Almost sure invariance principle for random piecewise expanding maps

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Research project

Part of the ongoing research project which deals with establishing limit theorems for random dynamical systems. Two approaches:

1. *martingale approach*: we will describe how to obtain almost sure invariance principle for random piecewise expanding maps using this technique;
2. *spectral approach*: heavily relies on the multiplicative ergodic theorem. We have obtained (local) central limit theorem and large deviations principle for random piecewise expanding maps. **will be briefly discussed in the talk of Cecilia González-Tokman.**

Almost sure invariance principle

Almost sure invariance principle is a strong result that ensures that the trajectories of a process can be matched with the trajectories of a Brownian motion. Contributions:

-originated in the work of **Philipp** and **Stout**;

-independent or weakly dependent one-dimensional processes (**Denker, Philipp, Hofbauer, Keller**)

-for higher dimensional processes (**Berkes, Philipp, Einmahl, Zaitsev**)

-hyperbolic dynamical systems (**Melbourne, Nicol, Gouëzel**)

-sequential dynamical systems (**Haydn, Nicol, Török, Vaienti**)

-random dynamical systems (**Kifer**' 98 mentioned without proof)

Piecewise expanding maps

Let $I=[0,1]$ denote the unit interval equipped with Borel σ-algebra $\mathcal{B}$ and a Lebesgue measure m. We say that $f \colon I \to I$ is a *piecewise expanding map* if there exists a partition

$$0=x_0<x_1<\ldots<x_{k-1}<x_k=1$$

and $\alpha>1$ such that:

1. restriction $f|_{(x_{i-1},x_i)}$ is a C^1 function which can be extended to a C^1 function on $[x_{i-1},x_i]$;
2. $|f'(x)|\geq \alpha$ for $x\in (x_{i-1},x_i)$;
3. $g(x)=\frac{1}{|f'(x)|}$ is a function of bounded variation.

Random piecewise expanding maps

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\sigma : \Omega \to \Omega$ be an invertible $\mathbb{P}$-preserving transformation. We will assume that $\mathbb{P}$ is ergodic. Let $f_\omega \colon [0,1] \to [0,1]$, $\omega \in \Omega$ be a collection of piecewise expanding maps on $[0,1]$. The associated *skew product transformation* $\tau \colon \Omega \times [0,1] \to \Omega \times [0,1]$ is defined by

$$\tau(\omega, x) = (\sigma(\omega), f_\omega(x)).$$

Let $\mathcal{L}_\omega$ be the transfer operator associated to f_ω. For each $n \in \mathbb{N}$ and $\omega \in \Omega$, set

$$f_\omega^n = f_{\sigma^{n-1}(\omega)} \circ \cdots \circ f_\omega \quad \text{and} \quad \mathcal{L}_\omega^{(n)} = \mathcal{L}_{\sigma^{n-1}(\omega)} \circ \cdots \circ \mathcal{L}_\omega.$$

Assumptions and consequences I

Let BV denote the space of all functions of bounded variation on $[0,1]$. Recall that BV is a Banach space with respect to the norm $\|\cdot\|_{BV} = \text{var}(\cdot) + \|\cdot\|_1$. We assume the following:

1. there exists $C > 0$ such that $\|\mathcal{L}_\omega \phi\|_{BV} \le C\|\phi\|_{BV}$ for $\phi \in BV$ and a.e. $\omega \in \Omega$;
2. there exist $K, \lambda > 0$ such that for a.e. $\omega \in \Omega$ and $n \in \mathbb{N}$,
$$\|\mathcal{L}_\omega^n \phi\|_{BV} \le Ke^{-\lambda n}\|\phi\|_{BV} \quad \text{for } \phi \in BV \text{ and } \int \phi\, dm = 0;$$
3. there exists $N \in \mathbb{N}$ such that for each $a > 0$ and any sufficiently large $n \in \mathbb{N}$, there exists $\alpha_* > 0$ such that
$$\operatorname{essinf} \mathcal{L}_\omega^{Nn} h \ge \alpha_* \|h\|_1, \quad \text{for every } h \in C_a \text{ and a.e. } \omega \in \Omega,$$

Assumptions and consequences II

where

$$C_a := \left\{\phi \in BV : \phi \geq 0 \quad \text{and} \quad \text{var}(\phi) \leq a \int \phi \, dm\right\}.$$

Then, there exists a unique acim μ (with respect to $\mathbb{P} \times m$) for τ with density h such that $h_\omega := h(\omega, \cdot) \in BV$ and

$$\text{esssup}_{\omega \in \Omega} \|h_\omega\|_{BV} < \infty.$$

Finally,

$$\text{essinf}\, h_\omega \geq \alpha_* \quad \text{for a.e. } \omega \in \Omega.$$

Examples

For f piecewise expanding, let $\delta(f) = \text{essinf}_{x\in[0,1]}|f'|$ and denote by $N(f)$ the number of intervals of monoticity of f. Assume that

$$\text{esssup}_\omega N(f_\omega) < \infty, \ \text{essinf}_\omega \delta(f_\omega) > 1, \ \text{esssup}_\omega |f''_\omega|_\infty < \infty$$

and $\text{esssup}_\omega \text{var}(1/|f'_\omega|) < \infty$. Under those conditions, our assumption 1. holds. On the other hand, 2. and 3. are implied by the following conditions:

1. there exists $N \in \mathbb{N}$ and $\alpha^N \in (0,1)$ and $K^N > 0$ such that

   $$\text{var}(\mathcal{L}_\omega^N \phi) \leq \alpha^N \text{var}(\phi) + K^N \|\phi\|_1 \quad \text{for } \phi \in BV \text{ and a.e. } \omega \in \Omega;$$

2. for every subinterval $J \subset I$ there exists $m = m(J)$ s.t. for a.e. $\omega \in \Omega$, $f_\omega^m(J) = I$.

