Spatial reasoning in mathematics

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Abstract. With the growing interest in spatial reasoning, stimulated by the development of powerful computer-based geometry and visualization packages, it is important to be clear about what is meant by spatial reasoning in mathematics. Starting from the point of various math educators, learning spatial thinking in mathematics has different aims than learning spatial thinking in other sciences.

Hence, although spatial skills may be intellectually interesting in themselves, the focus in this paper is placed on its relationship with teaching and learning geometry at the technical faculties. Furthermore, the course *Descriptive geometry with computer graphics*, which has evolved at the Faculty of Mining, Geology and Petroleum Engineering in Zagreb in conjunction with the recent developments in the modern geometry education, is described in detail. On the basis of the classical geometrical representation methods, the course focuses not only on the uprising of graphic-visual communication and developing learners' spatial visualization skills, which play a crucial role in engineering educations, but likewise on the development of learners' capacity with deductive reasoning and making use of aids and tools in mathematics education. Also, the effect of computer technology on geometry education is discussed according to the results of the SEFI – Mathematics Working Group (SEFI – stands for "European Society for Engineering Education"). The examples of student exercises will be given to show a large range of options offered within the course to make teaching of space mathematics innovating, more interactive and at the same time applicable to specific students' interests.

Keywords: spatial reasoning, teaching tools, computer graphics, higher education, e-learning

1. Introduction

As many educators emphasize, spatial reasoning, or spatial thinking, together with verbal reasoning, is one of particularly common modalities of human thoughts (Newcombe, 2010; K-12, 2014; JMC, 2001). While the verbal reasoning is the process of forming ideas by assembling symbols into meaningful sequences, spatial reasoning may be described as the process of forming ideas through the spatial relationship between objects (JMC, 2001, p. 55; Kovačević, 2016;).

Although many point out that spatial reasoning has always been a vital capacity for human action and thought (Sorby, 2009; Newcombe, 2010), some argue how it has not always been adequately supported in formal education (Jones and Tzekaki 2016; Davis, B. et al., 2015; Clements and Sarama, 2011). Fortunately, in recent years the situation is changing. But we may note that the starting research interest is growing outside the milieu of the mathematical community (Leopold, 2015; Sorby, 2009; Uttal and Cohen, 2012). Namely, the results of the transdisciplinary studies have found many evidences that spatial reasoning plays a vital role not only in schooling, across all grades and within most academic STEM subjects, and also beyond it: in later careers, as a support to key learning (Cheng and Mix, 2004; Davis B. et al., 2015; Newcombe, 2010; Uttal and Cohen, 2012). For example, Uttal and Cohen (2012) carried out a systematic meta-analysis of the most recent 25 years of research on spatial training and showed the malleability of spatial abilities and their effects.

Therefore, with the growth interest in spatial reasoning, it is important to be clear about what is meant by it in mathematics. Particularly now when many math educators see spatial reasoning as a vital component of learners' successful mathematical thinking and problem solving, and when the development of spatial ability is declared as one of the key goals of mathematics education all around the world, from pre-school to university level (Cheng and Mix, 2014; Davis et al., 2015, p. 3; Jones and Tzekaki, 2016; JMC, 2001; ICME-13, 2016; Newcombe, 2010; K-12, 2014; Milin Šipuš and Čižmešija, 2012).

However, in recent times, there is a considerable on-going debate among researches, teachers and educators on what spatial reasoning in mathematics is (also what it is not), aiming mostly at casual use of the various notions. Therefore, in the following section we shortly clear the path for specifying what is meant by it in this paper, mainly focusing on the geometry education level of technical engineers, and its role in connection to the use of ICT technology in math education.

2. What is spatial reasoning in mathematics?

Connecting spatial reasoning to math is not of recent date and it is not an intention of the author in this paper to describe in details any of its threads, but only to point out the myriad of approaches focusing onto the same topic of interest, spatial reasoning in mathematics (Davis et al. 2015; ICME-13, 2016) or, more precisely in this case, geometrical reasoning (Bishop, 1980; Clements and Battista, 1992; Kovačević, 2016; Stachel, 2015; JMC, 2001; K-12, 2014).

Let us start with one, perhaps the most important fact for this paper. Namely, according to the well-known Clements and Battista's *Handbook of research on mathematics teaching and learning: Geometry and spatial reasoning,*back in the 20^{th} century most mathematicians and mathematics educators seemed to include spatial reasoning directly as a part of a geometry curriculum, emphasizing in that way a strong relations spatial reasoning and school geometry had/have (Clements and Battista, 1992, p. 420; Bishop, 1980). Importantly though, the results of various studies highlight significant variation across the European countries in the historical design of mathematics curricula, and spatial geometry curricula in particular (Bishop, 1980; Davis et al. 2015, p. 48; JMC, 2001 p. 33; Lawrence, 2003). The main differences in treating and teaching spatial problem tasks in European countries are shown in Figure 1 within the comparison of the systems of graphical communication educations back in the 19^{th} century in France, Germany and Great Britain, taken from Lawrence (2002, p. 1278).

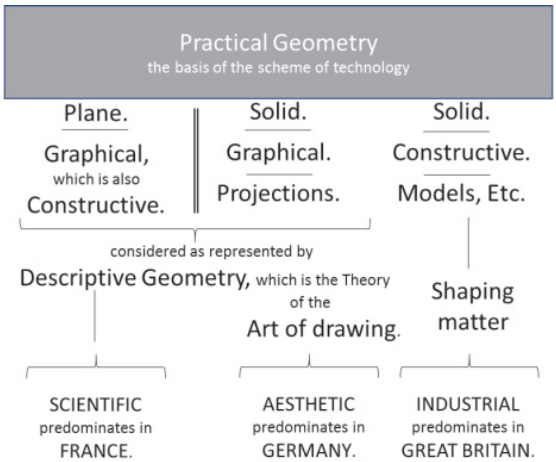


Figure 1. The history of spatial curricula in the 19^{th} century in some European countries.

