Analytic structure of an Ansatz nonperturbative quark propagator, and phenomenological restrictions

Talk presented at **ACHT 2016 workshop**
Non-perturbative aspects of Quantum Field Theory

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Hrvatska zaklada za znanost

Overview

Introduction

- Many QFT studies (on lattice, large majority of Schwinger–Dyson istudies, etc.) are **not done in the physical, Minkowski spacetime, but in 4-dim Euclidean space.**
- $\Rightarrow$ **Situation with Wick rotation (relating Minkowski with Euclid) must be under control, but this is highly nontrivial in the nonperturbative case – most importantly, the nonperturbative QCD.**
- **Do nonperturbative Green's functions permit Wick rotation?**
- **For solving Bethe-Salpeter equation and calculation of processes, extrapolation to complex momenta is necessary. $\Rightarrow$ Knowledge of the analytic behavior in the whole complex plane is needed.**
- **Very complicated matters $\Rightarrow$ studies of Ansatz forms are instructive and can be helpful to *ab initio* studies of nonperturbative QCD Green's functions. ... and vice versa of course ...**

How Schwinger–Dyson approach generates quark propagators

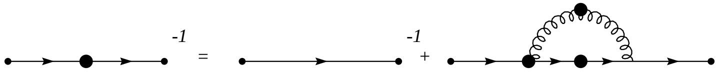
- Schwinger-Dyson (SD) approach: ranges from solving SD equations for Green's functions of non-perturbative QCD *ab initio*, to higher degrees of phenomenological modeling, *esp.* in applications including $T, \mu > 0$.

  e.g., [Alkofer, v.Smekal Phys. Rept. 353 (2001) 281], and [Roberts, Schmidt Prog.Part.Nucl.Phys. 45 (2000)S1]

- SD approach to quark-hadron physics = nonpertubative, covariant bound state approach with strong connections with QCD.

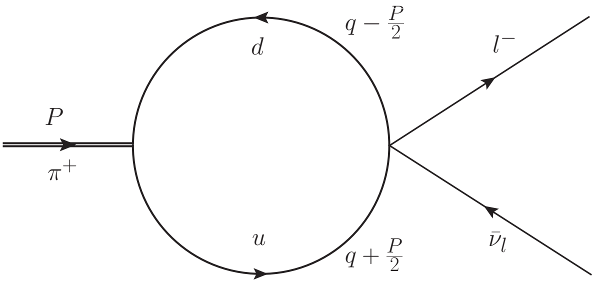
The "gap" Schwinger–Dyson equation for the quark propagator:

$$S^{-1}(p) = \not{p} - m - iC_F g^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu S(k) \Gamma^\nu(k,p) G_{\mu\nu}(p-k) \; . \tag{1}$$



Already just $S(p)$ enables calculating an observable - pion decay constant

... since close to the chiral limit, $\Gamma_\pi^{BS} \approx -\frac{2B(q^2)}{f_\pi}\gamma_5$ is a good **approximation**



$$f_\pi = i\frac{N_c}{2}\frac{1}{M_\pi^2}\int \frac{d^4q}{(2\pi)^4}\mathrm{tr}\left(\not{P}\gamma_5 S(q+\frac{P}{2})\left(-\frac{2B(q^2)}{f_\pi}\gamma_5\right)S(q-\frac{P}{2})\right)$$

$$\Gamma(\pi^+ \to e^+\nu_e) = \frac{1}{4\pi}G^2 f_\pi^2 \cos^2\theta_c(1-\frac{m_e^2}{M_\pi^2})^2 M_\pi m_e^2 \ .$$

Dynamically generated nonperturbative quark propagator

In principle, SD Eq. (1) yields the nonperturbatively dressed quark propagator

$$S(q) = \frac{A(q^2)\not{q} + B(q^2)}{A^2(q^2)q^2 - B^2(q^2)} = Z(-q^2)\frac{\not{q} + M(-q^2)}{q^2 - M^2(-q^2)} = -\sigma_V(-q^2)\not{q} - \sigma_S(-q^2)$$

[$M(x)$ = dressed quark mass function, $Z(x)$ = wave-function renormalization]

- One usually gets just a model solution for the propagator $S(q)$, since one usually simplifies SD Eq. (1) by approximations & modeling! For example:
- 1.) rainbow(-ladder) for the dressed quark–gluon vertex: $\Gamma^\mu(k,p) \to \gamma^\mu$,
- 2.) $A(k^2) = 1$, 3.) various model Ansätze for the dressed gluon propagator, $g^2 G^{\mu\nu}(k) \propto \alpha_s^{\mathrm{eff}}(k^2)/k^2$, so that, e.g., Eq. (1) yields

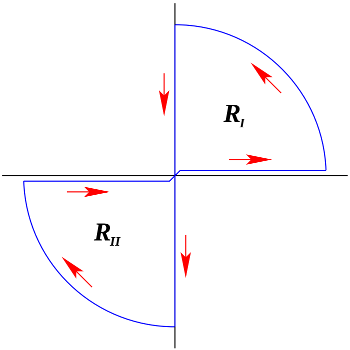
$$M(-p^2) = m + I(-p^2) = m + \int d^4k\, f(-k^2) g(-(k-p)^2)$$