Main result

We consider an observable $\psi \colon \Omega \times I \to \mathbb{R}$ such that

$$\mathrm{esssup}_{\omega \in \Omega} \|\psi_\omega\|_{BV} < \infty, \quad \text{where } \psi_\omega := \psi(\omega, \cdot).$$

Let

$$\tilde{\psi}_\omega := \psi_\omega - \int \psi_\omega \, d\mu_\omega, \quad \omega \in \Omega,$$

where $d\mu_\omega = h_\omega dm$. Finally, set

$$\Sigma^2 = \int \tilde{\psi}(\omega, x)^2 \, d\mu(\omega, x) + 2 \sum_{n=1}^{\infty} \int \tilde{\psi}(\omega, x) \tilde{\psi}(\tau^n(\omega, x)) \, d\mu(\omega, x).$$

Theorem

One of the following two alternatives holds:

1. *either* $\Sigma = 0$*, and this is equivalent to the existence of* $\phi \in L^2(\Omega \times [0,1])$ *such that*

$$\tilde{\psi} = \phi - \phi \circ \tau.$$

2. *or* $\Sigma^2 > 0$ *and in this case for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$ *and* $d > 0$*, by enlarging the probability space* $([0,1], \mathcal{B}, \mu_\omega)$ *if necessary, it is possible to find a sequence* $(Z_k)_k$ *of independent centered (i.e. of zero mean) Gaussian random variables such that*

$$\sup_{1 \le k \le n} \left| \sum_{i=1}^{k} (\tilde{\psi}_{\sigma^i \omega} \circ f_\omega^i) - \sum_{i=1}^{k} Z_i \right| = o(n^{1/4+d}), \quad \mu_\omega - a.s.$$

Idea of the proof

For $\omega \in \Omega$ and $k \in \mathbb{N}$, let

$$\mathcal{T}_\omega^k := (f_\omega^k)^{-1}(\mathcal{B}).$$

Set

$$M_n = \tilde{\psi}_{\sigma^n\omega} + G_n - G_{n+1} \circ f_{\sigma^n\omega}, \quad n \geq 0,$$

where $G_0 = 0$ and

$$G_{k+1} = \frac{\mathcal{L}_{\sigma^k\omega}(\tilde{\psi}_{\sigma^k\omega} h_{\sigma^k\omega} + G_k h_{\sigma^k\omega})}{h_{\sigma^{k+1}\omega}}, \quad k \geq 0.$$

Then,

1. $\mathcal{T}_\omega^{k+1} \subset \mathcal{T}_\omega^k$;
2. $M_n \circ f_\omega^n$ is $\mathcal{T}_\omega^n$-measurable;
3. $\mathbb{E}_\omega(M_n \circ f_\omega^n | \mathcal{T}_\omega^{n+1}) = 0$.

Theorem (Cuny and Merlevède)

Let $(X_n)_n$ *be a sequence of square integrable random variables adapted to a non-increasing filtration* $(\mathcal{G}_n)_n$. *Assume that* $\mathbb{E}(X_n|\mathcal{G}_{n+1}) = 0$ *a.s.,* $\sigma_n^2 := \sum_{k=1}^n \mathbb{E}(X_k^2) \to \infty$ *when* $n \to \infty$ *and that* $\sup_n \mathbb{E}(X_n^2) < \infty$. *Moreover, let* $(a_n)_n$ *be a non-decreasing sequence of positive numbers such that the sequence* $(a_n/\sigma_n^2)_n$ *is non-increasing,* (a_n/σ_n) *is non-decreasing and such that :*

1. $$\sum_{k=1}^{n} (\mathbb{E}(X_k^2|\mathcal{G}_{k+1}) - \mathbb{E}(X_k^2)) = o(a_n) \quad a.s.;$$
2. $$\sum_{n\geq 1} a_n^{-v}\mathbb{E}(|X_n|^{2v}) < \infty \quad \text{for some } 1 \leq v \leq 2.$$

Theorem (continued)

Then, enlarging our probability space if necessary, it is possible to find a sequence $(Z_k)_k$ *of independent centered (i.e. of zero mean) Gaussian variables with* $\mathbb{E}(X_k^2) = \mathbb{E}(Z_k^2)$ *such that*

$$\sup_{1 \le k \le n} \left| \sum_{i=1}^{k} X_i - \sum_{i=1}^{k} Z_i \right| = o((a_n(|\log(\sigma_n^2/a_n)| + \log \log a_n))^{1/2}), \quad a.s.$$

Set $X_n = M_n \circ f_\omega^n$ and $\mathcal{G}_n = \mathcal{T}_\omega^n$.

1.
$$\sigma_n^2 = \mathbb{E}_\omega\Big(\sum_{k=0}^{n-1} X_k\Big)^2 \sim \mathbb{E}_\omega\Big(\sum_{k=0}^{n-1} \tilde{\psi}_{\sigma^k\omega} \circ f_\omega^k\Big)^2 \sim n\Sigma^2;$$

2. using Sprindzuk theorem (or Gal-Koksma inequality),
$$\sum_{k=1}^{n} (\mathbb{E}(X_k^2|\mathcal{G}_{k+1}) - \mathbb{E}(X_k^2)) = o(n^{1/2+d}) \quad a.s.;$$

3. finally,
$$\sum_{n\geq 1} \frac{\mathbb{E}(|X_n|^4)}{n^{1+2d}} < \infty.$$

This yields an ASIP for $(M_n \circ f_\omega^n)_n$ which implies ASIP for $(\tilde{\psi}_{\sigma^n\omega} \circ f_\omega^n)_n$.