Hence, following these historical variations in educational trends, back in 20^{th} century many European countries put a strong emphasis in geometry curricula on the traditional Euclidean geometry (JMC, p. 31). Conversely, Croatia, as a historical part of Austro-Hungarian Empire, mostly followed the central European approach in spatial geometry within the subject descriptive geometry. Thus, as Stachel (2015) points out for the central European countries, descriptive geometry as a subject in the hierarchy of sciences, is placed somewhere within or next to the field of Mathematics, but also near to Architecture, Mechanical Engineering and Engineering Graphics (Stachel, 2015). Even today its specificity at many technical faculties in Croatia is focused on making mathematics more applicable to engineering education through the promotion of the spatial reasoning and its graphic representation within the area of projective and synthetic geometry (Horvatić-Baldasar and Hozjan, 2010; Weiss, 2015; Stachel, 2015; Weiss, 2015).

Furthermore, in connection to the historical development of the spatial reasoning in mathematics, an examination of the history of higher education in 20^{th} century revealed, as Horton (1955) interestingly puts it, that often the Euclidean geometry has been treated as a prerequisite to collegiate matriculation aiming at three basic needs that Euclidean geometry filled at the time: the necessity for higher education, the screening device of the unfit for higher education, and the development of a way of reasoning. Thereat often only the Euclidean plane geometry was studied. Naturally, some of the above-mentioned educational outcomes were questioned over the past decades, either by the mathematicians and math educators, either by the psychologists or some other educators, and there have been substantial changes in geometry education in the second half of the twentieth century over the countries (Davis et al., 2015 p. 48; JMC, 2001, p. 31; Lawrence, 2003; Weiss, 2015; Stachel, 2015). Unfortunately, the overall geometry content changes in primary and secondary education are being less visible in the 21^{st} century mostly because of the new "outcomes to competence oriented curriculum" (Horvatić-Baldazar and Hozjan, 2010; Kovačević, 2016). But, the *Working group on the teaching and learning geometry 11–19* in UK states for example that basic changes are mostly regarding the increasing emphasis on the "applicable" geometrical content embodied in coordinate geometry, vectors and transformations, at the expense of the "purer" mathematics of classical Euclid (JMC, 2001 p. 31).

Furthermore, it is important to note that nowadays, in connection to mathematics education, the importance of spatial reasoning is recognized beyond the limits of geometry, and the existing literature provides a firm basis for a conclusion that spatial ability and mathematics share cognitive processes beginning early in development (Cheng and Mix, 2014 p. 3; Davis et al. 2015; Jones and Tzekaki 2016; K-12, 2014, p. 3). So, spatial reasoning seems to become crucial at the very beginning of the math education.

On the other hand, it seems that the influence of the non-mathematical researches becomes larger in some areas of the mathematics, given the continuing expansion of the important role of mathematical education in science and contemporary society. Some argues that that may again become a misfortune for teaching and learning geometry at the higher education levels (Jones and Tzekaki 2016; Kovačević, 2016; Stachel, 2015; Weiss, 2015). Also, there are still some who follow the viewport of studies going after a seemingly paradoxical hypothesis: *even though spatial abilities are highly correlated with entry into a STEM field, they actually tend to become less important as a student progresses to mastery and ultimately expertise* (Uttal and Cohen, p. 157). In other words, some believe that spatial reasoning is of less importance as progress in a STEM field increases. But, even if the mentioned assumption turns out to be true for some science disciplines (or some areas of mathematics), there are still areas strongly relying onto spatial and visual abilities in their reasoning processes even in their expertise level (Weiss, 2015; Stachel, 2015; Gorjanc and Jurkin, 2015). Furthermore, it remains questionable, whether (and when) one should be focusing in mathematics on the development of the spatial ability per se, and when on the spatial reasoning, or on spatial thinking.

There are many didactical and cognitive problems in connection to the role of spatial reasoning in math that are still waiting to be solved (Jones and Tzekaki 2016; Davis et al. 2015).

For example, how learners' mathematical/spatial reasoning is influenced by the ways in which geometric objects are represented? Or, how can one analyze the spatial/reasoning processes involved in mathematical activity? Does mathematical activity require only one common cognitive process, or, indeed, certain very specific cognitive structures whose development must be taken care of in teaching? Namely, as Tartre (1990) pointed out in her study of the role of spatial orientation skill in the solution of mathematics problems, it is questionable whether any attempt to verbalize the processes involved in spatial thinking ceases to be spatial thinking. Also, a French psychologist R. Duval discovered many didactical problems when analyzing the cognitive model of mathematical, and particularly geometrical reasoning, as well as the use of, today inevitable, graphical representation in mathematics (Duval, 2002). He studied how visualization works towards understanding in mathematics, aiming thereby at the important fact pointed out by Sorby (2009) that the graphical expression in engineering field is both a form of communication and a means for analysis and synthesis. Duval further claims that representation in mathematics becomes usable only when it involves physical things or concrete situations (Duval, 2002, p. 333).

To conclude, although many new results regarding malleability of spatial reasoning are encouraging (Davis et al., 2015, p. 85), and the fact that spatial abilities can be improved through education and experience may suggest that spatial ability training can improve math performance (K-12, 2014, p. 6), our focus in this paper is not onto spatial ability per se but on the applicable geometry, or as some refer to it as "vision guided spatial reasoning", i.e. descriptive geometry (Stachel, 2015; Gorjanc and Jurkin, 2015; Kovačević, 2016).

Furthermore, in this paper the reader may spot the author's often mixing the use of terms "spatial reasoning", "spatial thinking" and "spatial ability", purely because of the recent review of the research literature. Therefore, only for the purpose of this paper, in the rest of the section we will briefly clarify these notions, primarily emphasizing their inevitable interrelation. Namely, while some papers explicitly distinguished the terms in question, others did not. Also, some researchers suggest how this tendency of "mixing notions" is particularly prominent in areas of the mathematics sciences associated with geometry whereas geometry is being marginalized in many mathematics curricula unlike 3D geometry and associated spatial reasoning that is, according to various researches, widespread over a number of applied areas (Clements and Sarama 2011; Davis et al. 2015, p. 12; Jones and Tzekekaki, 2016; Kovačević, 2016).

