Spectral characterization of nonuniform behaviour

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Uniform exponential dichotomy

We first recall the notion of a (uniform) exponential dichotomy.

Let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of bounded operators on a Banach space $X=(X,\|\cdot\|)$. For each $m,n\in\mathbb{Z}$ such that $m\geq n$, we define

$$\mathcal{A}(m,n)=\begin{cases}A_{m-1}\cdots A_n & \text{if } m>n,\\ \mathrm{Id} & \text{if } m=n.\end{cases}$$

We say that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a *uniform exponential dichotomy* if there exist projections P_m for $m\in\mathbb{Z}$ satisfying

$$P_{m+1}A_m=A_mP_m \quad \text{for } m\in\mathbb{Z} \tag{1}$$

such that each map $A_m|\ker P_m\colon \ker P_m\to\ker P_{m+1}$ is invertible and constants $\lambda, D>0$ such that:

$$\|\mathcal{A}(m,n)P_n\| \le De^{-\lambda(m-n)} \quad \text{for } m \ge n$$

and

$$\|\mathcal{A}(m,n)Q_n\| \le De^{-\lambda(n-m)} \quad \text{for } m \le n,$$

where $Q_n = \mathrm{Id} - P_n$ and

$\mathcal{A}(m,n) = (\mathcal{A}(n,m)|\ker P_m)^{-1} : \ker P_n \to \ker P_m$ for $m < n$.

Some consequences of the existence of uniform exponential dichotomy:

1. existence and regularity of invariant stable and unstable manifolds;
2. linearization of dynamics;
3. center manifold theory.

Nonuniform exponential dichotomy

We say that $(A_m)_{m\in\mathbb{Z}}$ admits a *nonuniform exponential dichotomy* if there exist projections P_m for $m \in \mathbb{Z}$ satisfying (1) such that each map $A_m| \ker P_m \colon \ker P_m \to \ker P_{m+1}$ is invertible and there exist constants $\lambda, D > 0$ and $\varepsilon \geq 0$ such that

$$\|\mathcal{A}(m,n)P_n\| \leq De^{-\lambda(m-n)+\varepsilon|n|} \quad \text{for} \quad m \geq n$$

and

$$\|\mathcal{A}(m,n)Q_m\| \leq De^{-\lambda(n-m)+\varepsilon|n|} \quad \text{for} \quad m \leq n,$$

where $Q_n = \mathrm{Id} - P_n$.

Example

Let $\mathcal{A}$ be a cocycle with generator A over ergodic measure preserving dynamical system $(X, \mathcal{B}, \mu, f)$ whose all Lyapunov exponents are nonzero. Then, for a.e. $x \in X$, the sequence $(A_n)_{n \in \mathbb{Z}}$ defined by $A_n = A(f^n(x))$, $n \in \mathbb{Z}$ admits a nonuniform exponential dichotomy.

We refer to:

L. Barreira and C. Valls, *Stability of Nonautonomous Differential Equations*, Springer, 2008,

for a detailed descriptions of consequences of the notion of nonuniform exponential dichotomy.

Exponential dichotomies for a sequence of norms

Let $\|\cdot\|_m$, $m \in \mathbb{Z}$ be a sequence of norms on X. We say that $(A_m)_{m\in\mathbb{Z}}$ in $B(X)$ admits an *exponential dichotomy with respect to the sequence of norms* $\|\cdot\|_m$ if: there exist projections $P_m \colon X \to X$ for each $m \in \mathbb{Z}$ satisfying (1) and such that each map $A_m| \ker P_m \colon \ker P_m \to \ker P_{m+1}$ is invertible and there exist constants $\lambda, D > 0$ such that for each $x \in X$ we have

$$\|\mathcal{A}(m,n)P_n x\|_m \le De^{-\lambda(m-n)}\|x\|_n \quad \text{for} \quad m \ge n$$

and

$$\|\mathcal{A}(m,n)Q_n x\|_m \le De^{-\lambda(n-m)}\|x\|_n \quad \text{for} \quad m \le n,$$

where $Q_n = \mathrm{Id} - P_n$.

Connection

Proposition

The following properties are equivalent:

1. $(A_m)_{m\in\mathbb{Z}}$ *admits a nonuniform exponential dichotomy;*
2. $(A_m)_{m\in\mathbb{Z}}$ *admits an exponential dichotomy with respect to a sequence of norms* $\|\cdot\|_m$ *satisfying*

$$\|x\| \le \|x\|_m \le Ce^{\varepsilon|m|}\|x\|, \quad m\in\mathbb{Z},\ x\in X$$

for some constants $C>0$ *and* $\varepsilon \ge 0$.

Admissibility

Let

$$l^\infty = \{\mathbf{x} = (x_m)_{m\in\mathbb{Z}} \subset X : \|\mathbf{x}\|_\infty := \sup_{m\in\mathbb{Z}} \|x_m\|_m < \infty\}.$$

Theorem

The following properties are equivalent:

1. *the sequence $(A_m)_{m\in\mathbb{Z}}$ admits an exponential dichotomy with respect to the sequence of norms $\|\cdot\|_m$;*
2. *for each $\mathbf{y} = (y_m)_{m\in\mathbb{Z}} \in l^\infty$, there exists a unique $\mathbf{x} = (x_m)_{m\in\mathbb{Z}} \in l^\infty$ such that*

$$x_{m+1} - A_m x_m = y_{m+1}, \quad m \in \mathbb{Z}.$$

Corresponding operator

We define a linear operator $T\colon \mathcal{D}(T) \subset l^\infty \to l^\infty$ by

$$(T\mathbf{x})_{m+1} = x_{m+1} - A_m x_m, \quad m \in \mathbb{Z},$$

where

$$\mathcal{D}(T) = \{\mathbf{x} \in l^\infty : T\mathbf{x} \in l^\infty\}.$$

Then, T is closed and thus $\mathcal{D}(T)$ is a Banach space with respect to the norm

$$\|\mathbf{x}\|_T := \|\mathbf{x}\|_\infty + \|T\mathbf{x}\|_\infty$$

and $T\colon (\mathcal{D}(T), \|\cdot\|_T) \to l^\infty$ is a bounded operator. Then, exponential dichotomy with respect to a sequence of norms $\|\cdot\|_m$ is equivalent to the invertibility of T.

