therefore, the lines AD_{BC} and CD_{AB} have the equations

$$AD_{BC}...y=\frac{(a^2+d^2)(b+c)-2(a^2+bc)}{d(d-2a)+a(b+c)-bc}x- -\frac{d^2(b+c-a)+bc(a-2d)}{d(d-2a)+a(b+c)-bc}a,$$

$$CD_{AB}...y=\frac{(c^2+d^2)(a+b)-2(c^2+ab)}{d(d-2c)+c(a+b)-ab}x- -\frac{d^2(a+b-c)+ab(c-2d)}{d(d-2c)+c(a+b)-ab}c.$$

These two lines intersect in the point (x_0,y_0):

$$x_0=\frac{N_x}{D_x},$$

$$y_0=\frac{N_y}{D_y}$$

where

$$N_x=abc(ab-ac+bc)-4ab^2cd-d^2(a^2b-ab^2- -a^2c-ac^2+bc^2-b^2c-4abc)-4acd^3+ +d^4(a-b+c),$$

$$D_x=a^2b^2-a^2c^2+b^2c^2-2d(a^2b+ab^2-a^2c-ac^2+ +b^2c+bc^2)+2d^2(2ab-2ac+2bc+b^2)-4bd^3+d^4,$$

$$N_y=a^2b^2c^2-4a^2bc^2d+2acd^2(2ab+ac+2bc-2b^2) -2d^3(a^2b-ab^2+a^2c+ac^2-b^2c+bc^2)+ +d^4(a^2-b^2+c^2),$$

$$D_y=a^2b^2-a^2c^2+b^2c^2-2d(a^2b+ab^2-a^2c-ac^2+ +b^2c+bc^2)+2d^2(2ab-2ac+2bc+b^2)-4bd^3+d^4.$$

which is incident with the line BD having the equation

$$y=(b+d)x-bd$$

if and only if $ab-2ac+bc+ad-2bd+cd=0$. According to Theorem 3 that happens precisely when $ABCD$ is a harmonic quadrangle.

Theorem 3 *Let $ABCD$ be an allowable quadrangle and* C *its circumscribed circle. If $C_{AB}=AB\cap\mathscr{C}$, $D_{AB}=AB\cap\mathscr{D}$, $A_{BC}=BC\cap\mathscr{A}$, $D_{BC}=BC\cap\mathscr{D}$, $A_{CD}=CD\cap\mathscr{A}$, $B_{CD}=CD\cap\mathscr{B}$, $B_{AD}=AD\cap\mathscr{B}$, $C_{AD}=AD\cap\mathscr{C}$ are considered, then the following statements are equivalent:*

1. *$ABCD$ is a harmonic quadrangle.*
2. *The second tangents to* C *at A_{BC} and A_{CD} intersect on AC.*
3. *The second tangents to* C *at B_{AD} and B_{CD} intersect on BD.*
4. *The second tangents to* C *at C_{AB} and C_{AD} intersect on AC.*
5. *The second tangents to* C *at D_{BC} and D_{AB} intersect on BD.*

Proof: Let us prove the equivalency of statements 1 and 5.
The lines

$$t_{D_{AB}}...y=2\frac{2ab-ad-bd}{a+b-2d}x-\left(\frac{2ab-ad-bd}{a+b-2d}\right)^2,$$

$$t_{D_{BC}}...y=2\frac{2bc-bd-cd}{b+c-2d}x-\left(\frac{2bc-bd-cd}{b+c-2d}\right)^2$$

are the tangents to C at D_{AB} and D_{BC} (different from $\mathscr{D}$). They intersect at the point

$$\left(\frac{1}{2}\left(\frac{2ab-ad-bd}{a+b-2d}+\frac{2bc-cd-bd}{b+c-2d}\right), \frac{(2ab-ad-bd)(2bc-cd-bd)}{(a+b-2d)(b+c-2d)}\right). \tag{12}$$

By doing some calculations, we obtain

$$\tfrac{1}{2}(b+d)\tfrac{(2ab-ad-bd)(b+c-2d)+(2bc-cd-bd)(a+b-2d)}{(a+b-2d)(b+c-2d)}-$$

$$-bd-\tfrac{(2ab-ad-bd)(2bc-cd-bd)}{(a+b-2d)(b+c-2d)}=$$

$$=\tfrac{(b-d)^2(ab+ad+bd+cd-2ac-2bd)}{(a+b-2d)(b+c-2d)}.$$

Therefore, the point with the coordinates (12) lies on the line BD from (3) when $ab+ad+bd+cd-2ac-2bd=0$, i.e. when $ABCD$ is a harmonic quadrangle.

