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SENSITIVITY TO CORRELATIONS IN STRUCTURAL PROBLEMS

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Summary

The paper presents some aspects of the sensitivity analysis in structural problems involving correlations between the design variables. Time invariant structural problems based on multivariate distribution models consistent with prescribed marginal distributions and correlations are considered. The Monte Carlo simulation methods and the conventional analytical methods are provided for the first order and the second order reliability procedures. The examples given in the paper confirm the relevance of the applied procedures.

Key words: *structural reliability, probability of failure, first order reliability methods, second order reliability method, Monte Carlo simulation methods, parametric sensitivity, correlation, sensitivity analysis*

OSJETLJIVOST NA KORELACIJE U STRUKTURNIM PROBLEMIMA

Sažetak

U članku se prikazuju neki aspekti analize osjetljivosti kod strukturnih problema koji uključuju korelacije medu projektnim varijablama. Razmatraju se vremenski invarijantni strukturni problemi opisani multivarijatnim modelima razdiobe sa zadanim marginalnim razdiobama i korelacijama. Primijenjeni su Monte Carlo simulacijski postupci te analitički postupci prvog i drugog reda za analize pouzdanosti i osjetljivosti. Primjerima se u članku potvrđuje upotrebljivost primijenjenih postupaka.

Ključne riječi: *pouzdanost, vjerojatnost oštećenja, postupci pouzdanosti prvog reda, postupci pouzdanosti drugog reda, Monte Carlo simulacija, parametarska senzitivnost, korelacije, analiza senzitivnosti*

1. Introduction

It is well established that sensitivity analysis is a standard part of any sound engineering procedure since it provides information on the stability (or robustness) of solution and also on the level of sophistication required in the selection and determination of the problem parameters and of the assumptions involved.

In structural reliability analysis efficient methods are available for computing sensitivities of different measures of reliability to changes in distribution and limit state function parameters.

In this paper, the sensitivity to correlation between the random variables in the structural model is investigated. The results of the sensitivity analysis of reliability measures, with respect to the correlation matrix are given as the sensitivity matrices and their appropriate norms. The calculation of the elements of these matrices imply evaluation of derivatives of reliability measures with respect to possibly numerous correlation coefficients.

Application of the finite difference method (FDM) to the sensitivity analysis is useful but inefficient. The presented numerical procedures for sensitivity computation in structural reliability problems with respect to correlations can be implemented within the Monte Carlo simulation (MCS) methods, the first order reliability methods (FORM) and the second order reliability methods (SORM). The sensitivity analysis could be added to the standard reliability analysis, without repeated evaluations of the structural model.

The reliability of time invariant structural problem is defined by multivariate distribution model using n random variables denoted as $\mathbf{X}^T = \{x_1, x_2, ..., x_n\}$. The space of the basic design variables is denoted as the X-space. Each of the variables is defined by its marginal cumulative distribution function (CDF) denoted as F_x, by its marginal probability density function (PDF) denoted as f_x. The (n x n) correlation matrix $\mathbf{R} = [\rho_{km}]$ is considered in general as a function of correlation coefficients $\rho_{km}, k, m = 1,2,...n$.

Multivariate distribution approach based on the Nataf model is applied. The tree principal steps when applying the Nataf's model to structural problems are as follows:

(1) The correlation matrix **R** is transformed to matrix **R'** whose elements are denoted as $\rho'_{km}, k, m = 1,2,...n$. The relation of ρ'_{km} and ρ_{km} is uniquely expressed as $\rho'_{km} = F_{km}\rho_{km}$, [1].

(2) Standard normal variables $\mathbf{Y}^T = \{y_1, y_2, ..., y_n\}$ are obtained by the marginal transformation of **X**: $F(x_i) = \Phi(y_i)$, for $i = 1,2,...,n$. The space of the Y-variables is denoted as the Y-space. The joint PDF $f(\mathbf{X})$ of the random vector **X** is expressed on the basis of the marginal PDF's $f_{x_i}(x_i)$ as follows:

$$f(\mathbf{X}) = \frac{\phi_n(\mathbf{Y}, \mathbf{R}')}{\phi_n(\mathbf{Y})} \prod_{i=1}^{n} f_{x_i}(x_i) \tag{1}$$

$\phi_n(\mathbf{Y}, \mathbf{R}')$ in eqn. (1) is the n-dimensional joint normal PDF of zero means, unit standard deviations and correlation matrix $\mathbf{R}'$:

$$\phi_n(\mathbf{Y}, \mathbf{R}') = \frac{1}{(2\pi)^{n/2} |\mathbf{R}'|^{1/2}} \exp\left(-\frac{1}{2} Q_c\right) \tag{2}$$

where Q_c in eqn. (2) is a quadratic form as shown:

$$Q_c = \mathbf{Y}^T \mathbf{R'}^{-1}\, \mathbf{Y} \tag{3}$$

The joint standard normal PDF for independent random variables in the denominator of eqn. (1) is defined as

$$\phi_n(\mathbf{Y}) \equiv \phi_n(\mathbf{Y},\mathbf{I}) = \prod_{i=1}^{n} \phi_i(y_i) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}Q_y\right) \tag{4}$$

where Q_y in eqn. (4) is a quadratic form as follows:

$$Q_y = \mathbf{Y}^T \mathbf{Y} \tag{5}$$

(3) Vector of variables **Y** can be related to independent standard normal variables $U^T = \{u_1, u_2, ..., u_n\}$ by transformation **Y**=**A U**. The space of the U-variables is denoted as the U-space. The matrix **A** can be obtained by applying the two widely used methods: the one based on Cholesky's decomposition and another is based on spectral decomposition of the correlation matrix **R**=**A A**^{T}.

Regarding the system reliability, each failure mode is considered as an elementary event, also denoted as a "component". The failure modes are defined by the failure function $g_i(\mathbf{X}) = 0,\ i = 1,2,...n_f$, where n_f is the number of failure modes. Failure states are defined by $g_i(\mathbf{X}) \le 0$, while the safe states are defined by $g_i(X) > 0$. Elementary set algebra is used to define structural system as an interaction of components in terms of intersections and unions of elementary failure events. The structural reliability can be formulated as the assessment of the system failure probability, either in the X, Y or in U-space, in the form of the following integrals:

$$P_f = \int_{Dx}^{\cdots}\!\int f(\mathbf{X})d\mathbf{X} = \int_{Dy}^{\cdots}\!\int \phi_n(\mathbf{Y},\mathbf{R'})d\mathbf{Y} = \int_{Du}^{\cdots}\!\int \phi_n(\mathbf{U})d\mathbf{U};\ d\mathbf{X} = \prod_{i=1}^{n} dx_i;\ \text{etc.} \tag{6}$$

The integration domain D_x in eqn. (6) is in general defined by the failure functions as: $g_i(\mathbf{X}) \le 0,\ i = 1,2,...n_f$. The integration domain D_y is obtained as a result of the marginal transformation of the design variables, see step (2). The definition of the domain D_u in the standard normal U-space requires additional transformation, see step (3). The structural reliability is defined using the eqn. (6) as shown:

$$S = 1 - P_f \tag{7}$$

The generalized safety index β_G is expressed as follows:

$$\beta_G = \Phi^{-1}(S) - \Phi^{-1}(P_f) \tag{8}$$

The sensitivities of the reliability measures (RM) such as P_f, S, β_G, etc. with respect to elements of **R** are developed in the sequel.

2. The sensitivity of reliability measures in the Nataf's model

The analytical step in sensitivity evaluation, when using the Nataf's model, is presented first.

2.1. The derivatives of the Nataf's transformation in steps (1) and (2)

Considering the steps (1) from the previous section, the derivatives of the Nataf's correlation matrix can be expressed as shown:

$$e_{km} = \frac{\partial \rho'_{km}}{\partial \rho_{km}} = F_{km} + \rho_{km}\frac{\partial F_{km}}{\partial \rho_{km}}, \qquad k,m = 1,2,...,n\,; \tag{9}$$

The expressions for $e_{km} = \partial\rho'_{km} / \partial\rho_{km}$ for commonly used two-parametric statistical distributions are presented in Appendix A.