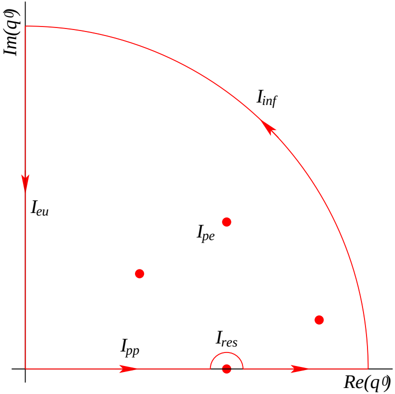
$$g(-k^2) = 4\pi C_F \frac{\alpha_s^{\mathrm{eff}}(-k^2)}{(-k^2)} \quad f(-k^2) = \frac{3i}{(2\pi)^4}\, \frac{M(k^2)}{k^2 - M^2(k^2)}$$

$$I(-p^2) = \int d^3k \int dk^0\, f(-(k^0)^2 + |\mathbf{k}|^2) g(-(k^0 - p^0)^2 + |\mathbf{k} - \mathbf{p}|^2)$$

Wick rotation

- Even with such approxim. & modeling, solving SD equations like Eq. (1), and related calculations (e.g., of f_π) with Green's functions like $S(q)$, are technically very hard to do in the physical, Minkowski space-time.
- Thus, additional simplification is sought by transforming to QFT in 4-dim. Euclidean space by the Wick rotation to the imaginary time component: $q^0 \to i\,q^0$.





The issue of singularities is more problematic than in the perturbative case!

Example: simple separable approximation for low-E QCD

$$g^2 G^{\mu\nu} = g^{\mu\nu} D(p-k) \approx g^{\mu\nu} \left[D_0 f_0(-p^2) f_0(-k^2) - D_1 (p\cdot k) f_1(-p^2) f_1(-k^2) \right]$$

produces complicated singularity structure in the complex plane $z = -k^2$

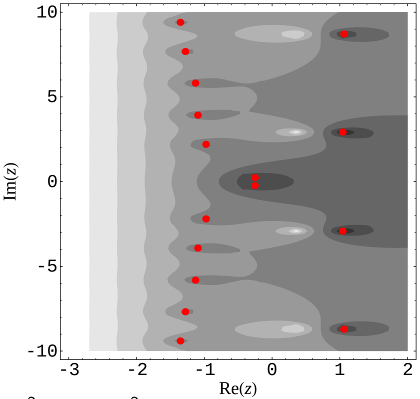
The "gap" SD equation $\Rightarrow$

$$\begin{aligned} A(p^2) &= 1 + a f_1(-p^2) \\ B(p^2) &= m + b f_0(-p^2) \end{aligned}$$

Typical phen.succes. Ansatz

$$\begin{aligned} f_0(x) &= e^{-x/\Lambda_0^2} \\ f_1(x) &= \frac{1+e^{-x_0/\Lambda_1^2}}{1+e^{-(x-x_0)/\Lambda_1^2}} \end{aligned}$$

produces **poles** which are **obstacles to (any) Wick rotation!**



The contour plot of $z \mapsto \log |A(z)^2 z + B(z)^2|$ close to the origin. The red points are solutions of the equation $A(z)^2 z + B(z)^2 = 0$ and extend much further than the depicted region.

General properties a quark propagator should have:

- $S(q) \to S_{\rm free}(q)$ because of asymptotic freedom
  $\Rightarrow \sigma_{V,S}(-q^2) \to 0$ for $q^2 \in \mathbb{C}$ and $q^2 \to \infty$
- $\sigma_{V,S}(-q^2) \to 0$ cannot be analytic over the whole complex plane
- positivity violating spectral density $\leftrightarrow$ confinement

Try Ansätze of the form (meromorphic parametrizations, like Alkofer&al. [1]):

$$S(p) = \frac{1}{Z_2} \sum_{j=1}^{n_p} r_j \left(\frac{\not{p} + a_j + ib_j}{p^2 - (a_j + ib_j)^2} + \frac{\not{p} + a_j - ib_j}{p^2 - (a_j - ib_j)^2} \right)$$

$$\text{Dressing f'nctns are thus} \qquad \sigma_V(x) = \frac{1}{Z_2} \sum_{j=1}^{3} \frac{2r_j(x + a_j^2 - b_j^2)}{(x + a_j^2 - b_j^2)^2 + 4a_j^2 b_j^2}$$

$$\text{(e.g., see Ref. [1]) and} \qquad \sigma_S(x) = \frac{1}{Z_2} \sum_{j=1}^{3} \frac{2r_j a_j(x + a_j^2 + b_j^2)}{(x + a_j^2 - b_j^2)^2 + 4a_j^2 b_j^2}$$

Constraints:

$$\sum_{j=1}^{n_p} r_j = \frac{1}{2} \qquad\qquad \sum_{j=1}^{n_p} r_j a_j = 0$$

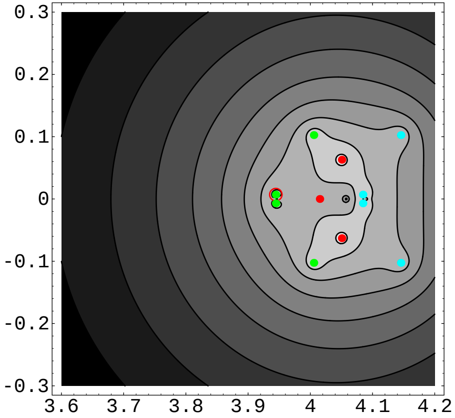
2CC quark propagator - with just 2 pairs of complex conj. poles

Parameters (yielding 2CC propagator of Alkofer & al.): $n_p = 2$, $a_1 = 0.351$, $a_2 = -0.903$, $b_1 = 0.08$, $b_2 = 0.463$, $r_1 = 0.360$, $r_2 = 0.140$, $Z_2 = 1$.

