Limit theorems for deterministic and random Lasota-Yorke maps using the spectral method

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Piecewise expanding maps

Let $I = [0,1]$ denote the unit interval equipped with Borel σ-algebra $\mathcal{B}$ and a Lebesgue measure m. We say that $T\colon I \to I$ is a *piecewise expanding map* if there exists a partition

$$0 = x_0 < x_1 < \ldots < x_{k-1} < x_k = 1$$

and $\alpha > 1$ such that:

1. restriction $T|_{(x_{i-1},x_i)}$ is a C^1 function which can be extended to a C^1 function on $[x_{i-1}, x_i]$;
2. $|T'(x)| \geq \alpha > 0$ for $x \in (x_{i-1}, x_i)$;
3. $g(x) = \frac{1}{|T'(x)|}$ is a function of bounded variation.

Deterministic setting

1. *transfer operator:* $L\colon L^1(m) \to L^1(m)$ defined by

$$Lf(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T'(y)|};$$

2. L is quasicompact of diagonal type on the BV space and consequently it has a nonnegative fixed point which gives the existence of *acim*;
3. under some additional assumptions acim is unique (and we denote it by μ) and mixing;
4. in this case $L = \Pi + N$ and we have the exponential decay of correlation result.

Central limit theorem

Assume that $\phi \colon I \to \mathbb{R}$ bounded observable in BV such that $\int_{[0,1]} \phi \, d\mu = 0$. For each $n \in \mathbb{N}$, let

$$S_n = \sum_{k=0}^{n-1} \phi \circ T^k.$$

Theorem (Rousseau–Egele, 1983)

We have that $\lim_{n\to\infty} \int_{[0,1]} \frac{S_n^2}{n} = \sigma^2$, *where*

$$\sigma^2 = \int_{[0,1]} \phi^2 \, d\mu + 2\sum_{n=1}^{\infty} \int_{[0,1]} \phi(\phi \circ T^n) \, d\mu < \infty.$$

If $\sigma^2 > 0$, *then* $\frac{S_n}{\sqrt{n}}$ *converges in distribution to* $N(0, \sigma^2)$.

Large deviation principle

Theorem

If $\sigma^2 > 0$, then there exists $\delta > 0$ and a strictly convex, continuous and nonnegative function $c\colon (-\delta, \delta) \to \mathbb{R}$ which vanishes only at 0 such that

$$\lim_{n\to\infty} \frac{1}{n} \log \mu(S_n > n\varepsilon) = -c(\varepsilon), \quad \textit{for } \varepsilon \in (0, \delta).$$

Ideas of the proofs

We define

$$L_\theta(g) = L(e^{\theta\phi}g), \quad \text{for } g \in BV \text{ and } \theta \in \mathbb{C}.$$

Since $\theta \mapsto L_\theta$ is analytic, for θ sufficiently close to 0,

$$L_\theta = \omega(\theta)\Pi(\theta) + N(\theta).$$

For CLT ($d\mu = f\, dm$):

$$\lim_{n\to\infty} \int_{[0,1]} e^{itS_n/\sqrt{n}}\, d\mu = \lim_{n\to\infty} \int_{[0,1]} L^n_{it/\sqrt{n}}(f)\, dm = \lim_{n\to\infty} \omega(it/\sqrt{n})^n$$
$$= e^{-t^2\sigma^2/2},$$

for $t \in \mathbb{R}$.

For LDP:

we first show that $\omega'(0) = 0$ and $\omega''(0) = \sigma^2$ and then that

$$\lim_{n\to\infty} \frac{1}{n} \log \int_{[0,1]} e^{\theta S_n}\, d\mu = \Lambda(\theta),$$

where $\Lambda(\theta) = \log \omega(\theta)$, for $\theta \in \mathbb{R}$ sufficiently close to 0.

Random Lasota-Yorke maps

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and assume that $\sigma \colon \Omega \to \Omega$ is invertible transformation that preserves $\mathbb{P}$. Furthermore, assume that $\mathbb{P}$ is ergodic. We now take the collection T_ω, $\omega \in \Omega$ of piecewise expanding maps. For $\omega \in \Omega$ and $n \in \mathbb{N}$, set

$$T_\omega^n = T_{\sigma^{n-1}\omega} \circ \ldots \circ T_{\sigma\omega} \circ T_\omega$$

and

$$L_\omega^n = L_{\sigma^{n-1}\omega} \circ \ldots \circ L_{\sigma\omega} \circ L_\omega.$$