The theorem given above provides the following geometric construction of the harmonic quadrangle: Let c be a circle inscribed to given triangle PQR. D is a point of contact of a tangent QR and the circle c. The line DP intersects the circle c in the point B. A and C are points of intersection of the lines BR and BQ with the circle c, respectively. The quadrangle $ABCD$ which the construction results with is the harmonic quadrangle. This claim follows from Theorem 3 by using the notation in a way: $\mathscr{D} = QR$, $D_{BC} = Q$ and $D_{AB} = R$. Figure 2 illustrates the described construction.

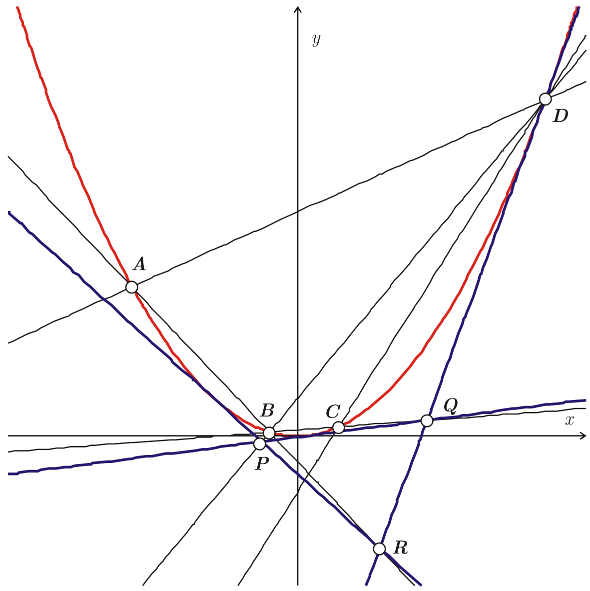

Figure 2: A construction of the harmonic quadrangle $ABCD$ from the given triangle PQR

There is an interesting result discussed in [1]: whole family of harmonic quadrangles can be obtained out of given harmonic quadrangle. Namely, let $ABCD$ be a harmonic quadrangle. Lines a', b', c', d' are taken in a way that are incident to vertices A, B, C, D, respectively, and make equal angles to sides AB, BC, CD, DA, respectively. The quadrangle formed by lines a', b', c', d' is a harmonic quadrangle as well. Furthermore, denoting obtained quadrangle by $A'B'C'D'$, the ratio of the corresponding sides of given quadrangle $ABCD$ and obtained quadrangle $A'B'C'D'$ is equal, Figure 3. Only in one case points A', B', C', D' coincide with one point P_1, the first *Brocard point*. In similar manner, the second Brocard point P_2 is obtained as well. In the latter case lines P_2A, P_2B, P_2C, and P_2D form the equal angles with the sides AD, DC, CB, and BA, respectively. Brocard points are of the form

$$P_1 = \left(k, 3k^2\right), P_2 = \left(-k, 3k^2\right). \qquad (13)$$

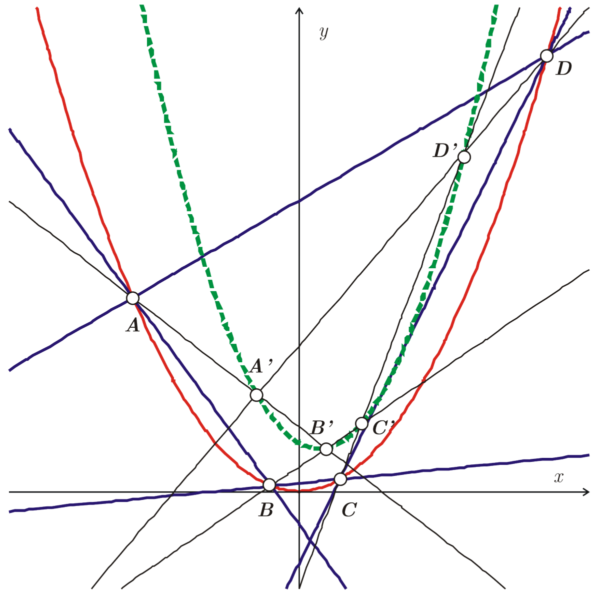

Figure 3: A harmonic quadrangle $A'B'C'D'$ obtained from the given harmonic quadrangle $ABCD$

In the sequel, we will also study on the harmonic quadrangle joint to given harmonic quadrangle. Similar investigations in the Euclidean case can be found in [3]. First let us prove lemma that follows:

Lemma 4 *Let $ABCD$ be an allowable cyclic quadrangle. There exist unique point Q_1 such that triangles Q_1AB and Q_1CD have equal angles (i.e. $\angle(Q_1A, AB) = \angle(Q_1C, CD)$, $\angle(AB, BQ_1) = \angle(CD, DQ_1)$). Similarly, there exist unique point Q_2 such that triangles Q_2BC and Q_2DA have equal angles.*