- **Naive Wick rotation impossible**
- **In calculated quantities, proliferation of poles from propagators:**
- **contour plot of the f_π integrand - the complex function**

  $q^0 \mapsto |\mathrm{trace}(q^0, \xi)|$
  for $\xi = 4.0$.

  Dots are the poles of this function.



2CC quark propagator Ansatz (the one by Alkofer & al.)

For $\xi = 0.5$ the integrals over q^0 are:

$$
\begin{aligned}
I_{\rm pp} &= 0.184102 \\
I_{\rm res} &= 0.0166534i \\
I_{\rm inf} &\approx 0 \\
I_{\rm eu} &= 0.0230837i \\
I_{\rm pe} &= 0.184102 + 0.0397371i
\end{aligned}
$$

Cauchy's residue theorem satisfied: $|I_{\rm pp} + I_{\rm res} + I_{\rm eu} - I_{\rm pe}| = 1.4 \cdot 10^{-9}$

$f_\pi = 0.071~\mathrm{GeV}$ = Euclidean result (from $I_{\rm eu}$), cannot be reproduced in Minkowski space (i.e., from $I_{\rm pp} + I_{\rm res}$).

Quark propagators with CC poles – "popular" ... **but, ∃ objections besides not alowing naive Wick rotation: possible problems for causality and unitarity of the theory.**
Also, Benić *et al.*, PRD 86 (2012) 074002, have shown that CC poles of the quark propagator cause thermodynamical instabilities at nonvanishing temperature and density.

Quark propagators with singularities only on the real axis

Investigations (e.g. [1]Alkofer et al., Phys.Rev. D70 (2004) 014014) based on the available *ab initio* SD and lattice results indicate: one should use Ansätze exhibiting singularities (isolated poles or branch cuts) only for real momenta.

An example of a propagator with singularities only on the real axis:

$$S(q) = \frac{A(-q^2)\not{q} + B(-q^2)}{A^2(-q^2)q^2 - B^2(-q^2)} = Z(-q^2)\frac{\not{q} + M(-q^2)}{q^2 - M^2(-q^2)}$$

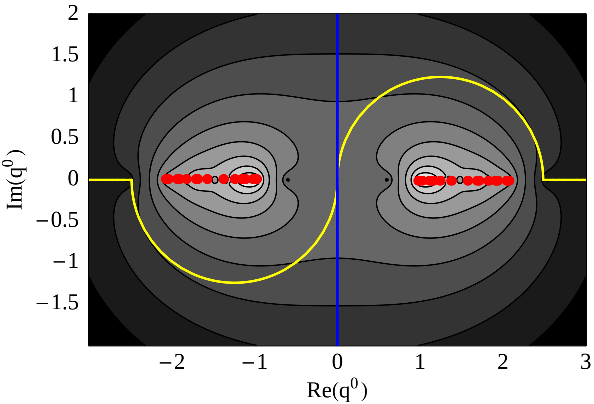
$$M(z) = \ln\left(\frac{(z+a_1)(z+a_2)(z+a_3)}{(z+b_1)(z+b_2)(b+b_3)}\right) \quad \& \quad Z(z) = 1$$

$$\sigma(z) = \frac{1}{z + M^2(z)} \ .$$

The pertinent set of parameters:

a_1	a_2	a_3	b_1	b_2	b_3
3.00278	1.78718	0.554466	2.92927	2.01400	0.401162

This example of propagator with singularities only on the real axis: f_π calculation



Contour plot of the integrand and path of integration in the complex q^0–plane.

$\Rightarrow$ $\boxed{2\frac{f_\pi - f_\pi}{f_\pi + f_\pi} \sim 10^{-5}}$

but $f_\pi = 67\ \mathrm{MeV}$ too small $\Rightarrow$ use **3R quark propagator**

3R quark propagator - 3 poles on the real axis

Parameters (yielding 3R propagator of Alkofer & al., and larger f_π): $n_p = 3$, $a_1 = 0.341$, $a_2 = -1.31$, $a_3 = -1.35919$, $b_1 = 0$, $b_2 = 0$, $b_3 = 0$, $r_1 = 0.365$, $r_2 = 1.2$, $r_3 = -1.065$, $Z_2 = 0.982731$.

$\Rightarrow$ No obstacles to Wick rotation $\Rightarrow$ π decay constant calculated equivalently

*** in the Minkowski space:**

$$f_\pi^2 = -i \frac{N_c}{4\pi^3 M_\pi^2} \int_0^\infty \xi^2 \, d\xi \int_{-\infty}^{+\infty} dq^0 \, B(q^2) \, \mathrm{tr} \left(\not{P} \gamma_5 S(q + \frac{P}{2}) \gamma_5 S(q - \frac{P}{2}) \right)$$

where $q^2 = (q^0)^2 - \xi^2$, $\xi = |\mathbf{q}|$, **and** $q \cdot P = M_\pi q^0$, **OR**

*** in the Euclidean space:**

$$f_\pi^2 = \frac{3}{8\pi^3 M_\pi^2} \int_0^\infty dx \, x \int_0^\pi d\beta \, \sin^2\beta \, B(q^2) \mathrm{tr} \left(\not{P} \gamma_5 S(q + \frac{P}{2}) \gamma_5 S(q - \frac{P}{2}) \right)$$

where $q^2 = -x$ **and** $q \cdot P = -iM_\pi \sqrt{x} \cos\beta$.