The associated skew-product transformation $\tau \colon \Omega \times I \to \Omega \times I$ is given by

$$\tau(\omega, x) = (\sigma\omega, T_\omega x).$$

Existence and uniqueness of ACIM, Buzzi 2000

Under some mild assumption we have that

$$\|L_\omega h\|_{BV} \le K(\omega)\|h\|_{BV}, \quad \text{for } h \in BV \text{ and } \log K \in L^1(\mathbb{P});$$

and there exists $N \in \mathbb{N}$ we have

$$var(L_\omega^N h) \le \alpha^N(\omega) var(h) + K^N(\omega)\|h\|_1,$$

for $h \in BV$ and a.e. $\omega \in \Omega$ and with

$$\int_\Omega \log \alpha^N(\omega)\, d\mathbb{P}(\omega) < 0.$$

If for each subinterval $J \subset I$ and for a.e. $\omega \in \Omega$, there exists $n(\omega) \in \mathbb{N}$ such that

$$\text{essinf}_{x\in[0,1]}(L_\omega^n \mathbf{1}_J) > 0, \quad \text{for } n \ge n(\omega),$$

there exists a unique acim (w.r.t. $\mathbb{P} \times m$) μ for τ such that $\pi_* \mu = \mathbb{P}$, where $\pi \colon \Omega \times I \to \Omega$ is a projection. We can regard μ as a collection of fiber measures μ_ω, $\omega \in \Omega$ on I. Also, one has fiberwise decay of correlation result.

We consider observables $\phi \colon \Omega \times I \to \mathbb{R}$ such that

$$\text{esssup}_{(\omega,x)} |\phi(\omega, x)| < \infty \quad \text{and} \quad \text{esssup}_\omega \, var(\phi(\omega, \cdot)) < \infty.$$

Moreover, we assume that

$$\int_{[0,1]} \phi(\omega, \cdot) \, d\mu_\omega = 0, \quad \omega \in \Omega.$$

We form Birkhoff sums

$$S_n(\omega, x) = \sum_{i=0}^{n-1} (\phi \circ \tau^i)(\omega, x) = \sum_{i=0}^{n-1} \phi(\sigma^i \omega, T_\omega^i x).$$

We are interested in the quenched type of limit theorems i.e. those that give an information about the asymptotic behaviour of Birkhoff sums w.r.t. to μ_ω for "typical" ω.

Previous work:

1. Kiffer, 1998: quenched limit theorems in random enviroment but not with spectral method;
2. Aimino, Nicol and Vaienti, 2014: spectral method but the base space is assumed to be a Bernoulli shift.

MET, Froyland/Lloyd/Quas, 2013.

Assume that $(\Omega, \mathcal{F}, \mathbb{P}, \sigma)$ is a probability space where Ω is a Borel subset of a separable, complete metric space. Furthermore, let B be a Banach space and $\mathcal{L} = L_\omega$, $\omega \in \Omega$ a family of bounded linear operators on B such that the map $\omega \mapsto L_\omega$ is $\mathbb{P}$-continuous. Then, for a.e. $\omega \in \Omega$, the following limits exist (and are independent on ω)

$$\Lambda(\mathcal{L}) := \lim_{n\to\infty} \frac{1}{n} \log \|L_\omega^n\|$$

and

$$\kappa(\mathcal{L}) := \lim_{n\to\infty} \frac{1}{n} \log ic(L_\omega^n),$$

where $ic(L_\omega^n) = \inf\{r > 0 :$ $L_\omega^n(B_B)$ can be covered with finitely many balls of radius $r\}$.

MET2

If $\kappa(\mathcal{L}) < \Lambda(\mathcal{L})$, then there exists $1 \le l \le \infty$ and a sequence of Lyapunov exponents

$$\Lambda(\mathcal{L}) = \lambda_1 > \lambda_2 > \ldots > \lambda_l > \kappa(\mathcal{L}) \quad (\text{if } 1 \le l < \infty)$$

or

$$\Lambda(\mathcal{L}) = \lambda_1 > \lambda_2 > \ldots \quad \text{and} \quad \lim_{n\to\infty} \lambda_n = \kappa(\mathcal{L}) \quad (\text{if } l = \infty);$$

and for $\mathbb{P}$-almost every $\omega \in \Omega$ there exists a unique splitting (called the *Oseledets splitting*) of B into closed subspaces

$$B = V(\omega) \oplus \bigoplus_{j=1}^{l} Y_j(\omega),$$

MET3

depending measurably on ω and such that:

1. For each $1 \leq j \leq l$, dim $Y_j(\omega)$ is finite-dimensional, Y_j is equivariant i.e. $L_\omega Y_j(\omega) = Y_j(\sigma\omega)$ and for every $y \in Y_j(\omega) \setminus \{0\}$,
$$\lim_{n\to\infty} \frac{1}{n} \log \|L_\omega^n y\| = \lambda_j.$$
2. V is equivariant i.e. $L_\omega V(\omega) \subseteq V(\sigma\omega)$ and for every $v \in V(\omega)$,
$$\lim_{n\to\infty} \frac{1}{n} \log \|L_\omega^n v\| \leq \kappa(\mathcal{L}).$$

In order to be able to apply MET, we will require that: Ω is a Borel subset of a separable, complete metric space and that

$$\text{the map} \quad \omega \to T_\omega \quad \text{has a countable range}$$

We also form a twisted cocycle. More precisely, for $\omega \in \Omega$ and $\theta \in \mathbb{C}$, we define

$$L_\omega^\theta(h) = L_\omega(e^{\theta\phi(\omega,\cdot)}h), \quad h \in BV.$$

Theorem

For $\theta \in \mathbb{R}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \int_{[0,1]} e^{\theta S_n(\omega,\cdot)}\, d\mu_\omega = \Lambda(\theta),$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ where $\Lambda(\theta)$ is a top Lyapunov exponent of the cocycle L_ω^θ, $\omega \in \Omega$.