2.1. Spatial reasoning, spatial ability or spatial thinking

For example, Clements and Battista (1992) used the first notion, "spatial reasoning" purely in connection to the specific *set of cognitive processes by which mental representation for spatial objects, relationships and transformations are constructed and manipulated* (p. 420). Interestingly, they further described the "school geometry" as the *study of those spatial objects, relationships and transformations that have been formalized (mathematized) and the axiomatic mathematical systems that have been constructed to represent them* (p. 420), mainly pointing at the traditional Euclidean geometry that was, for a long time, synonym for the school geometry in many countries (JMC, 2001, p. 31).

However, in connection to the spatial reasoning mentioned in the title of this paper, Clements and Battista further distinguished the use of the term "spatial thinking" in connection to the scientific mode of thought used to represent and manipulate information in learning and problem solving (p. 442). They probably aimed at the suggestions of some researches that *spatial ability* and *visual imagery* play vital roles in mathematical thinking (p. 443). Namely, spatial thinking was often perceived as one of different modes of thinking in mathematics. Its importance is recognized and emphasized also in the lifelong education in the definition of the mathematical competence as one of its eights key competences (EFQ, 2006; Kovačević, 2016). Some even argued, following Einstein's comments on thinking in images, that much of the thinking required in higher mathematics is spatial in nature (Duval, 2002; JMC 2001, p. 55; Newcombe 1980). But, as we have already pointed out, researchers had, and still have, their own different descriptions or subtle distinctions.

For example, Duval (2002) sees reasoning only as a part of visualization process, claiming further that representation and visualization are at the core of understanding in mathematics thinking (p. 312). But, he argues that representation becomes usable in mathematics only when it involves physical things or concrete situations (p. 333).

On the other hand, Jones and Tzekaki (2016) also emphasize the inevitable overlapping of geometrical visualization and spatial reasoning, whereby they take visualization to be *the capacity to represent, transform, generate, communicate, document and reflect on visual information* (p. 114), and they associate the process of geometrical reasoning to the deductive reasoning and proof (p. 124). Furthermore, regarding visualization, they have pointed out in their comprehensive review of recent research in geometry education that visualization is indispensable in proving and problem solving, but visual representations or processes they develop are not always effective in solving or proving relevant tasks (p. 117).

Newcombe (2010) is more focused on purely psychological aspect of the spatial thinking and in her studies spatial thinking is defined by the four tests (3D spatial visualization, 2D spatial visualization, mechanical reasoning and abstract reasoning) used to assess it (p. 31).

Thus, nowadays in some papers/studies various terms are used interchangeably, or with subtle distinction demands, aiming sometimes at "spatial reasoning", as a thinking process particularly important in the development of mathematical competence (EFQ, 2006; JMC, 2001; K-12, 2014; Kovačević, 2016; Stachel, 2015), and sometimes aiming at "spatial reasoning" as spatial skills (Clements and Battista, 1992; Sorby, 2009; Milin Šipuš and Čižmešija, 2012). Or sometimes even using the terms "thinking" and "reasoning" interchangeably (for example Davis et al., 2014, p. 5 or K-12, 2014, p. 3).

It is also important to note, from the mathematical point of view and in connection to the teaching and learning processes, that mathematical educators sometimes distinguish between the competences "thinking mathematically" and "reasoning mathematically", whereby the first competence includes *the recognition of mathematical concepts and an understanding of their scope and limitations* (Alpers et al., 2013, p. 13), and the second one *includes the constructions of chains of logical arguments and of transforming heuristic reasoning into proofs* (for details on general mathematical competencies for engineers see Alpers et al., 2013).

In this paper, the focus is on the development of the mathematical competence as a whole, and the term “spatial reasoning” is used merely to emphasize the spatial aspect of higher cognitive mode of thinking particularly significant in the teaching and learning of geometry at the technical faculties, more precisely in our case, of descriptive geometry, the mathematical subject in question discussed in the third section of this paper. Also, in this paper we shall continue to take “spatial reasoning in mathematics” to be the “geometrical reasoning” (Bishop, 1986; Jones and Tzekaki, 2016; Kovačević, 2016) aiming thereby not just on Euclidean spatial geometry but also on projective geometry (Lawrence, 2003; Stachel, 2015; Weiss, 2015) that deals with three-dimensional objects and their plane representations.

3. Descriptive geometry with computer graphics at the Faculty of Mining, Geology and Petroleum Engineering

Descriptive geometry has been a part of applicable geometry dealing with methods which aim to study 3D geometry and providing an important theoretical basis on which all the modern graphical communication was built. It enables insight into geometrical structure and metrical properties of spatial objects, processes and principles. Typical for it is the interplay between the 3D situation and its 2D representation, and between intuitive grasping and rigorous logical reasoning.

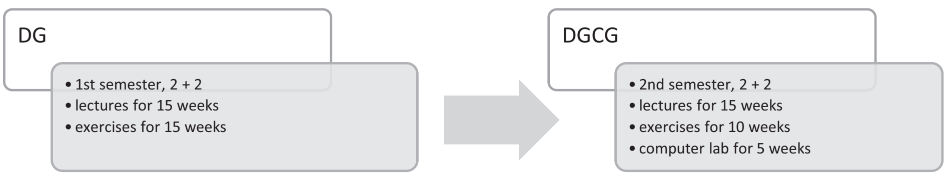


Figure 2. Descriptive geometry curriculum at the Faculty of Mining, Geology and Petroleum Engineering.

Descriptive geometry at the Faculty of Mining, Geology and Petroleum Engineering is focused on the developing a set of learning outcomes of basic knowledge of natural sciences and technical fields important for the scientific fields of mining, petroleum and geological engineering. It is currently taught within two obligatory courses: *Descriptive geometry* (DG) and *Descriptive geometry with computer graphics (DGCG)*, each within one semester (for about 180 students per each course) as it is shown in Figure 2.

3.1. Subject contents and methods of teaching

The last content changes within the courses DG and DGCG were made in 2013/2014. From the content point of view, there has been no substantial changes from the traditional subject content of descriptive geometry besides reducing the scope of course geometrical content primarily regarding the more complex geometrical structures (Horvatić-Baldazar and Hozjan, 2010). However, today emphasis is not being placed on the education of practical techniques, but on teaching the theory behind the specific techniques and the development of associated mathematical concepts. The subject is also responsible for establishing the foundation of mathematical representational systems and the use of various drawing tools important in graphical communication of engineers.