3R quark propagator Ansatz (Alkofer & al.)

Numerically, for $\xi = 0.5$

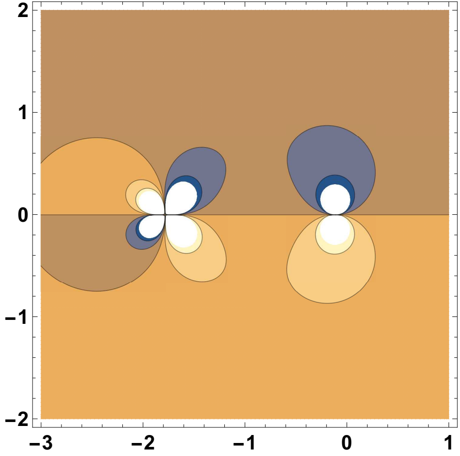
$$|I_{\rm pp} + I_{\rm res} + I_{\rm inf} + I_{\rm eu} - I_{\rm pe}| \sim 10^{-8}$$

Particular integrals for $\xi = 0.5$

$$\begin{aligned} I_{\rm pp} &\approx 4 \cdot 10^{-12} \,, \\ I_{\rm res} &= -0.0221314i \\ I_{\rm inf} &\approx 4 \cdot 10^{-12} \,, \\ I_{\rm eu} &= 0.0221314i \\ I_{\rm pe} &= 0 \end{aligned}$$

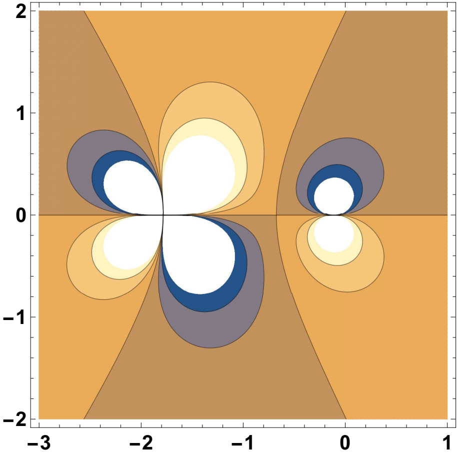
$\Rightarrow f_\pi = 0.072\ \mathrm{GeV}$ in both Euclidean and Minkowski space.

3R Quark Propagator



Contour plot of $\mathrm{Im}(\sigma_V(z))$ in the complex z-plane:
the first two poles on the real axis are very close - "glued together" .

3R Quark Propagator



A similar contour plot of $\mathrm{Im}(\sigma_S(z))$ in the complex z-plane.

3R Quark Propagator

Spectral representation of the 3R quark propagator is defined by the spectral density

$$\rho(\sigma^2) = \sum_{j=1}^{3} A_j^{-1} \delta(\sigma^2 - M_j^2)$$

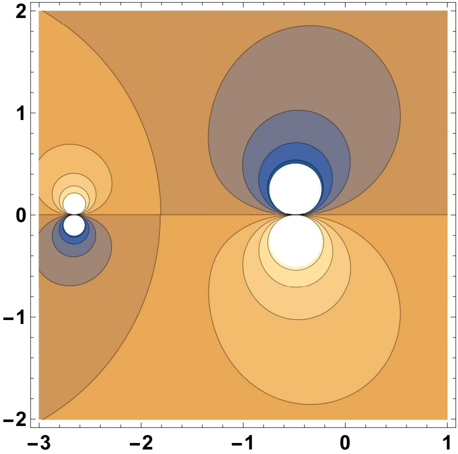
where $A_1 = 1.35$, $A_2 = 0.41$, $A_3 = -0.46$, $M_j = B_j / A_j$, $j = 1, 2, 3$.

Although the propagator functions A and B from $S^{-1}(q) = A(-q^2)\not{q} - B(-q^2)$ exhibit different structure of the poles,

$$\begin{aligned} A(x) &= \frac{\sigma_V(x)}{\sigma_S^2(x) + x\,\sigma_V^2(x)} \\ B(x) &= \frac{\sigma_S(x)}{\sigma_S^2(x) + x\,\sigma_V^2(x)} \end{aligned}$$

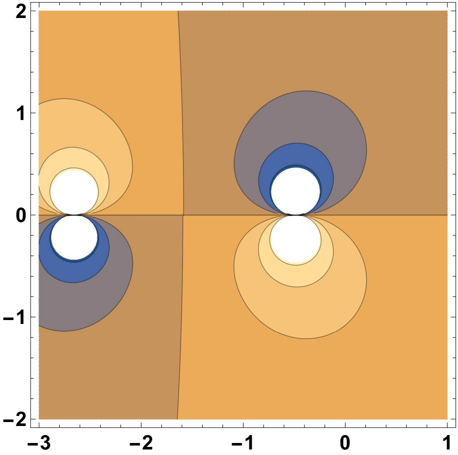
the poles are still on the real axis:

3R Quark Propagator



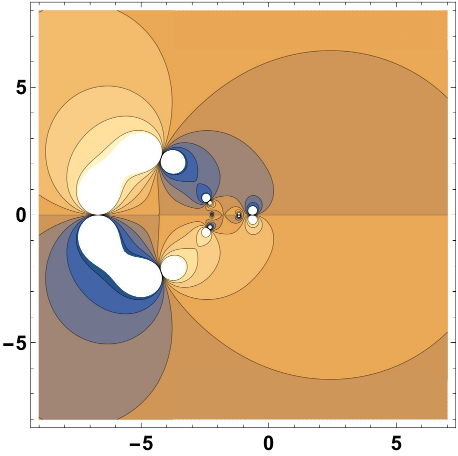
Contour plot of $\text{Im}(A(z))$ in the complex z-plane: all poles are still on the real axis.