Regularity of Λ

In order to establish the corresponding regularity property of Λ we introduce some additional assumptions. We require that:

1. norms of L_ω are uniformly bounded;
2. densities v_ω^0 are uniformly bounded from below ($d\mu_\omega = v_\omega^0\, dm$) away from zero;
3. there exists $D, \lambda > 0$ such that

$$\|L_\omega^n f\|_{BV} \le De^{-\lambda n}\|f\|_{BV},$$

for $f \in BV$, $\int f\, dm = 0$, $n \in \mathbb{N}$ and a.e. ω.

We briefly sketch the argument that shows that $\theta \mapsto \Lambda(\theta)$ is of C^2 on a neighborhood of 0.

Regularity of Λ

Key points:

1. we construct the top space as $v_\omega^0 + \mathcal{W}^\theta(\omega,\cdot)$ where $\mathcal{W}^\theta$ is a (unique) solution of $F(\theta,\mathcal{W}) = 0$, where

$$F(\theta,\mathcal{W}) = \frac{L^\theta_{\sigma^{-1}\omega}(v^0_{\sigma^{-1}\omega} + \mathcal{W}(\sigma^{-1}\omega,\cdot))}{\int (L^\theta_{\sigma^{-1}\omega}(v^0_{\sigma^{-1}\omega} + \mathcal{W}(\sigma^{-1}\omega,\cdot)))\,dm} - \mathcal{W}(\omega,\cdot) - v^0_\omega,$$

   where $\mathcal{W} \in \mathcal{S}$ and

$$\mathcal{S} := \{\mathcal{W} \colon \Omega \times I \to \mathbb{C} : \mathcal{W}(\omega,\cdot) \in BV,\ \text{esssup}_\omega \|W(\omega,\cdot)\|_{BV} < \infty\}.$$

2. $\Lambda(\theta) = \int \log|\int e^{\theta\phi(\omega,\cdot)}(v^0_\omega + \mathcal{W}^\theta(\omega,\cdot))\,dm|\,d\mathbb{P}(\omega)$;
3. for θ close to 0, the top Oseledets space of the twisted cocycle L^θ_ω is one-dimensional.

Also, $\Lambda'(0) = 0$ and $\Lambda''(0) = \Sigma^2$, where Σ^2 is a variance.

Theorem (Large deviation principle)

Assume that $\Sigma^2 > 0$. Then, there exists $\varepsilon_0 > 0$ and a function $c\colon (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ which is nonnegative, continuous, strictly convex, vanishing only at 0 and such that

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\omega(S_n(\omega, \cdot) > n\varepsilon) = -c(\varepsilon), \quad \text{for } 0 < \varepsilon < \varepsilon_0 \text{ and a.e. } \omega.$$

We can also obtain CLT.

Theorem (Central limit theorem)

If $\Sigma^2 > 0$, we have that

$$\lim_{n\to\infty} \int g(S_n(\omega, \cdot)/\sqrt{n})\, d\mu_\omega = \int g\, dN(0, \Sigma^2),$$

for g continuous and bounded and a.e. $\omega \in \Omega$.

Idea of the proof

We need to show that

$$\lim_{n\to\infty}\int e^{it\frac{S_n(\omega,\cdot)}{\sqrt{n}}}\,d\mu_\omega = e^{-\frac{t^2\Sigma^2}{2}}, \quad \text{for a.e. } \omega\in\Omega.$$

This follows by proving that:

1.
$$\lim_{n\to\infty}\int e^{it\frac{S_n(\omega,\cdot)}{\sqrt{n}}}\,d\mu_\omega = \lim_{n\to\infty}\prod_{j=0}^{n-1}\lambda_{\sigma^j\omega}^{\frac{it}{\sqrt{n}}},$$

   where

$$\lambda_\omega^\theta = \int L^\theta_{\sigma^{-1}\omega}(v^0_{\sigma^{-1}\omega}+\mathcal{W}^\theta(\sigma^{-1}\omega,\cdot))\,dm =: H(\theta,\mathcal{W}^\theta)(\omega);$$

2. by Taylor expansion of $\theta\to H(\theta,W^\theta)$ around 0:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\lambda_{\sigma^j\omega}^{\frac{it}{\sqrt{n}}} = -\frac{t^2\Sigma^2}{2}.$$

Local central limit theorem

Theorem (Local central limit theorem)

Assume that $\Lambda(it) < 0$ for $t \neq 0$. Then, for a.e. ω and every bounded interval $J \subset \mathbb{R}$, we have

$$\lim_{n\to\infty} \sup_{s\in\mathbb{R}} \left| \Sigma\sqrt{n}\mu_\omega(s + S_n g \in J) - \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2n\Sigma^2}} |J| \right| = 0,$$