Although both courses are taught with hand drawing (sketching as well), commercial graphic processing software is being used by students only in DGCG. Namely, after basics of geometry of projection (extended Euclidean space objects – affine and projective transformations) and of two- and three-dimensional objects (basic plane and space curves, surfaces, solids) are introduced in the DG, together with some basic descriptive geometry relations and constructive principles (perpendicular relationship, piercing points, plane intersections, intersection of two solids), the experience of Computer Aided Design (CAD) software is introduced as one of the educational objectives in DGCG.

However since the focus is on geometry, CAD software is used through geometric problem solving and modelling. In doing so, problem solving in descriptive geometry involves the planning and implementation of the 2D representations of 3D objects and the corresponding relations, both in the plane and in the space, using appropriate tools, methods and principles. Furthermore, modelling in descriptive geometry means transferring previously analyzed data in a simplified and idealized geometric shape. An example of such activities is decomposition of complex structures from the actual context into geometrical objects and recognition of relationship between objects used within the computer lab exercises.

There are some content overlapping in both courses (DG and DGCG) that allow simultaneous approach to the same problem situation using various descriptive geometry methods. During the teaching process, this overlap enable constant comparison of advantages and disadvantages of different methods and principles used within the subject.

It should also be noted that the future professionals of mining, geological and oil profile, in contrast to, for example mechanical engineer, require knowledge not only of classical orthogonal projections and axonometric, unavoidable for computer graphic, but of other methods of descriptive geometry, particularly the projections with elevation which is used for solving various mining and topographic problems in relation to engineering profession.

3.1.1. Exercises in Computer Lab

Most of the educational e-materials used within the computer lab exercises were made during the year 2012 on the joint project of four technical faculties of the University of Zagreb with twelve participating teachers. Within the project the repository consisting of about 50 five-min videos helping to learn basics of *Rhinoceros*

3D modelling was produced (for details on the project see Gorjanc and Jurkin, 2015). Further developments were made during the years 2015 and 2016. Namely, because of the implementations of higher standards of qualifications and occupations in mining and geology in accordance to the Croatian Qualification Framework (CROQF), additional course implementations were made within the project TARGET by raising the level of e-learning technology in both DG and DGCG.

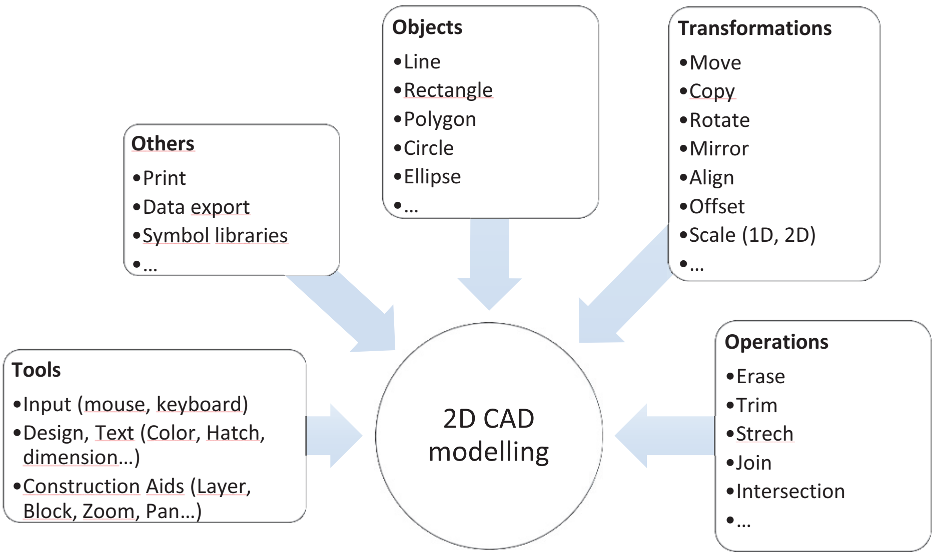


Figure 3. CAD areas for 2D modelling.

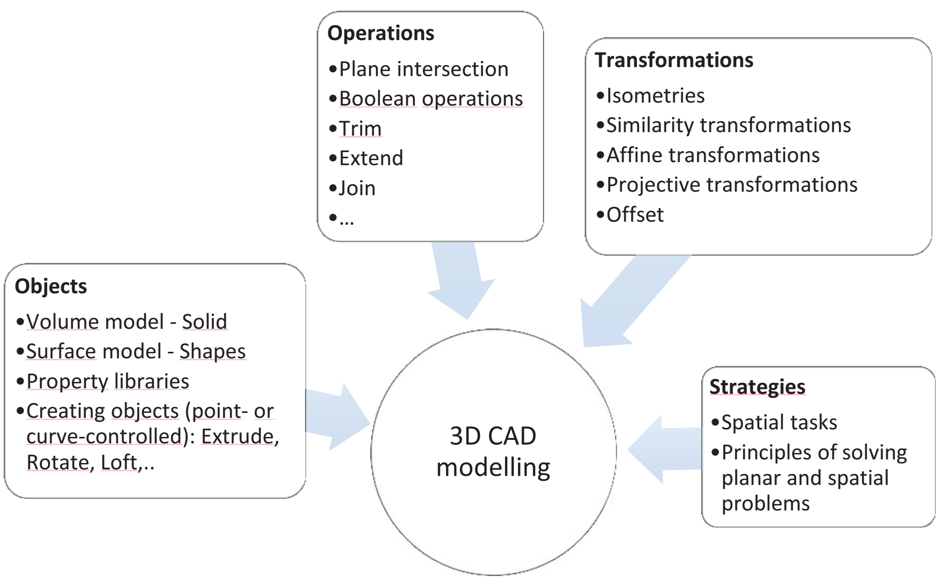


Figure 4. CAD areas for 3D modelling.