Similarly with B in the 3R Quark Propagator:



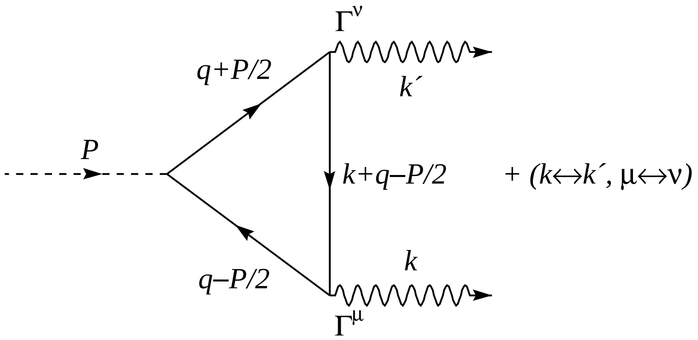
Contour plot of $\mathrm{Im}(B(z))$ in the complex z-plane: all poles are still on the real axis.

But a modest change of parameters spoils 3R Quark Propagator:



Contour plot of $\mathrm{Im}(B(z))$ in the complex z-plane. Parameters changed to $a_1 = -1.31 \;\; \rightarrow \;\; a_1 = -2 \Rightarrow$ complicated analytic structure, some poles of $B(z)$ are now moved off the real axis. Wick rotation does not go any more!

Transition form factor for flavorless pseudoscalar mesons



- Diagram for $\pi^0 \to \gamma\gamma$ decay, and for the $\gamma^\star\pi^0 \to \gamma$ process if $k'^2 \neq 0$
- ... also for η and η', but even just π^0 is challenging enough for now

Transition form factor

$$S_{fi} = (2\pi)^4 \delta^{(4)}(P - k - k')e^2\, \varepsilon^{\alpha\beta\mu\nu} \varepsilon_\mu^\star(k,\lambda)\varepsilon_\nu^\star(k',\lambda')\, T_{\alpha\beta}(k^2,k'^2)$$

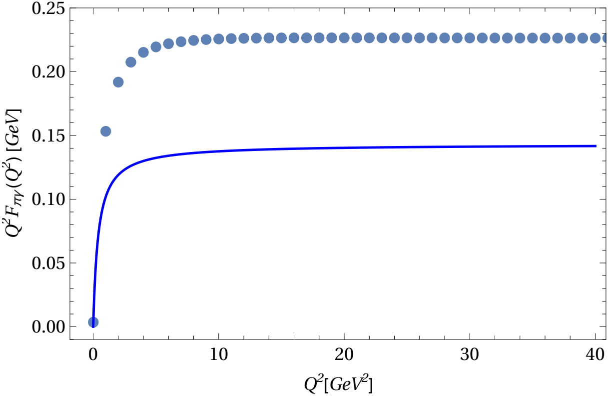
$$\begin{aligned}
T^{\mu\nu}(k,k') \;&=\; \varepsilon^{\alpha\beta\mu\nu}\, k_\alpha k'_\beta\, T(k^2,k'^2) \\
&= -N_c\, \frac{\mathcal{Q}_u^2 - \mathcal{Q}_d^2}{2} \int \frac{d^4q}{(2\pi)^4} \mathrm{tr}\{\Gamma^\mu(q - \frac{P}{2}, k + q - \frac{P}{2}) S(k + q - \frac{P}{2}) \\
&\times \Gamma^\nu(k + q - \frac{P}{2}, q + \frac{P}{2}) S(q + \frac{P}{2}) \left(-\frac{2B(q^2)}{f_\pi}\gamma_5\right) S(q - \frac{P}{2})\} \\
&+ (k \leftrightarrow k', \mu \leftrightarrow \nu)\ .
\end{aligned}$$

The π^0 transition form factor: $F_{\pi\gamma}(Q^2) = |T(-Q^2,0)|$

UV limit: asymptotically, $F_{\pi\gamma}(Q^2) \to \frac{2f_\pi}{Q^2}$ for $Q^2 \to \infty$

In the chiral limit, the π^0 decay amplitude to two real photons: $T(0,0) = \frac{1}{4\pi f_\pi}$