Considering step (2) in the Nataf model, the derivatives of the marginal transformation $x_i = F^{-1}_{km}\left[\Phi(y_i)\right]$ can be represented in a diagonal matrix **J**. The elements of **J** are easily obtained in the form of:

$$\mathbf{J} = \left[\frac{\partial x_i}{\partial y_i}\right] = \left[\frac{\phi(y_i)}{f_{x_i(x_i)}}\right], \qquad i = 1,2,...,n\,. \tag{10}$$

2.2. The derivatives of the multinormal distribution in step (2)

The derivatives of the multinormal PDF from eqn. (2) is obtained in a form of a product of the multinormal PDF and a function $d_{km}(.)$:

$$\frac{\partial \phi_n(\mathbf{Y},\mathbf{R'})}{\partial \rho'_{km}} = d_{km}(\mathbf{Y},\mathbf{R'})\phi_n(\mathbf{Y},\mathbf{R'}) \tag{11}$$

where $d_{km}(.)$ is defined as follows:

$$d_{km} = d_{km}(\mathbf{Y},\mathbf{R'}) = -\frac{1}{2}\left(\left|\mathbf{R'}\right|^{-1}\frac{\partial\left|\mathbf{R'}\right|}{\partial\rho'_{km}} + \frac{\partial Q_c}{\partial \rho'_{km}}\right) \tag{12}$$

It can be easily proved that the first term in the eqn. (11) reduces to the element $\overline{\rho}'_{km}$ of the Nataf's correlation matrix inverse $\mathbf{R'}^{-1}$. The derivation of the second term of eqn. (11) $\partial Q_v / \partial\rho'_{km}$ leads also to a simple expression, see eqn. (3) as:

$$\frac{\partial Q_c}{\partial \rho'_{km}} = \mathbf{Y}^T\frac{\partial \mathbf{R'}^{-1}}{\partial \rho'_{km}}\mathbf{Y} = -\mathbf{Y}^T\mathbf{R'}^{-1}\mathbf{r}_{KM}\mathbf{R'}^{-1}\mathbf{Y} \tag{13}$$

where $\mathbf{r}_{KM} = \left[\frac{\partial \mathbf{R'}}{\partial \rho'_{\mathbf{km}}}\right]$ in eqn. (13) is the derivative pointer matrix. The nonzero terms are at locations (k,m) or (m,k), due to $\rho'_{km} = \rho'_{mk}$. Introducing in eqn. (13) an auxiliary vector $\overline{\mathbf{Y}}$ defined as:

$$\overline{\mathbf{Y}} = \left\{\overline{y}_i\right\} = \mathbf{R'}^{-1}\mathbf{Y}; \qquad i = 1,...,n \tag{14}$$

the derivatives of Q_c in eqn. (13), using eqn. (14), can be expressed as:

$$\frac{\partial Q_c}{\partial \rho_{km}^{'}} = \overline{\mathbf{Y}}^T \mathbf{r}_{KM} \overline{\mathbf{Y}} = -2\overline{y}_k \overline{y}_m \tag{15}$$

and finally, taking into account $\rho_{km} = \rho_{mk}$:

$$d_{km} = -\overline{\rho}_{km}^{'} + \overline{y}_k \overline{y}_m \quad \text{or} \quad \mathbf{D} = [d_{km}] = -\mathbf{R'}^{-1} + \overline{\mathbf{Y}}\overline{\mathbf{Y}}^T \tag{16}$$

The vector $\overline{\mathbf{Y}}$ in eqn. (16) can be calculated even without inversion of matrix $\mathbf{R'}$, as a solution of the system of equations $\mathbf{R'}\overline{\mathbf{Y}} = \mathbf{Y}$ or $\mathbf{A}^T \overline{\mathbf{Y}} = \mathbf{U}$. The quadratic form in eqn. (3) can be rewritten using eqn. (14) as product $Q_c = \mathbf{Y}^T \overline{\mathbf{Y}}$.

2.3. The derivatives of the transformation Y=AU in step (3)

The derivatives of the random variables in the U-space with respect to ρ_{km} for a given **Y** (and corresponding **U**) can be in general expressed as shown:

$$\left.\frac{\partial \mathbf{U}}{\partial \rho_{km}}\right|_{\mathbf{U}*} = \left.\frac{\partial \mathbf{A}^{-1}}{\partial \rho_{km}} \mathbf{Y}\right|_{\mathbf{Y}*} = \left.\frac{\partial \mathbf{A}^{-1}}{\partial \rho_{km}} \mathbf{A}\mathbf{U}\right|_{\mathbf{U}*} = \left.-\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial \rho_{km}} \mathbf{U}\right|_{\mathbf{U}*} \tag{17}$$

Two methods used for calculation of derivatives are presented next.

(1) The transformation **Y=LU**, with **L=A** obtained from the Cholesky's decomposition of $\mathbf{R'} = \mathbf{L}\mathbf{L}^T$, transforms the quadratic from Q_c, see eqn. (3), to the form $Q_u = \mathbf{U}^T\mathbf{U}$. **L** is a lower-triangular (*n* x *n*) matrix, with elements λ_{ij}, whose inverse is denoted **M** with elements μ_{ij}, *i*=1,2,…,*n*, *j*=1,2,…,*i*, see Appendix B.

L and **M** always exist since **R'** is symmetric and positive definite. Analytically, the transformation **T** and its inverse $\mathbf{T}^{-1}$ based on Cholesky's decomposition for *j*=1,2,…,*n*, are expressed as:

$$\mathbf{T}(\mathbf{U},\mathbf{R'}): x_j = F_{x_j}^{-1}\left[\Phi(y_j)\right] = F_{x_j}^{-1}\left[\Phi\left(\sum_{i=1}^{j} \lambda_{ji} u_i\right)\right] \tag{18}$$

$$\mathbf{T}^{-1}(\mathbf{X},\mathbf{R'}): u_j = \sum_{i=1}^{j} \mu_{ji} y_i = \sum_{i=1}^{j} \mu_{ji} \Phi^{-1}\left[F_{x_i}^{-1}(x_i)\right] \tag{19}$$

The differentiation applied on eqns. (18) and (19) locally in the point $\mathbf{U}^*$ and corresponding $\mathbf{Y}^*$ and $\mathbf{X}^*$ gives:

$$\left.\frac{\partial x_j}{\partial \rho_{km}}\right|_{\mathbf{U}^*} = e_{km} \frac{\phi(y_j^*)}{f_{x_j}(x_j^*)} \sum_{i=1}^{j} u_i^* \frac{\partial \lambda_{ji}}{\partial \rho_{km}^{'}} \tag{20}$$

$$\left.\frac{\partial u_j}{\partial \rho_{km}}\right|_{\mathbf{X}^*} = e_{km} \sum_{i=1}^{j} \Phi^{-1}\left[F_{x_j}(x_i^*)\right] \frac{\partial \mu_{ji}}{\partial \rho_{km}^{'}} \tag{21}$$

The terms $\partial \lambda_{ji} / \partial \rho_{km}^{'}$ and $\partial \mu_{ji} / \partial \rho_{km}^{'}$ are given in Appendix B.

(2) The transformations of quadratic form Q_c to principal axes, see eqn. (3), can be performed by spectral decomposition of matrix $\mathbf{R'} = \mathbf{V}\Lambda\mathbf{V}^T$. In this case $\mathbf{Y} = \mathbf{V}\Lambda^{1/2}\mathbf{U}$, where $\mathbf{V}$ is the (n x n) square ortonormal matrix with elements v_{ji}, containing the eigenvectors $\mathbf{v}_i = \{v_j\}_i$, $j = 1,2,...,n$. Note that each of the equations is a solution of an eigenproblem $\mathbf{R'v}_i = \lambda_i \mathbf{v}_i$. Eigenvalues λ_i are elements of a diagonal matrix Λ.

Analytically the Nataf transformation and its inverse based on spectral decomposition for j=1,2,…,n is expressed as follows:

$$\mathbf{T}(\mathbf{U},\mathbf{R'}) : x_j = F_{x_j}^{-1}\left[\Phi(y_j)\right] = F_{x_j}^{-1}\left[\Phi\left(\sum_{i=1}^{n} v_{ji}\lambda_i^{1/2}u_i\right)\right] \tag{22}$$

$$\mathbf{T}^{-1}(\mathbf{X},\mathbf{R'}) : u_j = \lambda_j^{-1/2}\sum_{i=1}^{n} v_{ij}y_i = \lambda_j^{-1/2}\sum_{i=1}^{n} v_{ji}\Phi^{-1}\left[F_{x_i}(x_i)\right] \tag{23}$$

The differentiation applied on eqn. (23) in the point $\mathbf{U}^*$ and corresponding $\mathbf{Y}^* = \boldsymbol{A}\boldsymbol{U}^* = (\mathbf{Y})_{U=U^*}$ gives:

$$\left.\frac{\partial \mathbf{U}}{\partial \rho_{km}}\right|_{\mathbf{U}^*} = \frac{\partial}{\partial \rho'_{km}}\left(\Lambda^{-1/2}\mathbf{V}^T\mathbf{Y}^*\right) = \left(-\frac{1}{2}\Lambda^{-3/2}\frac{\partial \Lambda}{\partial \rho'_{km}}\mathbf{V}^T + \Lambda^{-1/2}\frac{\partial \mathbf{V}^T}{\partial \rho'_{km}}\right)\mathbf{Y}^* \tag{24}$$