For the purpose of the courses DG and DGCG, and with a view to facilitating the initial work with commercial CAD programs, following mostly modern methodical principles regarding the teaching of descriptive geometry, 2D-CAD and 3D-CAD parts are divided into areas shown in Figure 3 and Figure 4. Only some of the mentioned topics were covered within the courses, and the overall subject contents regarding computer graphic in DGCG part are shown in Table 1.

Table 1. Subject contents of CAD modelling within DGCG at the Faculty of Mining, Geology and Petroleum Engineering

1^{st} time-block		**2^{nd} time-block**	
	Guidance, Interface of Rhinoceros 3D	1^{st} week	3D – CAD modelling: Operations, Transformations, Extrusion Working planes and working axes
1^{st} week	2D – CAD modelling: Tools, Editing operation Coordinate systems: WCS, UCS, Objects, Transformations	2^{nd} week	Application of descriptive geometry procedures to solve spatial problems (Regular polyhedral, Shortest distance problems, Terrain and layer)
2^{nd} week	3D – CAD modelling: Guidance, Viewports, View control, Objects, Editing operations	3^{rd} week	Testing – individual tasks (overall assignments)

Hence, exercises in computer lab (90 min time per slot per 5 week) were divided into two time-blocks $(2+3$ weeks$)$. The basic guidance information about the interface of *Rhinoceros 3D*, views, viewports and work with layers are provided through e-materials by using e-learning platform *Merlin*, the system based on the learning management system *Moodle*. The students were encouraged to learn interface basics on their own by watching videos at home. After that, in each time-block, for each unit, first brief lectures are given concerning related topic and then the operational methods of *Rhinoceros 3D*, as a representative of Computer Aided Design (CAD) software, are taught using simple examples. During the course, and particularly in between the time-blocks, students were given time to improve their abilities with ICT on their own by further watching videos at home or in the computer lab, under teacher assistance if needed.

Thus, in the 2^{nd} 3-week teaching block, students had to use their knowledge to solve problems typical for descriptive geometry on their own, after solving one or two similar tasks on the spot with teacher. Details of the assignments are shown in the following section.

Importantly, from 2005 – 2012 *AutoCAD* was used as the main graphic processing software, but since 2013 *Rhinoceros* is being introduced as a CAD representative. These software changes within the course were made mostly for two reasons. Firstly, to facilitate the teaching and learning process in the course in accordance with recent changes in the undergraduate study programs. These changes allowed, in the same working time, more accent to be placed onto basic knowledge of mathematics and mathematical principles in problem solving, and not on the computer work with more and more complex commercial software at the very beginning of the study. Namely, rapid changes in commercial software industry cause constant changes in software interfaces and increase of the complexity of application programming tools. Hence, there is an inevitable move of educational focus from purely developing the students' mathematical competence also to the teaching and learning specific software-version techniques and procedures. In order to facilitate the synergy between various educational goals, a more geometry friendly commercial software *Rhinoceros 3D* was chosen as a CAD representative. And secondly, as a result of harmonization of educational standards on various technical faculties at the University of Zagreb within the previously mentioned project, a basic repository providing adequate e-materials was made using *Rhinoceros 3D*. This repository facilitates the educational process for both, teachers and students. For a number of teachers, the joined database of preparatory materials is available for use, and for students, a large amount of e-materials, including video-lessons, allowed them to individually choose the time and place for qualitative learning.

Before reporting on the activities used within the course DGCG, let us discuss some didactical principles of importance for the chosen activities.

3.1.2. Some didactical principles

Firstly, it is important to note that there is a growing number of mathematicians and mathematics educators that find well-known thinking frameworks like the Van Hiele's or Piaget's ones helpful only in the first access to geometry by young children (Davis et al., 2014), but unfit when it comes to teaching geometry at higher educational level, such as high school or university levels, (ICME-13, 2016; Kuzniak et al, 2007).

Furthermore, although both courses, DG and DGCG at the Faculty of Mininig, Geology and Petroleum Engineering combine different didactical principles in the teaching process, for the purpose of this paper and in connection to the teaching descriptive geometry, particularly interesting is the Duval's theory on figural apprehension in mathematical reasoning, especially in geometrical reasoning and work with geometric drawings and computers (Duval, 2002; ICME-13, 2016; Jones, 1998; Kuzniak and Richard, 2014).

Duval also proposes the synergy of three cognitive process necessary for proficiency in geometry which fulfil specific epistemological functions. Those are: visualization, construction and reasoning (see Figure 5, from Jones, 1998). His work on cognitive process level, important for geometry and mathematics as well, was further adapted by Kuzniak and Richard (2014).

All three processes are included in geometrical reasoning and can be performed separately. However, Duval emphasizes, visualization doesn't necessarily depend on construction and it doesn't always help reasoning. The reasoning process, on the other hand, can be developed in an independent way of two other processes included. In the Figure 5 below, each arrow represents a direction in which one kind of cognitive process can support another kind in any geometrical activity. As it can be seen, the construction process, that is in the focus when it comes to application and practice, depends only on connections between relevant mathematical properties and the constraints of the tools being used and cannot be directly supported by visualization.

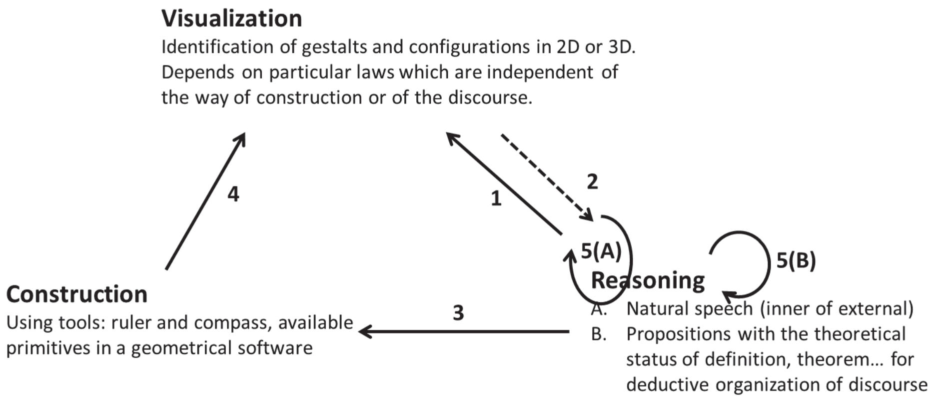


Figure 5. The underlying cognitive interactions involved in geometrical activity.