Transition form factor



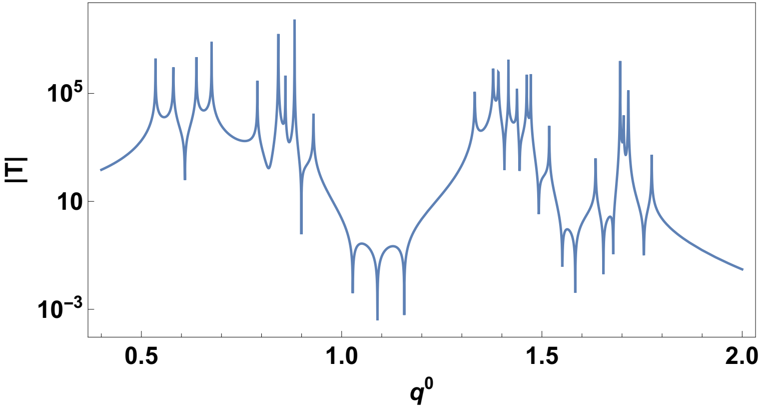
- Black dots: π^0 transition form factor calculated using 3R QP *Ansatz*
- Blue curve: the Brodsky-Lepage interpolation formula [2, 3] $F_{\pi\gamma}(Q^2) = (1/4\pi^2 f_\pi) \times 1/(1 + Q^2/8\pi^2 f_\pi^2)$ with $f_\pi = 72\ \mathrm{MeV}$.

Transition form factor

Loop integration:

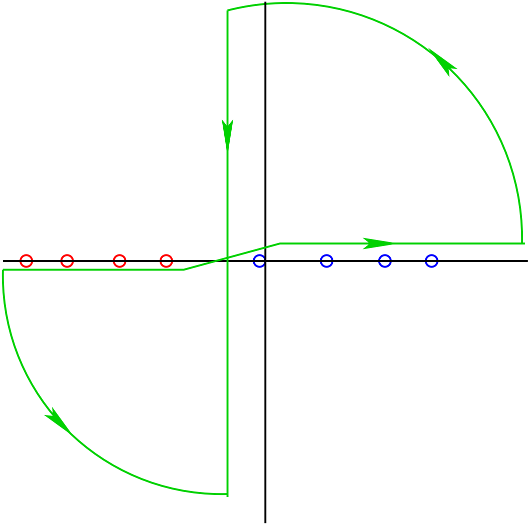
$$q = (q^0,\ \xi\ \sin\vartheta\cos\varphi,\ \xi\ \sin\vartheta\sin\varphi,\ \xi\ \cos\varphi) \qquad (\xi = |\mathbf{q}|)$$

$$Q^2 = 0,\ \xi = 0.5,\ \vartheta = \pi/3 \qquad \text{(values giving } |T\text{ integrand}| \text{ below)}$$

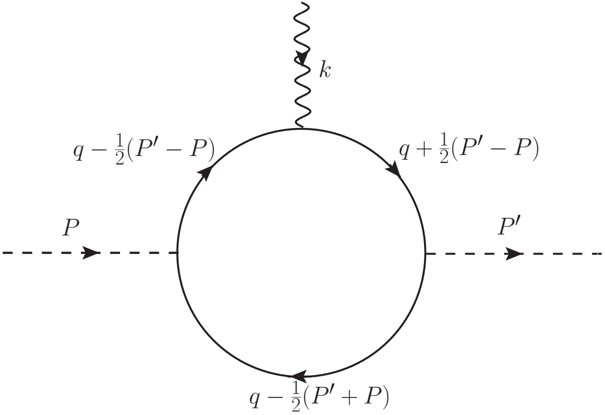


Transition form factor

Loop integration: q^0 complex plane



Electromagnetic form factor



Electromagnetic form factor

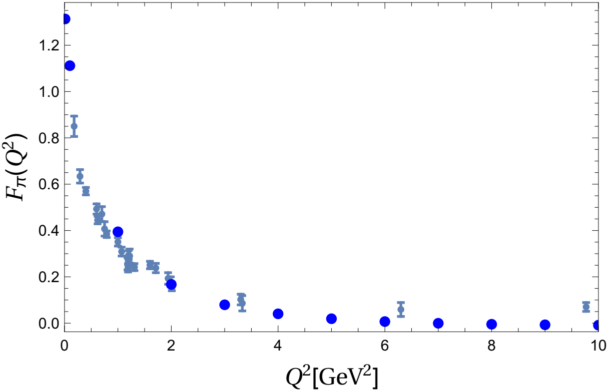
$$\langle\pi^+(P')|J^\mu(0)|\pi^+(P)\rangle = (P^\mu + P'^\mu)F_\pi(Q^2) = i(\mathcal{Q}_u - \mathcal{Q}_d)\frac{N_c}{2}\int\frac{d^4q}{(2\pi)^4}\times$$

$$\times\mathrm{tr}\Big\{\bar\Gamma(q-\tfrac{P}{2},P')S(q+\tfrac{1}{2}(P'-P))\Gamma^\mu(q+\tfrac{1}{2}(P'-P),q-\tfrac{1}{2}(P'-P))$$
$$\times S(q-\tfrac{1}{2}(P'-P))\Gamma(q-\tfrac{1}{2}P',P)S(q-\tfrac{1}{2}(P+P'))\Big\}$$

- $\Gamma^\mu(p',p)$ – dressed quark γ vertex, modeled by Ball-Chiu vertex
- The proper perturbative QCD asymptotics cannot be expected with this Ansatz ... but in the future this will be the goal:

$$F_\pi(Q^2) = 16\pi\frac{\alpha_s(Q^2)}{Q^2}f_\pi^2 \propto \frac{1}{Q^2\ln(Q^2)} \quad \text{for} \;\; Q^2\to\infty$$

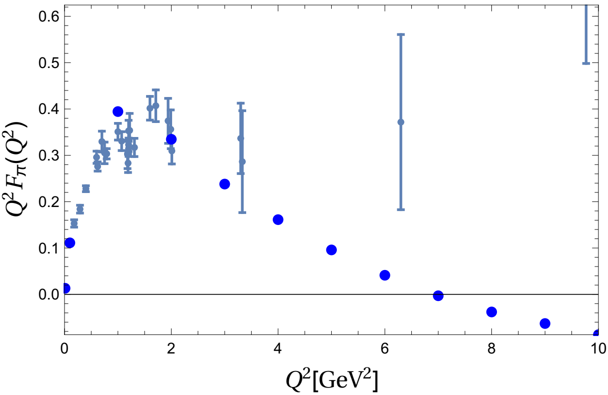
Electromagnetic form factor



Charged pion electromagnetic form factor.