The terms of derivatives in eqns. (24), are obtained by the perturbation method. The results are given in the sequel. Spectral decomposition of correlation matrix $\mathbf{R'}$ is as follows:

$$\mathbf{R'} = \mathbf{V}\Lambda\mathbf{V}^T = \sum_{i=1}^{n}\lambda_i \mathbf{v}_i \mathbf{v}_i^T \tag{25}$$

The derivatives of eigenvectors and eigenvalues are as shown:

$$\Lambda_{KM}\frac{\partial \Lambda}{\partial \rho'_{km}} = \left[\frac{\partial \lambda_i}{\partial \rho'_{km}}\right] = \left[2(v_k)\ (v_m)_i\right] \tag{26}$$

where (v_k) is the k-th element of I-th eigenvector.

$$\frac{\partial \mathbf{V}}{\partial \rho'_{km}} = \left[\frac{\partial \mathbf{v}_i}{\partial \rho'_{km}}\right] = \mathbf{V}\mathbf{N}_{KM} \tag{27a}$$

and $\mathbf{N}_{KM} = \lfloor n_{ij} \rfloor$ is a skew-symmetric matrix with n_{ij}=0 and

$$n_{ij} = \frac{(v_m)_i(v_k)_j + (v_k)_i(v_m)_j}{\lambda_j - \lambda_i + a}; \qquad \left(a = 0 \text{ if } \lambda_i \neq \lambda_j\right) \tag{27b}$$

A more general method is presented in [2].eqn (24) can be written ass:

$$\left.\frac{\partial \mathbf{U}}{\partial \rho_{km}}\right|_{\mathbf{U}^*} = -e_{km}\Lambda^{-1/2}\left[\mathbf{N}_{KM} + \frac{1}{2}\Lambda^{-1}\Lambda_{KM}\right]\Lambda^{1/2}\mathbf{U}^* = e_{km}\mathbf{W}_{KM}\mathbf{U}^* \tag{28}$$

2.4. The sensitivity of the reliability measures to correlation

(a) The derivative π_{km} of the failure probability to the correlation coefficient ρ_{km} can be obtained from eqn. (6) in the X-space as:

$$\pi_{km} = \frac{\partial P_f}{\partial \rho_{km}} = e_{km} \int_{D_x}^{\cdots} \int \frac{\partial \phi_n [\mathbf{Y}(\mathbf{X}), \mathbf{R}']}{\partial \rho'_{km}} \frac{1}{\prod_{I=1}^{m} [y_i(x_i)]} \prod_{i=1}^{n} f_{x_i}(x_i) d\mathbf{X} \tag{29}$$

and alternatively in Y-space as shown:

$$\pi_{km} = e_{km} \int_{D_y}^{\cdots} \int \frac{\partial \phi_n [\mathbf{Y}, \mathbf{R}']}{\partial \rho'_{km}} d\mathbf{Y} = e_{km} \int_{D_y}^{\cdots} \int d_{km} (\mathbf{Y}, \mathbf{R}') \phi_N (\mathbf{Y}, \mathbf{R}') d\mathbf{Y} \tag{30}$$

The derivative B_{km} of β_G to correlation coefficient ρ_{km} can be obtained from eqns. (8) and (29) or (30), as shown:

$$B_{km} = \frac{\partial \beta_G}{\partial \rho_{km}} = -\pi_{km} / \phi(\beta_G) \tag{31}$$

The derivative of the structural system reliability defined in eqn. (7) can be obtained on the basis of eqns. (29,30) as follows:

$$\frac{\partial S}{\partial \rho_{km}} = -\frac{\partial P}{\partial \rho_{km}} = -\pi_{km} \tag{32}$$

(b) The results of the sensitivity analysis with respect to elements of the correlation matrix **R** can be naturally presented in the sensitivity matrices **S**. Sensitivity matrices are formed as a term wise product of the matrix of derivatives of the reliability measures $[\partial(RM)/\partial \rho_{KM}]$ and a matrix of multiplicators $\mathbf{s} = [s_{km}]$, as:

$$\mathbf{S} = [S_{km}] = \left[\frac{\partial(RM)}{\partial \rho_{km}} s_{km} \right] \tag{33a}$$

Using the derivatives of the failure probability and of the generalized safety index from eqn. (30,31), the corresponding sensitivity matrices read:

$$\pi^s = [\pi_{km} s_{km}] = \left[\frac{\partial(P_f)}{\partial \rho_{km}} s_{km} \right] \tag{33b}$$

$$\mathbf{B}^s = [B_{km} s_{km}] = -\pi^s / \phi(\beta_G) \tag{33c}$$

Some cases of multiplicators are of interest in the sensitivity analysis:

$s_{km} = 1$	rate of change (derivative) of RM
$s_{km} = \Delta\rho_{km}$	increment of RM due to perturbation, i.e the most unfavorable deviation of ρ_{km}
$s_{km} = \Delta\rho_{km} / RM$	logarithmic derivative of RM
$s_{km} = \rho_{km}$	first order approximation of change of RM between corr. and uncorr. case.

(Note that: (1) if the coefficients ρ_{km} are functions of parameters p, factors s_{km} would include terms $\partial\rho_{km}(p)/\partial(p)$, (2) if correlation (k,m) is impossible, the term $\partial(RM)/\partial\rho_{km}$ should be omitted by setting s_{km}=0.

Sensitivity matrices can also be used for sensitivity estimates via their different measures or norms $L_i(\mathbf{S})$ e.g.:

$L_\infty = \max_{k,m}|S_{km}|$ - gives the most influential correlation coefficient;

$L_{row} = \max_{k,m}\left(\sum_m |S_{km}|\right)$ - the row norm; identifies the most influential random variable;

$L_p = \left(\sum_k \sum_m |S_{km}|^p\right)^{1/p}$ - gives the total variability due to correlation (p=1,2);

$\Delta(RM) = \sum_k \sum_m S_{km}^{\Delta\rho}$ - gives the total change in RM due to $s_{km} = \Delta\rho_{km}$.

For the comparisons normalized forms of the sensitivity matrices can be use:

$$\mathbf{S}' = \frac{1}{L_i(\mathbf{S})}\mathbf{S} \tag{34}$$

In the sequel, the application of simulation and analytical methods to calculate sensitivity matrices in an efficient manner will be investigated in order to expand the standard reliability analysis by the sensitivity analysis with respect to numerous correlation coefficients.

3. Analysis of sensitivity to correlation by simulation methods

The simulation procedure is demonstrated on the crude MCS method.

3.1. The failure probability estimates by simulation

(a) By the following substitution:

$$c_\phi(\mathbf{Y},\mathbf{R}') = \frac{\phi_n(\mathbf{Y},\mathbf{R}')}{\phi_n(\mathbf{Y})} \tag{35}$$

the eqn (1) can be rewritten as shown:

$$f(\mathbf{X}) = c_\phi[\mathbf{Y}(\mathbf{X}),\mathbf{R}']\prod_{i=1}^{n} f_{x_i}(x_i) \tag{36}$$

In the zero-one indicator MCS, the state indicator function is defined as $I[g(\mathbf{X})] = 1$ for **X** in the failure set, otherwise zero. The failure probability in the X-space see eqn. (6), cab be expressed by means form the state indicator function as follows:

$$P_f = \int_{(all\ X)}^{\cdots}\int I[g(\mathbf{X})]c_\phi[\mathbf{Y}(\mathbf{X}),\mathbf{R}']\prod_{i=1}^{n} f_{x_i}(x_i)d\mathbf{X} \tag{37a}$$

and alternatively in Y-space as shown:

$$P_f = \int_{(all\ X)}^{\cdots}\int I[g(\mathbf{Y})]c_\phi[\mathbf{Y},\mathbf{R}']\phi_n(\mathbf{Y})d\mathbf{Y} \tag{37b}$$

The MCS could also take place in the U-space, but additional transformations would be needed as described in step (3) of the Nataf's transformation presented in the previous section.