Proof: Let p and q be lines through A and B, respectively, with $\angle(p, AB) = m$ and $\angle(AB, q) =$

n. The equations of p and q are

$$p \quad ... \quad y = (a+b-m)x + am - ab,$$
$$q \quad ... \quad y = (a+b+n)x - bn - ab.$$

Thus, $Q_1 = p \cap q$ has coordinates

$$\left(\frac{am+bn}{m+n}, \frac{a^2m+b^2n+mn(a-b)}{m+n}\right) \tag{14}$$

Analogously, let p' and q' be lines through C and D, respectively, with $\angle(p', CD) = m$ and $\angle(CD, q') = n$. Then the point $Q_1' = p' \cap q'$ is given with

$$\left(\frac{cm+dn}{m+n}, \frac{c^2m+d^2n+mn(c-d)}{m+n}\right) \tag{15}$$

Looking for m and n such that Q_1 and Q_1' coincide, we obtain unique solution

$$m = \frac{(b-d)(a-b+c-d)}{a-b-c+d}, \qquad n = \frac{(a-c)(-a+b-c+d)}{a-b-c+d}, \tag{16}$$

which turns (14) into

$$Q_1 = \left(\frac{ad-bc}{a-b-c+d}, 2\left(\frac{ad-bc}{a-b-c+d}\right)^2 + \frac{ac(b-d)-bd(a-c)}{a-b-c+d}\right). \tag{17}$$

In similar manner, we get

$$Q_2 = \left(\frac{ab-cd}{a+b-c-d}, 2\left(\frac{ab-cd}{a+b-c-d}\right)^2 + \frac{bd(c-a)-ac(b-d)}{a+b-c-d}\right), \tag{18}$$

as well.

Theorem 4 *Let $ABCD$ be a harmonic quadrangle, and let Q_1, Q_2 be the points, such that $\angle(Q_1A, AB) = \angle(Q_1C, CD)$, $\angle(AB, BQ_1) = \angle(CD, DQ_1)$, and $\angle(Q_2B, BC) = \angle(Q_2D, DA)$, $\angle(BC, CQ_2) = \angle(DA, AQ_2)$, respectively. The quadrangle $Q_1M_{AC}Q_2M_{BD}$ is also a harmonic quadrangle.*

Proof: Due to $ac = bd = -k^2$, (17) and (18) turn into

$$Q_1 = \left(\frac{ad-bc}{a-b-c+d}, 2\left(\frac{ad-bc}{a-b-c+d}\right)^2 + k^2\right),$$

$$Q_2 = \left(\frac{ab-cd}{a+b-c-d}, 2\left(\frac{ab-cd}{a+b-c-d}\right)^2 + k^2\right).$$

These two points are incident with the circle

$$y = 2x^2 + k^2,$$

which is also passing through M_{AC}, M_{BD}, U, P_1 and P_2, [1].

It is left to prove $d(Q_1, M_{AC}) \cdot d(Q_2, M_{BD}) = -d(M_{AC}, Q_2) \cdot d(M_{BD}, Q_1)$. That can be easily checked from:

$$d(Q_1, M_{AC}) = \frac{a+c}{2} - \frac{ad-bc}{a-b-c+d} = = \frac{(a-c)(a-b+c-d)}{2(a-b-c+d)},$$

$$d(Q_2, M_{BD}) = \frac{b+d}{2} - \frac{ab-cd}{a+b-c-d} = = \frac{(b-d)(-a+b-c+d)}{2(a+b-c-d)},$$

$$d(M_{AC}, Q_2) = \frac{ab-cd}{a+b-c-d} - \frac{a+c}{2} = = \frac{(a-c)(-a+b-c+d)}{2(a+b-c-d)},$$

$$d(M_{BD}, Q_1) = \frac{ad-bc}{a-b-c+d} - \frac{b+d}{2} = = \frac{(b-d)(-a+b-c+d)}{2(a-b-c+d)}.$$

Let us notice that due to (11) and (3), points W, Q_1 and V, Q_2 are parallel ones. Therefore, Q_1 and Q_2 can be obtained as the points of intersection of the circle $y = 2x^2 + k^2$ and isotropic lines through diagonal points W and V, respectively.