- Experimental points: a compilation from Zweber [4]
- Round blue points: “3R Quark Propagator” *Ansatz*.

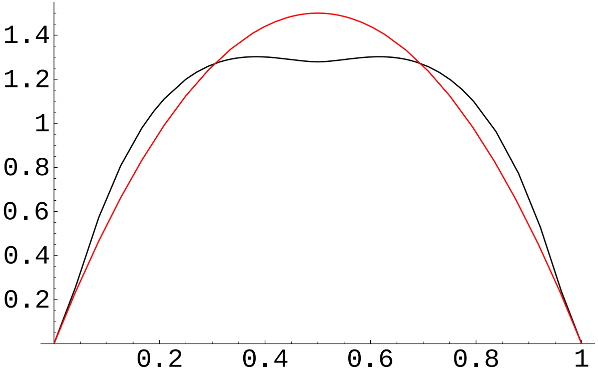
Electromagnetic form factor



Charged pion electromagnetic form factor $\times\mathbf{Q^2}$.

- Experimental points: a compilation from Zweber [4]
- Round blue points: "3R Quark Propagator" *Ansatz*.

Pion distribution amplitude



- Red curve: The asymptotic pion distribution amplitude (PDA) $\phi_\pi^{\rm as}$
- Black curve: the PDA calculated from the BS amplitude χ [5]:

$$\phi_\pi(u) = i\frac{N_c}{8\pi f_\pi}\mathrm{tr}\left(\gamma^+\gamma_5\int\frac{dq_-}{2\pi}\int\frac{d^2q_\perp}{(2\pi)^2}\chi(q,P)\right)$$

Pion distribution amplitude

- For high values of Q^2:

$$F_{\pi\gamma}(Q^2) = \frac{2f_\pi}{3Q^2} \int_0^1 \frac{du\,\phi_\pi(u)}{1-u}$$

- The asymptotic form of the pion distribution amplitude gives

$$\int_0^1 \frac{du\,\phi_\pi^{\mathrm{as}}(u)}{1-u} = 3$$

- This actual ϕ_π gives

$$\int_0^1 \frac{du\,\phi_\pi(u)}{1-u} = 3.07$$

- Asymptotic form of the transition form factor:

$$F_{\pi\gamma}(Q^2) \sim \frac{2f_\pi}{Q^2}$$

Summary

- The knowledge of the analytic behavior in the whole complex plane is needed.
- But even for physically motivated dressed quark mass function $M(p^2)$, with good analytic structure, it is very hard to predict and control the analytic structure of the corresponding nonperturbative quark propagator.
- $\Rightarrow$ the analytic structure of for quark propagators in the nonperturbative regime of QCD has been investigated for certain Ansätze.
- **Among nonperturbative propagator forms, only those allowing the Wick rotation and enabling equivalent calculations in Minkowski and Euclidean spaces seem meaningful for reproducing phenomenology.**
- Building in the perturbative behavior = an especially challenging task ... but it was difficult even to adjust the parameters in the considered Ansätze to get even qualitative agreement with phenomenology. (But further progress in the next talk, by D. Kekez.)

ADDITIONAL SLIDES

Extra figures

Axial-anomalous processes $\eta \to 4\pi^0$ & various $\eta' \to 4\pi$

An example of a propagator with singularities only on the real axis

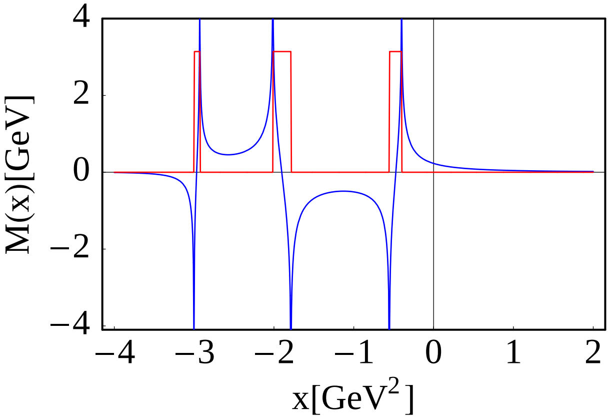
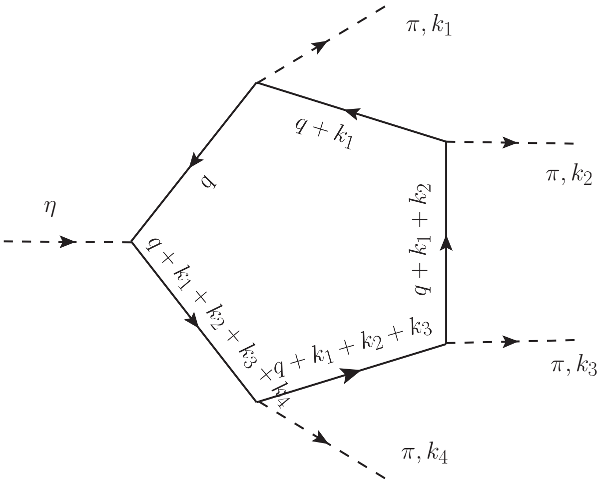


Figure : Real (blue) and imaginary (red) part of the function $x \mapsto M(x)$.