(b) The mean of the estimator $P_f^{'}$ of the failure probability of a correlated problem from eqn. (37a) can be expressed as the expectation with respect to $f_{x_i}(x_i)$ distributions as shown:

$$E\left(P_f^{'}\right)=\frac{1}{N}\sum_{i=1}^{N} c_{\phi}\left[\mathbf{Y}\left(\mathbf{X}^i\right),\mathbf{R}'\right] I\left[g\left(\mathbf{X}^i\right)\right]=P_f \tag{38}$$

(c) The variance of the estimator $P_f^{'}$ is as follows:

$$Var\left[P_f^{'}\right]=\frac{1}{N}\left\{\int\cdots\int c_{\phi}^2(.)I[g(\mathbf{X})]^2\prod_{i=1}^{n}f(x_i)d\mathbf{X}-P_f^2\right\} \tag{39}$$

The upper bound for the failure probability estimate of the original correlated problem and its variance, can be related to the estimates of the failure probability P_f^0 and its variance $Var\left[P_f^0\right]=P_f^0\left(1-P_f^0\right)/N$ of a hypothetically uncorrelated problem as:

$$P_f<\frac{1}{N}\sum_{i=1}^{N}c_{\phi\max}I\left[g\left(\mathbf{X}^i\right)\right]=c_{\phi\max}\frac{1}{N}\sum_{i=1}^{N}I\left[g\left(\mathbf{X}^i\right)\right]=c_{\phi\max}P_f^0 \tag{40}$$

$$Var\left[P_f^{'}\right]<\frac{1}{N}\left[c_{\phi\max}P_f-P_f^2\right]<\frac{1}{N}c_{\phi\max}P_f<c_{\phi\max}^2Var\left[P_f^0\right] \tag{41}$$

The value of $c_{\phi\max}$ in eqns. (40,41) can be obtained by applying a deterministic constraint optimization or by a trial simulation. The number of samples N_p for the failure probability estimates $P_f^{'}$ of a correlated problem can be related to the number of samples N_0 of a hypothetically uncorrelated problem used to obtain the prescribed coefficient of variation $COV=\left[\left(1-P_f^0\right)/\left(N_0P_f^0\right)\right]^{1/2}$, on the basis of eqns. (40,41) as follows:

$$COV<\left[\frac{1}{N}\left(c_{\phi\max}\frac{1}{P_f}-1\right)\right]^{1/2} \tag{42}$$

$$N_p<COV^{-2}\left(c_{\phi\max}\frac{1}{P_f}-1\right)=\frac{N_0P_f^0}{1-P_f^0}\left(c_{\phi\max}\frac{1}{P_f}-1\right)\approx N_0c_{\phi\max} \tag{43}$$

3.2. The estimates of the derivative values by simulation

(a) The eqn. (29) can be rewritten using eqn. (36) as follows:

$$\pi_{km}=\frac{\partial P_f}{\partial\rho_{km}}=e_{km}\int\cdots\int_{(all\ X)}c_{\phi}(.)I[g(\mathbf{X})]\prod_{i=1}^{n}f_{x_i}(x_i)d\mathbf{X} \tag{44}$$

The mean of the derivative estimate of the failure probability to correlation coefficient from eqn. (44) $\pi_{km}^{'}$ can be expressed as the expectation with respect to $f_{x_i}(x_i)$ distributions as shown:

$$E\left(\pi_{km}^{'}\right)=\frac{1}{N}e_{km}\sum_{i=1}^{N}c_{\phi}(.)d_{km}(.)I\left[g\left(\mathbf{X}^i\right)\right]=\pi_{km} \tag{45}$$

Comparing eqn. (45) to eqn. (38), it is easy to recognize that calculation of matrix of derivatives $\pi = [\pi_{km}]$ in eqn. (45), requires only additional multiplication of the $c_\phi(\mathbf{Y}_i, \mathbf{R}')$ defined in eqn. (35) by $e_{km} d_{km}(\mathbf{Y}_i, \mathbf{R}')$ from eqn. (16) for each $\mathbf{Y}_i$. Note that $c_\phi(.)$ is calculated the anyway in estimation of failure probability together with $\overline{\mathbf{Y}}_i = \mathbf{R}'^{-1}\,\mathbf{Y}_i$

(b) The variance of the estimator π'_{km} of the eqn. (44) is as follows:

$$Var[\pi'_{km}] = \frac{1}{N} e^2_{km} \left\{ \int_{(all\ X)}^{\cdots} \int [c_\phi(.) d_{km}(.)]^2 I[g(\mathbf{X})]^2 \prod_{i=1}^{n} f_{x_i}(x_i) d\mathbf{X} - \pi^2_{km} \right\} \tag{46}$$

The upper bounds of the derivative estimate and its variance from eqns. (45,46) can be related to failure probabilities P_f and P_f^0 , as well as to their variances as it is shown by:

$$|\pi_{km}| < |d_{km}|_{\max} e_{km} P_f < c_{\phi \max} |d_{km}|_{\max} e_{km} P_f^0 \tag{47}$$

$$\begin{aligned} Var[\pi'_{km}] &< \frac{1}{N} e^2_{km} \left\{ c_\phi d_{km}\big|_{\max} |\pi_{km}| - \pi^2_{km} \right\} < \\ &< \frac{1}{N} e^2_{km} \left\{ c_\phi d_{km}\big|_{\max} |d_{km}|_{\max} P_f - \pi^2_{km} \right\} < e^2_{km} \left[c_\phi d_{km} \right]^2 Var[P_f^0] \end{aligned} \tag{48}$$

The value of $|d_{km}|_{\max}$, as well as the value of $|c_\phi d_{km}|_{\max}$, in eqns. (47, 48) can be obtained by applying a deterministic constraint optimization procedure or by a preliminary trial simulation. The coefficient of variation of the sensitivity factor π_{km} can be expressed on the basis of eqns. (47, 48), as follows:

$$COV < e_{km} \left\{ \frac{1}{N} \left[|c_\phi d_{km}|_{\max} \frac{1}{|\pi_{km}|} - 1 \right] \right\}^{1/2} \tag{49}$$

The number of samples N_s to get the prescribed coefficient of variation for the estimates of the sensitivity factor π_{km} is derived from eqn. (49) and can be presented as follows:

$$\begin{aligned} N_s &< \frac{e_{km}}{COV^2} \left[|c_\phi d_{km}|_{\max} \frac{1}{|\pi_{km}|} - 1 \right] = \frac{N_0 P_f^0}{1 - P_f^0} e_{km} \left[|c_\phi d_{km}|_{\max} \frac{1}{|\pi_{km}|} - 1 \right] \approx \\ &\approx N_0 \frac{P_f^0}{|\pi_{km}|} e_{km} |c_\phi d_{km}|_{\max} \end{aligned} \tag{50}$$

The eqn. (50) indicates that the prescribed coefficient of variation for the sensitivity factor is attainable with the number of samples N_s related to the number of samples N_0 for a hypothetically uncorrelated problem.

The presented procedures are applicable to the crude Monte Carlo integration procedures, as well as to sampling procedures and other variance reduction techniques, either in the original random variable space, or in standard normal space.

4. Analysis of sensitivity to correlations by analytical methods

The conventional sensitivity analysis within FORM is provided first [3], giving a versatile insight into effects of correlation to reliability problem. Next, an efficient procedure for sensitivity calculation is presented. Finally, the sensitivity analysis using the 'fitting' method [4], especially within SORM, is investigated.

4.1. The conventional analytical sensitivity analysis in FORM

(a) Most current methods of FORM and SORM transform the original problem into standard normal space and fit the approximate failure surface in this transformed space. Using the quadratic form defined in the U-space $Q_u = \mathbf{U}^T\mathbf{U}$, the FORM safety index β_i form the I-th failure mode can be defined as shown:

$$\beta_i = \min_{g[T(\mathbf{U},\mathbf{R})]_{\le 0}} Q_u^{1/2} = \left(\mathbf{U}^{*T}\mathbf{U}^*\right)^{1/2} = \sum_{k=1}^{n} \alpha_k^* u_k^* \tag{51}$$

The solution to the optimization problem in eqn. (51) is the most probable failure point (design point, β-point) $\mathbf{U}=\mathbf{U}^*$, or $\mathbf{Y}^* = (\mathbf{Y})_{\mathbf{U}=\mathbf{U}^*}$ or $\mathbf{X}^* = (\mathbf{X})_{\mathbf{U}=\mathbf{U}^*}$.