Theorem 5 *A diagonal point $Q_1Q_2 \cap M_{AC}M_{BD}$ of the quadrangle $Q_1M_{AC}Q_2M_{BD}$ is a parallel point to the diagonal point U of the quadrangle $ABCD$. This point is incident with a line P_1P_2*

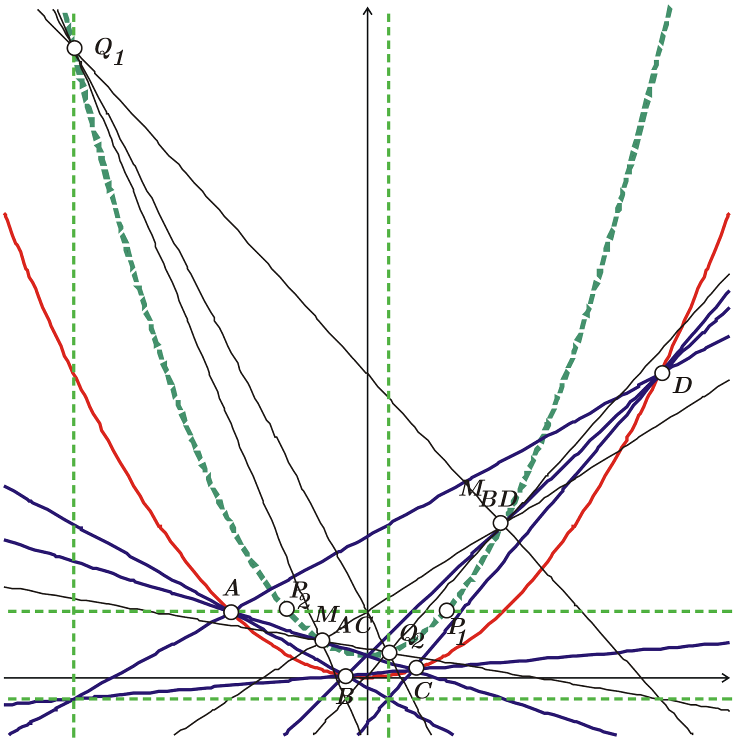

Figure 4: The harmonic quadrangle $Q_1M_{AC}Q_2M_{BD}$ associated to the harmonic quadrangle $ABCD$

as well, the connection line of Brocard points of ABCD. The other two diagonal points of the quadrangle $Q_1M_{AC}Q_2M_{BD}$ are incident to the line VW from (11).

Proof: Let us prove the first claim. We will show that the point of intersection of the lines Q_1Q_2 and $M_{AC}M_{BD}$ lies on the y-exis, more precisely that the point $(0,3k^2)$ lies both on Q_1Q_2 and $M_{AC}M_{BD}$. By direct calculation, we get the equation of the line Q_1Q_2:

$$y=2\left(\frac{ad-bc}{a-b+c-d}+\frac{ab-cd}{a+b-c-d}\right)x- \\ -2\frac{(ad-bc)(ab-cd)}{(a-b-c+d)(a+b-c-d)}+k^2, \tag{19}$$

i.e.

$$y=2\left(\frac{ad-bc}{a-b+c-d}+\frac{ab-cd}{a+b-c-d}\right)x- \\ -2\frac{a^2bd-acd^2-ab^2c+bc^2d}{a^2-b^2+c^2-d^2}+k^2. \tag{20}$$

Due to $ac=bd=-k^2$ we get that Q_1Q_2 has the equation of the form

$$y=2\left(\frac{ad-bc}{a-b+c-d}+\frac{ab-cd}{a+b-c-d}\right)x+3k^2. \tag{21}$$

On the other hand, the equation of the line $M_{AC}M_{BD}$ is

$$y=\frac{b^2+d^2-a^2-c^2}{b+d-a-c}\left(x-\frac{a+c}{2}\right)+\frac{a^2+c^2}{2},$$

i.e.

$$y=(a+b+c+d)\left(x-\frac{a+c}{2}\right)+\frac{a^2+c^2}{2}.$$

Thus, the line $M_{AC}M_{BD}$ has the equation of the form

$$y = (a+b+c+d)x + 3k^2. \qquad (22)$$

Obviously, the coordinates of the point $(0, 3k^2)$ satisfy both equations (21) and (22).

Let us now prove the second claim. The connection line of the two diagonal points $Q_1M_{AC} \cap Q_2M_{BD}$, $Q_1M_{BD} \cap M_{AC}Q_2$ is the polar line of the diagonal point $Q_1Q_2 \cap M_{AC}M_{BD}$ with respect to the circle $y = 2x^2 + k^2$. Therefore, its projective coordinates are

$$\begin{bmatrix} k^2 & 0 & -\frac{1}{2} \\ 0 & 2 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3k^2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}k^2 \\ 0 \\ -\frac{1}{2} \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} k^2 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, it is the line $y = -k^2$.

4. CONCLUSIONS

In this paper we studied the properties of the harmonic quadrangle in the isotropic plane. We gave a few characterizations of harmonic quadrangles among the cyclic ones, showed a construction of a such quadrangle, and studied a new harmonic quadrangle associated to a given one.

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