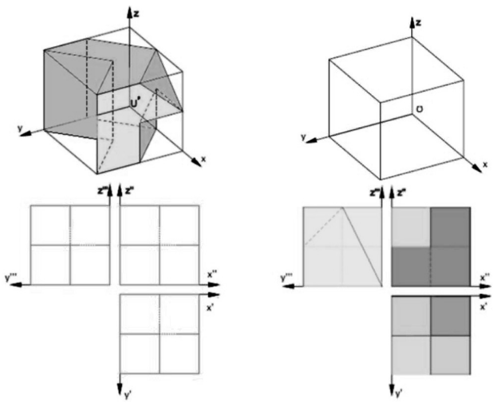
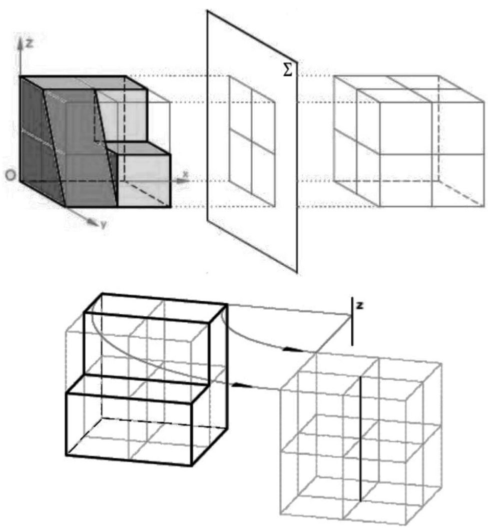
Hence, in order to achieve an ultimate goal of mathematical education of engineers, which is according to SEFI group to make engineers mathematically competent (Alpers et al., 2013, p. 65), activities focusing on a particular cognitive process are often included in the course, whether students are to work individually (at home or in the class) or in pairs, in parts of lessons with individual or mixed interaction.

3.1.3. Example of activities

Practicing visualization tasks

Over the years we have noticed a growing number of students at the Faculty of Mining, Geology and Petroleum Engineering having trouble with simple visualization of basic geometric objects based on the given data. Since these visuospatial abilities are prerequisite for their further study in technical fields, we have considered a set of different visualization tasks that are offered to students in the class at the very beginning of the course, lasting 15 to 20 minutes. These tasks mostly serve students to detect, if there are, their basic visuospatial problems and to encourage them to work on it. Further visualization tasks, aimed at further improvement of student visuospatial skills, are offered to students individually for home-based practice. Examples of visualization tasks are given in Table 2.

Table 2. Examples of visualization tasks.

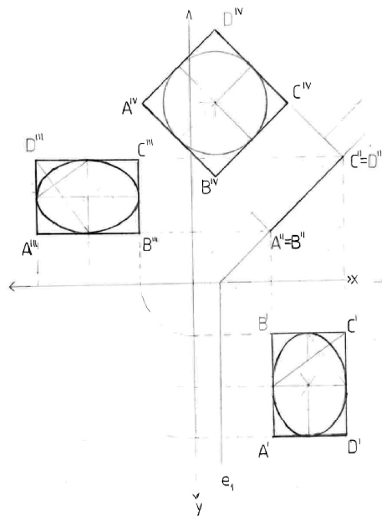
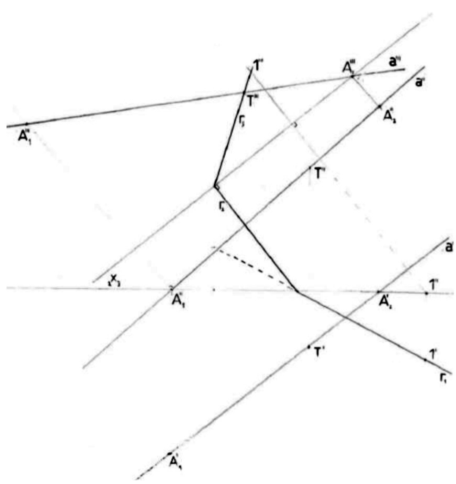
	**Orthogonal projection** a) Given the object, draw its principal views (top, front, left side) in the prepared grid (left image). b) Given three principal views of an object, determine its image in the prepared coordinate system (right image).
	**Congruent transformations** a) Given the mirror plane Σ, draw a mirror image of the given object in the prepared grid. b) Rotate the object around the z-axis and draw its image in the prepared grid.

Positioning and metrical tasks

Since geometry originated from practical needs, the geometric courses necessarily combines not only mathematical content but, in our case, mining and topographic content specific to geology, mining and petroleum engineering. Many of the problems included are based on construction tasks and tasks on sets (loci) of points with certain properties. Within each course, DG and DGCG, three individual geometrical problems are given to each student. Four of them are focused purely onto mathematical content, and only two combine specific geological contents with mathematical concepts.

Examples of two individual mathematical tasks were given in Table 3 were hand drawings are made by students. Both examples require students first to transform some spatial problems into a graphical one. They should think of representing spatial objects by a two dimensional drawings by means of some pictorial overview (representing mathematical entities). The requirement of the use of point coordinates emphases the mathematical understanding of spatial relations as well as the understanding of projection method, which in this case is the Monge projection. During the course the same construction problems are solved by applying different projection methods and by CAD.

Table 3. Examples of student individual mathematical tasks by using Monge projection.

	**Example 1.** Draw principal views (top, front ad left side) of the square and its $ABCD$ $\lceil a = 4 \rceil$ inscribed circle if it is perpendicular to plane π_2, makes an angle of with 45° and its π_1, two sides are perpendicular to having π_2 its bottom foreground vertex $A(3, 6, 2)$. Choose any profile plane π_4 so that the square $ABCD$ will be projected on that plane in its true size.
	**Example 2.** There are given plane $P(2, -1, 2.5)$ and line $a \equiv A_1A_2[A_1(-1, 4, 0), A_2(4, 0, 4.5)]$. Using the Monge projection determine the intersection of line a with plane P.