The α_k^* is the component of the unit normal vector α to the failure surface g_i directed towards the failure set (or the direction cosine) in the design point $\mathbf{U}^*$ in the U-space defined as follows:

$$\alpha_k^* = \alpha_k\big|_{\mathbf{U}^*} = -\left.\frac{\partial g_i/\partial u_k}{\left[\sum_j \left(\partial g_i/\partial u_j\right)^2\right]^{1/2}}\right|_{\mathbf{U}^*} = \frac{u_k^*}{\beta_i} = \left(\frac{\partial \beta_i}{\partial u_k}\right)_{\mathbf{U}^*} \tag{52}$$

The probability of failure P_f, as presented in eqn. (6) for the i-th failure mode, can be approximately expressed for closely linear failure surfaces by means of safety index from eqn. (51):

$$P_f = \Phi(-\beta_i) \tag{53}$$

Conventional approach for sensitivities to correlation of FORM estimates for a single failure mode, e.g. [3], yields to the following approximate (see [4]) values for derivatives:

$$\frac{\partial \beta_i}{\partial \rho_{km}} = \frac{e_{km}}{2\beta_i}\left(\frac{\partial Q_u}{\partial \rho_{km}}\right)_{\mathbf{U}=\mathbf{U}^*} = e_{km}\sum_{j=1}^{n} \alpha_j^* \frac{\partial u_j^*}{\partial \rho_{km}'} \tag{54a}$$

The error of replacing the derivative of β of true design point with the derivative of β of original design point w.r.t ρ_{km} is of order $\Delta\rho_{km}^2$ as shown in [4] and the derivative should be used in this context.

The eqn. (54a) can be rewritten in matrix form using eq. (17) and (51) as follows:

$$\frac{\partial \beta_i}{\partial \rho_{km}} = \frac{e_{km}}{\beta_i}\left(\mathbf{U}^T \frac{\partial \mathbf{U}}{\partial \rho_{km}} \right)_{\mathbf{U}^*} = -\frac{e_{km}}{\beta_i} \mathbf{U}^{*T} \mathbf{A}^{-1} \frac{\partial \mathbf{A}}{\partial \rho_{km}} \mathbf{U}^* \tag{54b}$$

In addition, the derivative of the gradient vector $\nabla \mathbf{g}_u = \left\{ \partial g_i \big/ \partial u_k \right\}$ in β-point reads:

$$\left. \frac{\partial (\nabla \mathbf{g}_u)_i}{\partial \rho_{km}} \right|_{\mathbf{U}^*} = e_{km} \frac{\partial \mathbf{A}^T}{\partial \rho_{km}} \mathbf{J}^* (\nabla \mathbf{g}_x)_i \big|_{\mathbf{X}^*} \tag{54c}$$

The derivatives of the failure probability w.r.t correlation coefficient is easily obtained using eqns. (54) as follows

$$\frac{\partial P_f}{\partial \rho_{km}} = -\phi(\beta_i) \frac{\partial \beta_i}{\partial \rho_{km}} \tag{55}$$

(b) Three approaches are used to calculate $\partial \beta_i / \partial \rho_{km}$:

(1) Derivatives of the safety index β_I to correlation coefficients ρ_{km} , k,m=1,2,…,n, based on Cholesky's decomposition **(A=L)** applied on FORM concept, see eqns. (54a) and (21), using Nataf's model, can be presented as follows:

$$\frac{\partial \beta_i}{\partial \rho_{km}} = e_{km} \sum_{j=1}^{n} \alpha_j^* \sum_{l=1}^{j} y_j^* \frac{\partial \mu_{jl}}{\partial \rho'_{km}} \tag{56}$$

The derivatives of the failure surface gradient vector components can be obtained using eqn. (54c) as follows:

$$\frac{\partial}{\partial \rho_{km}} \left(\frac{\partial g_i}{\partial u_j} \right)_{\mathbf{U}^*} = e_{km} \sum_{l=j}^{n} \frac{\partial \lambda_{lj}}{\partial \rho'_{km}} \frac{\partial x_l}{\partial y_l} \left. \frac{\partial g_i}{\partial x_l} \right|_{X^*;Y^*} \tag{57}$$

The terms $\partial \mu / \partial \rho$ and $\partial \lambda / \partial \rho$ are given in Appendix B. The terms $\dfrac{\partial x_i}{\partial y_i}$ are defined in eqn. (10).

(2) Derivatives of the safety index β_i to correlation coefficients ρ_{mn}, m,n=1,2,…,n, based on spectral decomposition $\left(\mathbf{A} = \mathbf{V}\Lambda^{1/2}\right)$ applied to FORM concept, see eqn. (28), using Nataf's model, can be presented as follows:

$$\frac{\partial \beta_i}{\partial \rho_{km}} = -\frac{e_{km}}{\beta_i} \mathbf{U}^{*T} \Lambda^{1/2} \left[\mathbf{N} + \frac{1}{2} \Lambda^{-1} \frac{\partial \Lambda}{\partial \rho_{km}} \right] \Lambda^{1/2} \mathbf{U}^* = -\frac{e_{km}}{\beta_i} \mathbf{U}^{*T} \mathbf{W} \mathbf{U}^* \tag{58}$$

(3) An efficient procedure form calculation of all the derivatives at once, convenient also for larger problems, can be obtained using the alternative expression of FORM safety index β_i in eqn. (51), as follows:

$$\beta_i = \left(\mathbf{Y}_i^{*T} \mathbf{R}'^{-1} \mathbf{Y}_i^* \right)^{1/2} \tag{59}$$

Using earlier considerations in section 2.2 and eqn. (15), with $\overline{\mathbf{Y}}^{*}=\mathbf{R}^{-1}\mathbf{Y}^{*}$, the derivative can be obtained as follows for the i-th failure surface:

$$\frac{\partial \beta_i}{\partial \rho_{km}}=\frac{e_{km}}{2\beta_i}\left(\mathbf{Y}_i^{*T}\frac{\partial \mathbf{R}'^{-1}}{\partial \rho_{km}}\mathbf{Y}_i^{*}\right)=-\frac{e_{km}}{2\beta_i}\overline{\mathbf{Y}}_i^{T}\mathbf{r}_{KM}\overline{\mathbf{Y}}_i=-\frac{e_{km}}{\beta_i}\left(\overline{y}_m\right)_i\left(\overline{y}_k\right)_i \tag{60}$$

The elements of the sensitivity matrices $\mathbf{B}_i^s$ and Π_i^s for the i-th failure mode, can therefore be presented simply as follows:

$$B_{ikm}=\frac{\partial \beta_i}{\partial \rho_{km}}=-\frac{e_{km}}{\beta_i}\left(\overline{y}_k\,\overline{y}_m\right)_i \tag{61a}$$

$$\Pi_{ikm}=\frac{\partial P_{fi}}{\partial \rho_{km}}=-\phi(\beta_i)B_{ikm} \tag{61b}$$

Note that the procedure (3) is simpler and faster than the procedures (1) and (2) which require derivatives of transformation matrices, particularly for larger problems. Procedures (1) and (2) give better insight into the local changes in the vicinity of the design point.

(c) Regarding the structural system analysis, the asymptotic values of the sensitivity factors to distributional parameters for fully dependent and fully independent series system, as well as for parallel systems, are considered, [3].

The sensitivity estimate of system reliability may be quite inaccurate. Improvements for series systems using bimodal joint probabilities of failure P_{ij}, based on Ditlevsen's upper bound P_u , can be obtained as shown:

$$\pi_{km}=\frac{\partial P_u}{\partial \rho_{km}}=\sum_{i=1}^{n_f}\frac{\partial P_i}{\partial \rho_{km}}-\sum_{i=2}^{n_f}\frac{\partial}{\partial \rho_{km}}\left(\max_{j<i} P_{ij}\right) \tag{62}$$

The joint failure probability P_{ij} in eqn. (62) can be expressed by means of a single integral over the bivariate normal PDF $\phi_2\left(\beta_i,\beta_j;z\right)$ with zero mean values, unit variances and correlation coefficient z, with the upper integration bound equal to the mode correlation coefficient γ_{ij}, as follows:

$$P_{ij}=\Phi\left(-\beta_i\right)\Phi\left(-\beta_j\right)+\int_0^{\gamma_{ij}}\phi_2\left(\beta_i,\beta_j;z\right)dz \tag{63}$$

The derivatives of the joint failure probability P_{ij} to correlation coefficient ρ_{km} can be expressed as shown:

$$\chi_{ijkm}=\frac{\partial P_{ij}}{\partial \rho_{km}}=C_{ij}B_{ikm}+C_{ji}B_{jkm}+\phi_2\left(\beta_i,\beta_j;\gamma_{ij}\right)H_{ijkm} \tag{64a}$$

where the derivative terms B_{ikm} and B_{jkm} are given in eqn. (61a). The H_{ijkm} terms in eqn. (64a) for the derivatives of the mode correlation coefficients γ_{ij} w.r.t. correlation coefficients ρ_{km} are obtained via derivatives of the cosine of the angle between the normalized gradient vectors α_i and α_j in the design points $\mathbf{U}_i^{*}$ and $\mathbf{U}_j^{*}$:

$$H_{ijkm}=\left.\frac{\partial \gamma_{ij}}{\partial \rho_{km}}\right|_{\mathbf{U}_i^*\mathbf{U}_j^*}=e_{km}\left\{\frac{\left(\overline{y}_k\right)_i\left(\overline{y}_m\right)_j}{\beta_i\beta_j}+\frac{\left(\overline{y}_m\right)_i\left(\overline{y}_k\right)_j}{\beta_i\beta_j}-\gamma_{ij}\left[\left(\frac{\overline{y}_m\,\overline{y}_k}{\beta^2}\right)_i\left(\frac{\overline{y}_m\,\overline{y}_k}{\beta^2}\right)_j\right]\right\} \tag{64b}$$