Admissibility II

Instead of spaces (l^∞, l^∞) we could also use the following pairs:

$$Y_1 = l^p \quad \text{and} \quad Y_2 = l^q \quad \text{for} \quad 1 \le q \le p < +\infty,$$

$$Y_1 = l^\infty \quad \text{and} \quad Y_2 = c_0,$$

$$Y_1 = c_0 \quad \text{and} \quad Y_2 = l^p \quad \text{for} \quad 1 < p < +\infty,$$

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$$Y_1 = l^\infty \quad \text{and} \quad Y_2 = l^p \quad \text{for} \quad 1 < p < +\infty,$$

where

$$c_0 := \{\mathbf{x} = (x_m)_{m\in\mathbb{Z}} \subset X : \lim_{|m|\to\infty} \|x_m\|_m = 0\}.$$

Dynamics on the half-line

Theorem

The following properties are equivalent:

1. *the sequence* $(A_m)_{m\geq 0}$ *admits an exponential dichotomy with respect to the sequence of norms* $\|\cdot\|_m$*;*
2. *there exists a closed subspace* $Z \subset X$ *such for each* $\mathbf{y} = (y_m)_{m\geq 0} \in l^\infty$ *with* $y_0 = 0$*, there exists a unique* $\mathbf{x} = (x_m)_{m\geq 0} \in l^\infty$ *such that* $x_0 \in Z$ *and*

$$x_{m+1} - A_m x_m = y_{m+1}, \quad m \geq 0.$$

Robustness

Theorem

Let $(A_m)_{m\in\mathbb{Z}}$ and $(B_m)_{m\in\mathbb{Z}}$ be two sequences in $B(X)$ such that:

1. *the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy;*
2. *there exists $c > 0$ such that*

$$\|A_m - B_m\| \leq ce^{-\varepsilon|m|}, \quad m \in \mathbb{Z}.$$

If $c > 0$ is sufficiently small, then the sequence $(B_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy.

Parametrized robustness

Assume that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy. Let I be a Banach space and assume that $B_n\colon I \to B(X)$, $n \in \mathbb{Z}$ is a sequence of maps. Then, if B_n are small, for each $\lambda \in I$ the sequence $(A_n + B_n(\lambda))_{n\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy. Moreover, if:

1. B_n are Lipschitz, then the associated projections are also Lipschitz;
2. B_n are smooth, then the associated projections are also smooth.

Trichotomy

Theorem

The following properties are equivalent:

1. (a) the sequence $(A_m)_{m\geq 0}$ admits an exponential dichotomy on $\mathbb{Z}^+$ with respect to the sequence of norms $\|\cdot\|_m$, $m\geq 0$ and projections P_m^+, $m\geq 0$;
   (b) the sequence $(A_m)_{m\leq 0}$ admits an exponential dichotomy on $\mathbb{Z}^-$ with respect to the sequence of norms $\|\cdot\|_m$, $m\leq 0$ and projections P_m^-, $m\leq 0$;
   (c) $X = \operatorname{Im} P_0^+ + \operatorname{Ker} P_0^-$;
2. for each $\mathbf{y}=(y_m)_{m\in\mathbb{Z}}\in l^\infty$, there exists $\mathbf{x}=(x_m)_{m\in\mathbb{Z}}\in l^\infty$ such that

$$x_{m+1}-A_m x_m = y_{m+1}, \quad m\in\mathbb{Z}.$$

Shadowing

Let f be a C^1-diffeomorphism of a compact Riemannian manifold M. We say that f has a *Lipschitz shadowing property* if there exist $d_0 > 0$ and $L > 0$ such that for any sequence $(x_n)_{n\in\mathbb{Z}} \subset M$ such that $d(f(x_n), x_{n+1}) \leq d \leq d_0$ for every $n \in \mathbb{Z}$, there exists $x \in M$ such that $d(f^n(x), x_n) \leq Ld$ for every $n \in \mathbb{Z}$.

Example

1. Anosov diffeomorphism has Lipschitz shadowing property;
2. every structurally stable diffeomorphism has Lipschitz shadowing property.

Shadowing II

Theorem (Pilyugin-Tikhomirov, 2010)

Every diffeomorphism that has Lipschitz shadowing property is structurally stable.

Idea of the proof: We need to verify that for any $x \in M$,

$$T_x M = S(x) + U(x),$$

where

$$S(x) = \{v \in T_x M : \lim_{n \to \infty} \|Df^n(x)v\| = 0\}$$

and

$$U(x) = \{v \in T_x M : \lim_{n \to \infty} \|Df^{-n}(x)v\| = 0\}.$$

Then, using Lipschitz shadowing one varifies that for each $\mathbf{y} = (y_n)_{n \in \mathbb{Z}} \in l^\infty$, there exists $\mathbf{x} = (x_n)_{n \in \mathbb{Z}} \in l^\infty$ such that

$$x_{n+1} - A_n x_n = y_{n+1}, \quad n \in \mathbb{Z},$$

where $A_n = Df(f^n(x))$, $n \in \mathbb{Z}$. Using the theorem on the trichotomy slide, we obtain the desired conclusion.

Some geneneralizations: Todorov, D.