Reading the first example, students have to reason about the properties of the square and corresponding properties of the projection method (mathematical thinking) in order to solve the problem, thinking of various possibilities of square position in space (modelling mathematically). In the second example, after graphically representing the given objects, to solve the problem student must understand the spatial relations among objects and then choose one of the constructive principles taught within the course to determine the piercing point. The task is solved by adding additional profile view in which the given plane appears as a line.

Student Self-Assessment Test

Within the e-learning platform *Merlin* students were given three tests as a student self-assessments tasks, available for them to be taken anywhere, at any time. The tests were created by a randomized selection of questions out of a larger question bank so that students could do each test many times.

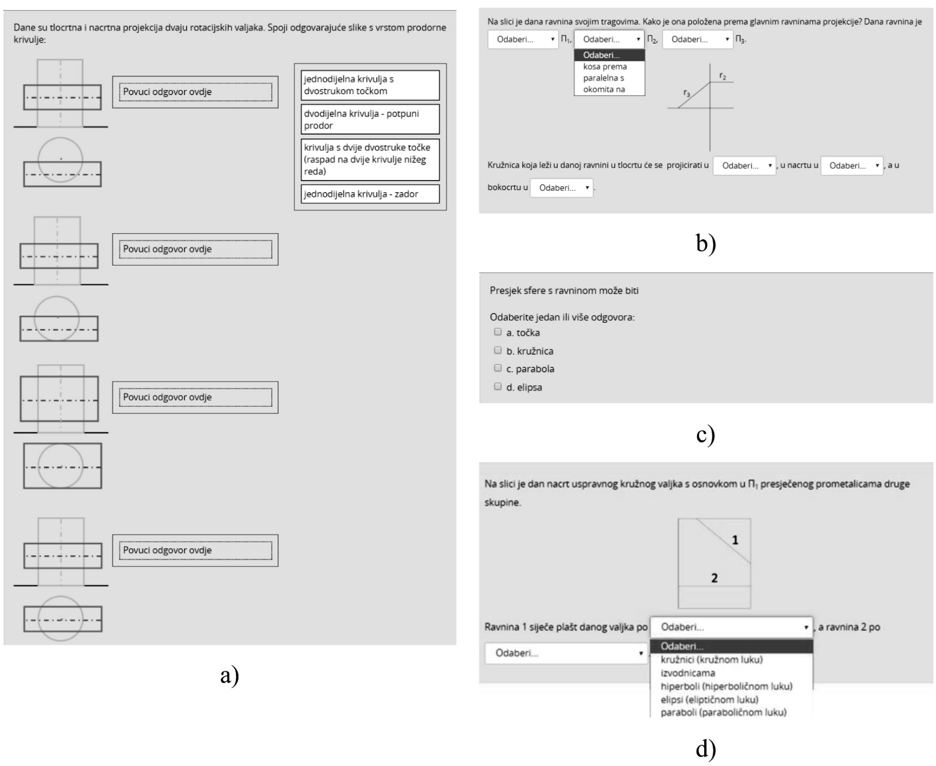


Figure 6. Some task examples of different self-assessment tests.

The tests were optional and their results were not used for students' grading, but to give students an opportunity to identify where their knowledge was weak and to revise their work. Due to the specific course interdisciplinary learning outcomes, computer-supported on-line assessments were very simple and short, mostly consisting of a number of multiple-choice questions some of which are shown in Figure 6, combining graphic representation and applying mathematical knowledge within concrete graphical situation.

Also, some of the given tasks required students' "pure" factual mathematical knowledge demonstration. The last year results pointed out that students were less successful when the formulation of questions was similar, but not exact, to those in the textbook. However, the number of students taking the tests was not representative, since the test was at this phase optional. Once a suitable set of questions are imported in the system, the system will provide students continuous on-line sport.

We may also note that teachers can also benefit from using such tests. For example, the overall students' weak spots could be detected, due to the specific concept-oriented tests, further providing teachers with valuable sources where to put additional focus when teaching.

Computer Lab Assessments

The subject descriptive geometry at the Faculty of Mining, Geology and Petroleum Engineering is organized so as to follow ideas of two sets of learning outcomes (CROQF proposal – level 6) of what student is supposed to acquire:

- basic knowledge of the natural sciences in the area of mining, geology and petroleum engineering
- technical knowledge in the area of mining, geology and petroleum engineering.

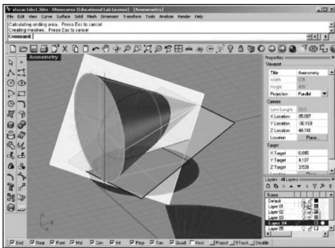
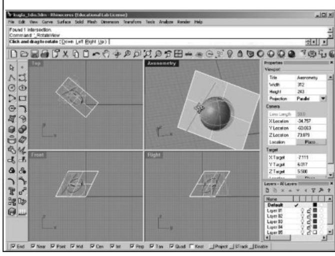
The 1^{st} time-block tasks in the computer lab were more or less similar to those taught at various engineering drawing course aiming at the development of the ICT skills, i.e. in this case CAD modelling skills, important for the technical engineers in their future jobs. Those tasks were mostly focused onto the development of the construction (using tools) and visualization processes, already highlighted in the Duval's cognitive model of geometrical reasoning.

However, the 2^{nd} time-block tasks are more problem-oriented, focusing onto reasoning process. In other words, the simple tasks were chosen with a specific goal: to emphasize the importance of mathematical knowledge in the problem solving activities. To solve this problems, student first had to understand that mathematics can do the job. Only after that some computer modelling, using tools, should take place.

Some of the tasks examples are given in the following table 2. Sometimes the same problem tasks were solved by various descriptive geometry method; either using Monge's projection, or projection with height, or axonometric projection, or by using CAD program. The tasks given in Table 4 contribute to the achievement of two course learning outcomes of what student is supposed to be able to demonstrate after he/she has completed the course DGCG:

- to apply basic mathematical knowledge in solving spatial problems
- to use appropriate software to address the technical and mathematical problems.