The terms C_{ij}, C_{ji} and a development of $H_{ijkm} = \partial\gamma_{ij} / \partial\rho_{km}$ are given in Appendix C. Alternatively, less accurate method using "conditional safety indices" [3] can be applied to avoid the numerical integration. Finally, the local derivatives of the failure probability to the correlation matrix **R**, based on Ditlevsen's upper bound are:

$$\pi_{km} = \sum_{i=1}^{n_f} \Pi_{ikm} - \sum_{i=2}^{n_f} \left(\chi_{ijkm} \Big| \max_{j<i} P_{ij} \right) \tag{65}$$

The derivatives defined by eqns. (61a,b, 64a,b, 65) within the FORM concept can be used in the following sensitivity matrices:

$\mathbf{B}^s = \left[B_{ikm} s_{km}\right] = \left[\dfrac{\partial\beta_i}{\partial\rho_{km}} s_{km}\right]$ - sensitivities of safety indices

$\mathbf{H}^s = \left[H_{ijkm} s_{km}\right] = \left[\dfrac{\partial\gamma_{ij}}{\partial\rho_{km}} s_{km}\right]$ - sensitivities of bimodal correlation coefficient γ_{ij}

$\Pi^s = \left[\Pi_{ikm} s_{km}\right] = \left[\dfrac{\partial P_i}{\partial\rho_{km}} s_{km}\right]$ - sensitivities of "i"-th mode failure probability

$\chi^s = \left[\chi_{ijkm} s_{km}\right] = \left[\dfrac{\partial P_{ij}}{\partial\rho_{km}} s_{km}\right]$ - sensitivities of joint failure probabilities (for modes i and j)

$\pi^s = \left[\pi_{km} s_{km}\right] = \left[\dfrac{\partial P_u}{\partial\rho_{km}} s_{km}\right]$ - sensitivity of failure probability (upper bound)

These sensitivity matrices, jointly used, enable identification of most significant parameters involving correlations on the structural reliability problems.

4.2. The sensitivity analysis using the "fitting" method

The method is based on the fitting of each of the failure surfaces to a set of points in the standard normal space, [4]. A change in a parameter changes the transformation into the standard normal space. The failure probability based on the new failure surfaces is used to estimate the sensitivity. The set of n+1 arbitrarily selected non-collinear points in the vicinity of the design point is used to define the linearized failure surface in the original space. The same set of points is also used to get the linearized failure surface in the new space, using the inverse transforms. The inverse transforms when the Cholesky's decomposition is applied for all j=1,2,…,n, are easily derived from eqn. (13) for a prescribed finite difference $\Delta\rho$ as follows:

$$\mathbf{T}^{-1}_{\rho+\Delta\rho} : u_j^{*+\Delta\rho} = \sum_{i=1}^{j} \mu_{ij}^{\rho+\Delta\rho} \Phi^{-1}\left[F_{x_i}(x_i)\right] \tag{66}$$

For the spectral decomposition, the inverse transformation from eqn. (17) is as shown:

$$\mathbf{T}^{-1}_{\rho+\Delta\rho} : u_j^{*+\Delta\rho} = \left(\lambda_i^{\rho+\Delta\rho}\right)^{-1/2} \sum_{i=1}^{n} \nu_{ij}^{\rho+\Delta\rho} \Phi^{-1}\left[F_{x_i}(x_i)\right] \tag{67}$$

The transformations in eqns. (66, 67) can be provided by repeated matrix manipulations for a changed parameter $\Delta\rho$.

The elements of the transformation matrices for incremented values of correlation coefficients in eqns. (66,67) can be assessed by using the decomposition matrix derivatives given in Appendix B and section 2.2.2., as shown:

$$\mu_{ij}^{\rho+\Delta\rho} \approx \mu_{ij}^{\rho} + \left(\partial\mu_{ij}/\partial\rho\right)\Delta\rho \tag{68}$$

$$\nu_{ij}^{\rho+\Delta\rho} \approx \nu_{ij}^{\rho} + \left(\partial\nu_{ij}/\partial\rho\right)\Delta\rho \text{ and } \lambda_{i}^{\rho+\Delta\rho} = \lambda_{i}^{\rho} + \left(\partial\lambda_{i}/\partial\rho\right)\Delta\rho \tag{69}$$

The coordinates of the point in the transformed standard normal space also can be assessed by using the derivatives from eqns. (14) and (18), if available, as shown:

$$u_{j}^{\rho+\Delta\rho} \approx u_{j}^{\rho} + \frac{\partial u_{j}}{\partial\rho}\Delta\rho \tag{70}$$

For a small increment in the correlation coefficient, the analytical method can be used to obtain a new linear approximation for each component. These new linear approximations can be used to estimate the changed failure probability and the sensitivity of the structural system. For Ditlevsen's bound in eqn. (61) in general the followings valid:

$$\frac{\Delta P_{u}}{\Delta\rho} = \frac{P_{u}^{\rho+\Delta\rho} - P_{u}^{\rho}}{\Delta\rho} = \sum_{i=1}^{n_f}\frac{P_{i}^{\rho+\Delta\rho} - P_{i}^{\rho}}{\Delta\rho} - \sum_{i=2}^{n_f}\frac{1}{\Delta\rho}\left(\max_{j<i} P_{ij}^{\rho+\Delta\rho} - \max_{j<i} P_{ij}^{\rho}\right) \tag{71}$$

The eqn. (71) can be solved by recalculating the upper Ditlevsen's bound for the incremented value of the correlation coefficient ρ.

The advantage of the method [2] is that it can be used to efficiently and accurately compute sensitivities of safety measures with respect to the correlation coefficients form SORM estimates. The method for computing the sensitivities for SORM is similar to that for FORM. The points used in the original standard normal space to fit the second order surface are transformed into the new standard normal space and a new second order surface is used to compute the sensitivity of the SORM estimates. The failure probability corresponding to this new second-order surface can be used to compute the sensitivity of the SORM estimates.

5. EXAMPLES FOR SENSITIVITY ANALYSIS OF RELIABILITY MEASURES TO CORRELATION COEFFICIENTS

5.1. Example 1: Linear failure functions

A problem with two dependent random variables, defined by Gaussian marginal distributions x_1: $N(\mu_1 = 8,\ \sigma_1 = 2)$; x_2: $N(\mu_2 = 5,\ \sigma_2 = 1)$ and by the correlation coefficient ρ_{12} between the design variables, is considered first. The two linear limit state functions are defined as $g_1(x_1, x_2) = x_1 - x_2$ and $g_2(x_1, x_2) = x_1 + x_2 - 9$. FORM results considering failure probabilities and sensitivity factors for selected values of correlation coefficient $\rho = \rho_{12}$ are presented in Table 1.

Table 1 FORM results for different values of r_{12} - Example 1

Tablica 1. FORM rezultati za razne vrijednosti r_{12} - Primjer 1

ρ_{12}	p_1	p_2	p_{12}	γ_{12}	p_f	$\partial p_1/\partial \rho$	$\partial p_2/\partial \rho$	$\partial p_{12}/\partial \rho$	$\partial \gamma/\partial \rho$	$\partial p_f/\partial \rho$
.00	.090	.037	.020	.600	.109	-.087	+.058	+.012	.000	-.0415
.05	.085	.039	.020	.605	.107	-.089	+.058	+.009	.020	-.0415
.25	.067	.051	.020	.612	.099	-.097	+.057	+.002	.102	-.0415
.50	.042	.066	.018	.655	.088	-.103	+.055	-.012	.249	-.0354
.75	.017	.079	.012	.750	.083	-.089	+.052	-.037	.562	.0000
.95	.003	.089	.003	.923	.089	-.043	+.049	-.042	1.329	+.0492
.999	.001	.091	.001	.998	.091	-.027	+.049	-.003	1.766	+.0500

Note restriction on $\Delta\rho_{km}$ - value, from [4], in applying derivative terms. Next, the prediction of the sample size for the MCS assessment of the sensitivities is considered. For $\rho_{12} = 0.5$, the maximal values of $c_{\phi\max} = 18$, $d_{12,\max} = 4$ and $\lfloor c_\phi d_{12} \rfloor_{\max} = 72$ are obtained in a trial simulation procedure. The number of samples needed to obtain the prescribed level of accuracy of the sensitivity factor is predicted according to eqn. (50) as $N_s < N_0 73(.109/.0354) = 220 N_0$. The variance of the sensitivity factor estimates is predicted to be less then 2.6/N, where N is the actual sample size.