Axial-anomalous processes $\eta \to 4\pi^0$ & various $\eta' \to 4\pi$



$\eta, \eta' \to 4\pi$

$$
\begin{aligned}
S_{fi} &= (2\pi)^4 \delta^{(4)}(P - k_1 - k_2 - k_3 - k_4)\, \mathrm{tr}(I_{\mathrm{color}}) \\
&\times \frac{\cos\phi_P}{2^5} \mathrm{tr}\left((\sqrt{\frac{2}{3}}\lambda^0 - \frac{1}{\sqrt{3}}\lambda^8)\lambda^3\lambda^3\lambda^3\lambda^3 \right) \left(-\frac{2M_q}{f_\pi} \right)^5 \\
&\times (-) \int \frac{d^4q}{(2\pi)^4} \mathrm{tr}\Big(\gamma_5\, iS(q)\gamma_5\, iS(q+k_1)\gamma_5\, iS(q+k_1+k_2)\gamma_5 \\
&\times iS(q+k_1+k_2+k_3)\gamma_5\, iS(q+k_1+k_2+k_3+k_4) \Big) \\
&+ \text{23 permutations}
\end{aligned}
$$

$\eta, \eta' \to 4\pi$

Loop integration gives the Passarino–Veltman five-point function E_0

$$\int d^4q \, \mathrm{tr}\Big(\gamma_5 \, iS(q)\gamma_5 \, iS(q+k_1)\gamma_5 \, iS(q+k_1+k_2)\gamma_5$$

$$\times iS(q+k_1+k_2+k_3)\gamma_5 \, iS(q+k_1+k_2+k_3+k_4)\Big)$$

$$= 4i\pi^2 M_q \, \varepsilon_{\mu_1\mu_2\mu_3\mu_4} k_1^{\mu_1} k_2^{\mu_2} k_3^{\mu_3} k_4^{\mu_4} E_0(a)$$

where $a = (M_\pi^2, M_\pi^2, M_\pi^2, M_\pi^2, (k_1+k_2)^2, (k_2+k_3)^2, (k_3+k_4)^2, (k_4+k_5)^2, (k_5+k_1)^2, M_q^2, M_q^2, M_q^2, M_q^2, M_q^2)$

- E_0 calculated using Golem95 library
- phase space integration, Kumar and Vegas

$\eta, \eta' \to 4\pi$

	PDG2011	$M_q = 0.3$	$M_q = 0.4$
$\eta \to 4\pi^0$	$< 6.9 \cdot 10^{-7}$	$1.31 \cdot 10^{-27}$	$2.12 \cdot 10^{-32}$
$\eta' \to 4\pi^0$	$< 5 \cdot 10^{-4}$	$6.68 \cdot 10^{-7}$	$2.03 \cdot 10^{-8}$
$\eta' \to \pi^-\pi^+2\pi^0$	$< 2.6 \cdot 10^{-3}$	$1.50 \cdot 10^{-3}$	$5.13 \cdot 10^{-3}$
$\eta' \to 2(\pi^-\pi^+)$	$< 2.4 \cdot 10^{-4}$	$8.59 \cdot 10^{-4}$	$7.11 \cdot 10^{-5}$

Table : η and $\eta' \to 4\pi$ branching ratios. The $\eta' \to \pi^-\pi^+2\pi^0$ branching ratios are calculated with $M_\pi = (M_{\pi^0}^{\mathrm{exp}} + M_{\pi^\pm}^{\mathrm{exp}})/2$.

$\eta, \eta' \to 4\pi$

	BESIII	$M_q = 0.3$	$M_q = 0.4$
$\eta \to 4\pi^0$	N/A	$1.31 \cdot 10^{-27}$	$2.12 \cdot 10^{-32}$
$\eta' \to 4\pi^0$	N/A	$6.68 \cdot 10^{-7}$	$2.03 \cdot 10^{-8}$
$\eta' \to \pi^-\pi^+2\pi^0$	$(1.82 \pm 0.39) \cdot 10^{-4}$	$1.50 \cdot 10^{-3}$	$5.13 \cdot 10^{-3}$
$\eta' \to 2(\pi^-\pi^+)$	$(8.53 \pm 0.94) \cdot 10^{-5}$	$6.58 \cdot 10^{-4}$	$2.24 \cdot 10^{-3}$

Table : η and $\eta' \to 4\pi$ branching ratios.

- $\eta' \to 4\pi^0$, $\eta' \to \pi^-\pi^+2\pi^0$, and $\eta' \to \pi^-\pi^+2\pi^0$ branching ratios are calculated using $M_\pi = M_{\pi^0}^{\mathrm{exp}}$, $M_\pi = (M_{\pi^0}^{\mathrm{exp}} + M_{\pi^\pm}^{\mathrm{exp}})/2$, and $M_\pi = M_{\pi^\pm}^{\mathrm{exp}}$, respectively.
- BESIII column contains the results of BESIII collaboration

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