Table 4. Examples of learning activities in the 2^{nd} time-block in the computer laboratory.

	**Zadatak 4:** • Konstruirajte presjek rotacijskog stošca [os SV, S(6,0,6), V(6,10,6), r=5, osnovica u Π_2] ravninom P(4,-2,3). • Konstruirajte tangentu u točki T presječne krivulje, ako ta točka leži na donjem dijelu stošca, a drugo probodište njezine izvodnice je točka N_2(7,0,–). Tangentu konstruirajte kao presječnicu tangencijalne ravnine stošca i ravnine presjeka. • Razvijte plašt stošca s presječnom krivuljom.
	**Zadatak 7:** • Konstruirajte presjek kugle [S(-4,4,4), r=3] ravninom P(-6.5,-5.5,5.5). • Odredite tangencijalnu ravninu kugle i tangentu presječne krivulje u točki T(–,–,5.5) na lijevoj strani plohe.

To successfully solve them, student should be able to:

- understand the use of the basic orthographic views of elementary solids
- understand the basic concepts of geometric congruence transformations in space
- recognize examples of an affine transformations (oblique parallel projection)
- recognize and use the plane-intersections of elementary solids
- understand and distinguish the concepts of tangent line and tangential plane
- understand the basics of 3D co-ordinate geometry.

4. Conclusion

With the expansion of higher education, the number of pupils entering technical faculties in Croatia that have had limited experience in their formal education in spatial activities (be that the spatial abilities or the spatial reasoning/spatial thinking process) has been growing. Also, the number of pupils that have finished gymnasiums programs and have entered various technical faculties is at the moment increasing in Croatia. At the Faculty of Mining, Geology and Petroleum Engineering there is already up to 40 % of gymnasium graduates. Consequently, many undergraduates are not properly equipped to deal with a large amount of spatial content used within their scientific courses. The author's many years of experience in teaching geometry at tertiary level have shown that many students often lack "spatial experience" not only in the case when dealing with basic 3D objects and relations, but even with 2D objects when they are placed in space. And naturally, when students lack experience it is hard to sort their knowledge into any system of a knowledge in a logical order.

Hence, although there is a never ending discussion what comes first, theory or practice, author firmly follows conviction that the basis for learning practical geometry at the tertiary level should be clarifying and fixing in mind basic geometrical concepts and principles, and only then applying these "knowledge-tool" for solving specific engineering construction problems. Geometry education still can provide *both*: a means of developing learners' spatial visualization skills *and* a vehicle for developing their capacity with deductive reasoning and proving.

Namely, much of basic science in technical fields requires good mathematical knowledge and skills, not only in numeracy, but also in dealing with spatial reasoning, intimately related to geometry. A broader geometrical education, including knowledge of various curves and solid/surface shapes (and their visual 2D representatives), projection methods (orthogonal projection, parallel projection, central projection...) and different congruence and non-congruence 3D transformations, is needed to provide some of the foundations upon which mathematical understanding could be built.

Furthermore, visual aspect of geometry also underpins much of information technology and lately relies a lot on computer graphics demanding higher ICT skills of both, teachers and students. These strong links between geometry and technology are also important because geometry needed for proficiency in many technical fields exceeds far beyond traditional Euclidean space geometry and deeply enters the area of affine and projective geometry, which in general is not taught at many technical faculties in Croatia.

Thus, regarding the spatial reasoning in mathematics we may conclude, accompanying numerous educators, the following:

- one unified and wide accepted definition of spatial reasoning does not exist
- there is a converging agreement on the importance and malleability of (visuo)spatial reasoning among researchers in various scientific fields (psychology, mathematics, technology, engineering, didactics...) for it can support learning and communication
- and most importantly, regarding its connection to mathematics, we may follow Kuzniak who said that "... *it appeared that rather than focusing on thinking first, it would be more efficient to define and study* ***what kind of geometrical work was at stake*** *in geometry teaching and learning. In this trend, studying geometrical thinking remains a basic and fundamental problem but drawn by geometry understanding in a school context rather than in a laboratory environment*." (Kuzniak et al., 2007, p. 956)

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Prostorno rasuđivanje u matematici

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Sažetak. Uz sve veći interes za prostornim rasuđivanjem, potaknut razvojem snažnih vizualizacijskih programa i računalne geometrije, važno je razjasniti što se podrazumijeva pod prostornim rasuđivanjem u matematici. Polazeći od gledišta različitih matematičkih edukatora, ostvarenje istog obrazovnog cilja *razvijanje prostornog rasuđivanja* ne ostvaruje se uvijek na isti način u matematici kao i u drugim znanostima.

Dakle, premda prostorne sposobnosti mogu biti same po sebi intelektualno zanimljive, u ovom se radu fokus stavlja na njihovu povezanost s poučavanjem i učenjem geometrije na tehničkim fakultetima. Nadalje, detaljno će se opisati kolegij *Nacrtna geometrija s računalnom grafikom* koji je nastao na Rudarsko-geološko-naftnom fakultetu u Zagrebu slijedeći suvremene trendove razvoja geometrijskog obrazovanja. Koristeći između ostalog i tradicionalne geometrijske metode reprezentacije, kolegij se ne usmjerava samo na podizanje razine grafičke i vizualne komunikacije i razvijanje prostornih sposobnosti pojedinca, koje imaju ključnu ulogu u obrazovanju inženjera, već i na razvoj sposobnosti deduktivnog rasuđivanja te korištenje različitih alata i pomagala u matematičkom obrazovanju inženjera. U radu se raspravlja i o utjecaju računalne tehnologije na geometrijsko obrazovanje slijedeći smjernice SEFI – matematičke radne skupine. Dani su i različiti primjeri studentskih vježbi kako bi se prikazale brojne mogućnosti koje se nude studentima kroz razvoj inovativnih i interaktivnih obrazovnih metoda istovremeno primjenjivih u učenju matematike prostora i usklađenih sa specifičnim studentskim interesima.

Ključne riječi: prostorno rasuđivanje, nastavna pomagala, računalna grafika, visoko obrazovanje, e-učenje