Finally, a simulation experiment of 100 independent MCS was carried through to obtain the mean and the variance of the estimates the sensitivity factors. The convergence rate of the experiment with the 95% confidence intervals and the variance upper bound prediction $\mathrm{Var(s)}^{\mathrm{upp}}$ compared to the experimentally obtained variance Var(s) are presented in Table 2.

Table 2 Crude MCS results for $r_{12} = 0.5$ - Example 1

Tablica 2. Rezultati grube Monte Carlo simulacije za $r_{12} = 0.5$ - Primjer 1

N	p_f	Var(p_f)	c.o.v.	$\partial \rho_f/\partial \rho$	Var	$\mathrm{Var}_{\mathrm{upp}}$	c.o.v.
500	.092±.003	3.6×10^{-4}	.21	-.031±.005	2.2×10^{-3}	(5.2×10^{-3})	1.34
1000	.090±.002	2.3×10^{-4}	.17	-.031±.005	1.2×10^{-3}	(2.6×10^{-3})	1.26
2000	.089±.001	1.0×10^{-4}	.11	-.033±.004	5.9×10^{-4}	(1.3×10^{-3})	0.73
5000	.088±.001	3.4×10^{-5}	.07	-.036±.002	2.3×10^{-4}	(5.2×10^{-4})	0.44
10000	.088±.001	1.8×10^{-5}	.05	-.036±.001	1.1×10^{-4}	(2.6×10^{-4})	0.29

Comments on results of Example 1:

- MCS method, see Table 2, gives coinciding results to FORM, see Table 1, i.e. the exact result.
- The convergence rate of the sensitivity factor is slower then the rate of convergence of the failure probability itself, see Table 2, and follows the predictions based on eqn. (50).
- The total sensitivity to correlation can be assessed from Table 1 as follows: $P_f(\rho = .75) - P_f(\rho = 0) = -0.026$.

5.2. Example 2: Component reliability, nonlinear failure function

Component reliability with three dependent random variables given by marginal distributions and correlation matrix is considered.

Distributions: x_1: lognormal $(\mu_1 = 500,\ \sigma_1 = 100)$

x_2: lognormal $(\mu_2 = 2000,\ \sigma_2 = 400)$

x_3: uniform $(\mu_3 = 5,\ \sigma_3 = 0.5)$

Deterministic parameter: t=1.0

Correlation matrix: $$\mathbf{R}=\begin{bmatrix}1.0 & 0.3 & 0.2\\ 0.3 & 1.0 & 0.2\\ 0.2 & 0.2 & 1.0\end{bmatrix}$$

Limit state function: $$g(x_1,x_2,x_3)=t_1-\frac{x_1}{1000x_3}-\left(\frac{x_1}{2000x_3}\right)^2$$

Crude MCS result for the probability of failure with the 95% confidence interval, obtained in a simulational experiment of 100 independent runs with 10000 samples each, is as shown:

$$P_f=0.0342\pm0.0005,\ \text{c.o.v.=0.05},\ \left(\beta_G=1.822\right).$$

FORM results for the failure probability and for the safety index are as shown: $P_f=0.0381,\ \beta=1.772$

The prediction of the upper bounds of the sample size and of the variance according to eqns. (50,48) are presented in Table 3.

Table 3 Simple size and variance prediction – Example 2
Tablica 3. Predviđanje veličine i varijance uzorka –Primjer 2

k	m	ρ_{km}	ρ'_{km}	$\partial\rho'_{km}/\partial\rho_{km}$	$c_{\phi\max}$	$(c_\phi d_{km})_{\max}$	π_{km}	P_f^0	$N_s<$	$Var(\pi_{km})<$
2	1	.300	.304	1.0078	20	60	+.0427	.0468	65No	2.6/N
3	1	.200	.206	1.0330	20	60	-.0738	.0468	38No	4.4/N
3	2	.200	.206	1.0330	20	30	-.0388	.0468	36No	1.1/N

Table 4 presents the results of the sensitivity analysis using FORM procedure for component reliability and compared to the MCS experiment of 100 independent runs of 10000 samples each.

Table 4 Sensitivity of component probability of failure – Example 2
Tablica 4. Senzitivnost vjerojatnosti oštećenja komponente na korelacije – Primjer 2

		FORM		MONTE CARLO SIMULATION			
k	m	$\partial\beta_G/\partial\rho_{km}$	$\partial P_f/\partial\rho_{km}$	$\partial P_f/\partial\rho_{km}$	Var	Var^{upp}	c.o.v.
2	1	-.515	+.0426	+.037±.001	4.3 x 10^{-5}	26 x 10^{-5}	.18
3	1	+.890	-.0738	-.060±.001	1.7 x 10^{-5}	44 x 10^{-5}	.07
3	2	+.468	-.0388	-.034±.001	1.2 x 10^{-5}	11 x 10^{-5}	.10

Next, the total sensitivity to correlations is checked by repeated FORM calculations for a hypothetically uncorrelated problem and for a highly correlated problem giving results:

$$\beta_{uncorr}=1.67,\quad (P_f)_{uncorr}=0.047;\qquad \beta_{hicorr}=2.47,\ (P_f)_{hicorr}=0.007;$$

$$\Delta\beta_{tot}=\beta_{hicorr}-\beta_{uncorr}=0.80,\qquad (\Delta P_f)_{glo}=(P_f)_{hicorr}-(P_f)_{uncorr}=-.040;$$

The results can be compared to $\Delta(P_f)=-0.070$ and $\Delta(\beta)=0.843$ in FORM, and to $\Delta(P_f)=-0.057$ in MCS.

5.3. Example 3: System reliability

The system reliability of a problem with three limit state functions and seven dependent random variables defined by marginal distributions and correlation coefficients is considered.

Margin distributions:

x_1, x_2 : Weibull $(\mu = 134,\ \sigma = 23)$ x_3 : Uniform $(\mu = 160,\ \sigma = 35)$

x_4, x_5 : Weibull $(\mu = 150,\ \sigma = 30)$ x_6 : Weibull $(\mu = 65,\ \sigma = 20)$

x_7: uniform $(\mu_3 = 50,\ \sigma_3 = 15)$, Deterministic parameter: t=5.0

Following correlation coefficients are given:

ρ_{12}=.4, ρ_{13}=.2, ρ_{14}=.2, ρ_{15}=.2, ρ_{23}=.4, ρ_{24}=.2, ρ_{25}=.2, ρ_{34}=.4, ρ_{35}=.2, ρ_{45}=.4, and ρ_{67}=.4,

Limit state functions under considerations are as follows:

$$g_1(X) = x_1 + x_2 + x_4 + x_5 - x_6 t$$

$$g_2(X) = x_1 + 2x_3 + 2x_4 + x_5 - x_6 t - x_7 t$$

$$g_3(X) = x_2 + 2x_3 + x_4 - x_7 t$$

Table 5 presents the sensitivity factors based on upper probability bound, using the FORM procedures for the system reliability and compared to the direct numerical calculation, using finite difference of $\Delta\rho = 0.01$ (FDM), as well as the crude MCS results from an experiment of 50 runs with 30x10000 samples each.

Table 5 Sensitivity of system reliability to correlations– Example 2
Tablica 5. Senzitivnost sistemske pouzdanosti na korelacije – Primjer 2

k	m	ρ_{km}	$\partial\beta_G/\partial\rho_{km}$ FORM	(FDM)	FORM	$\partial P_f/\partial\rho_{km}$ MCS	c.o.v.	(FDM)
2	1	0.4	-.029	(-.029)	+.0040	+.0034±.0012	1.45	(+.0040)
3	1	0.2	-.040	(-.041)	+.0055	+.0047±.0006	0.52	(+.0056)
4	1	0.2	-.074	(-.072)	+.0101	+.0077±.0007	0.36	(+.0099)
5	1	0.2	-.056	(-.055)	+.0077	+.0074±.0018	1.01	(+.0076)
3	2	0.4	+.030	(+.030)	-.0040	-.0028±.0006	0.97	(-.0040)
4	2	0.2	-.025	(-.022)	+.0033	+.0030±.0007	1.13	(+.0031)
5	2	0.2	-.037	(-.036)	+.0051	+.0050±.0021	1.71	(+.0050)
4	3	0.4	-.161	(-.163)	+.0220	+.0190±.0010	0.20	(+.0223)
5	3	0.2	-.054	(-.056)	+.0073	+.0061±.0005	0.35	(+.0076)
5	4	0.4	-.097	(-.095)	+.0133	+.0115±.0011	0.43	(+.0130)
7	6	0.4	-.180	(-.176)	+.0245	+.0217±.0006	0.10	(+.0240)
Total:			-.730	(-.665)	+.0975	-.0862		(+.0975)

FORM result for hypothetically uncorrelated problem are as shown:

$$P_{Suncorr} = 0.035 \quad \text{and} \quad \beta_{Guncorr} = 1.81$$

FORM result for hypothetically highly correlated problem $(\rho_{km} = 0.98, \text{ for all k,m})$ are:

$$P_{Shicorr} = 0.111 \quad \text{and} \quad \beta_{Ghicorr} = 1.21$$

The total sensitivities to correlation based on upper bounds are as follows:

$$\beta_{hicorr} - \beta_{uncorr} = -0.60, \quad P_{Shicorr} - P_{Suncorr} = 0.076.$$

The FORM results are presented also in the matrix form as shown:

$$\pi^s = \begin{bmatrix} .0000 & +.0040 & +.0055 & +.0102 & +.0077 & .0000 & .0000 \\ +.0040 & .0000 & -.0041 & +.0033 & +.0052 & .0000 & .0000 \\ +.0055 & -.0041 & .0000 & +.0219 & +.0074 & .0000 & .0000 \\ +.0102 & +.0033 & +.0219 & .0000 & +.0133 & .0000 & .0000 \\ +.0077 & +.0052 & +.0074 & +.0133 & .0000 & .0000 & .0000 \\ .0000 & .0000 & .0000 & .0000 & .0000 & .0000 & +.0246 \\ .0000 & .0000 & .0000 & .0000 & .0000 & +.0246 & .0000 \end{bmatrix}$$

Following measures and norms, see eqns. (33), are obtained from the matrix π^s:
Maximal derivations are

$$\Delta(P_s) = 0.099, \ +\Delta(P_s) = 0.1031, \ -\Delta(P_s) = 0.0206, L_1\text{=0.1072}, L_2\text{=0.00223},$$

The most influential correlation coefficient is ρ_{67} and the most influential variable is x_4 from

$$L_\infty = 0.0246 \ (k = 6, m = 7), \ L_{row} = 0.0487 \ (k = 4)$$

The sensitivity of mode correlation coefficient γ to variable correlation coefficients ρ_{67}, obtained by FORM is demonstrated in the matrix form as follows:

$$\mathbf{H}_{67} = [H_{ij67}] = \left[\frac{\partial \gamma_{ij}}{\partial \rho_{67}}\right] = \begin{bmatrix} 0.00 & & (sym) \\ +0.09 & 0.00 & \\ +0.28 & +0.06 & 0.00 \end{bmatrix}$$

The effect of the coefficient ρ_{67} to the probability upper bound P_u and to the sensitivity factor obtained by FORM and by MCS experiment in 50 runs wit x 30x10000 samples, are given in Table 6.

Table 6 Sensitivity of the system probability of failure – Example 3
Tablica 6. Senzitivnost vjerojatnosti oštećenja sistema na korelacije – Primjer 3

	FORM		MONTE CARLO SIMULATIONS				
ρ_{67}	P_u	$\partial P_u / \partial \rho_{67}$	P_f	c.o.v.	$\partial P_f / \partial \rho_{67}$	Var	c.o.v.
.00	.061	.028	.0455±.0004	0.04	.0198±.0004	2.4 x 10^{-6}	0.08
.20	.066	.026	.0501±.0005	0.05	.0211±.0005	2.9 x 10^{-5}	0.08
.40	.071	.024	.0543±.0006	0.06	.0271±.0006	4.8 x 10^{-5}	0.10
.60	.076	.023	.0586±.0007	0.07	.0235±.0007	8.1 x 10^{-5}	0.12
.80	.080	.021	.0651±.0014	0.08	.0286±.0026	1.2 x 10^{-4}	0.38
.95	.083	.020	.0696±.0051	0.30	.0463±.0300	1.7 x 10^{-2}	2.83

5.4. The practical example

The effect of correlation between the still water bending moment and the wave bending moment on a tanker structure is investigated, [5]. This correlation arises because of a week dependence of the wave bending moment on the weight distribution.

The limit state function is given in the form:

$$g(x_u, SM, \sigma_{cr}, x_{sw}, M_{sw}, x_w, x_s, M_w) = x_u SM\sigma_{cr} - x_{sw} M_{sw} - x_w x_s M_w$$

The distributions of random variables are given in Table 7.

Table 7 Distribution of random variables in the practical example
Tablica 7. Distribucije slučajnih varijabli u praktičnom primjeru

Var	Distribution	Mean value	C.O.V.	Description
SM	Lognormal	$4.658\text{x}10^5$ m cm	0.04	Effective section modulus
M_{sw}	Normal	$1.813\text{x}10^6$ kNm	0.40	Stillwater bending moment
M_w	Gumbel	$4.855\text{x}10^6$ kNm	0.09	Weve-induced bending moment
σ_{cr}	Lognormal	17.0 kN/cm^2	0.07	Critical stress

The model uncertainty is defined by random variables, see Table 8.

Table 8 Distributions of model uncertainties and parameters
Tablica 8. Distribucija neizvjesnosti i parametara modela u praktičnom primjeru

Var	Distribution	Mean	C.O.V.	Uncertainty due to:
x_u	N	1.0	0.15	Strength
X_{sw}	N	1.0	0.05	Still watter bending moment
X_w	N	0.9	0.15	Wave bending moment due to linear analysis
X_s	N	1.15	0.03	Nonlinearities in sagging

The FORM results of repeated calculations reported in Ref. [5] with correlation coefficient of ρ=0.0, 0.02, 0.05 and 0.08, are β=2.25, 2.23, 2.18 and 2.13 respectively. The sensitivity factor can be assessed as $\Delta\beta / \Delta\rho = (2.13 - 2.25)/0.8 = -0.15$.

The equivalent result for sensitivity factors can be obtained immediately in the FORM procedure according to the analytical procedure presented in the paper as follows:

$$\frac{\partial\beta}{\partial\rho} = -0.15 \quad \text{and} \quad \frac{\partial P_f}{\partial\rho} = 0.049 .$$

6. CONCLUSIONS

- Derivatives of Nataf correlation matrix form commonly used two-parametric statistical distributions are given. Derivative of multinormal PDF is conveniently split into the product of PDF with a simple function of correlation matrix and coordinates of point considered, for further use in simulation procedures.
- The presented Monte Carlo simulation procedure for estimation of sensitivity factors to correlations using derivatives of multinormal PDF with respect to correlation coefficients gives accurate results, but generally requires much more samples than the estimation of the failure probability itself. The paper presents guidelines for sample size upper bound prediction of sensitivity factor estimation in terms of the sample size of the reliability calculation.
- The paper also considers the derivatives of the transformation matrices applied to numerical procedures. The derivatives of the Cholesky's decomposition matrix are available in a form of two recursive procedures. The derivatives of the eigenvalues and eigenvector in the spectral decomposition are available using a perturbation method. The Cholesky's decomposition is numerically more efficient than the spectral decomposition.

But the spectral decomposition renders information about sensitivities in terms of principal axes.

- The conventional numerical approach in FORM for the sensitivity analysis of componental reliability measures w.r.t correlations based on derivatives of transformation matrices is developed. Sensitivity estimation within the conventional approach of system reliability measures w.r.t correlations in FORM is formulated as an upper bound and proven in this sense sufficiently accurate.
- The paper presents a comprehensive and numerically efficient method for sensitivity analysis in FORM. It require neither the derivatives of the transformation matrices nor the recalculation of the transformation matrices and enables a direct calculation of sensitivity matrices for component and system reliability measures with respect to all correlation coefficients simultaneously.
- The procedure for sensitivity analysis of component and system failure probabilities base on the "fitting" method is directly straightforward by applicable to correlation coefficients, either in FORM or in SORM. The "fitting" method is based on the recalculation of the transformation matrices for incremented values of correlation coefficients, being in this sense numerically accurate but quite inefficient.
- The sensitivity matrices w.r.t correlation coefficients or their parameters are available as the intermediate results of the failure probability calculation. These matrices, used jointly, enable efficient identification of most significant correlation related parameters in the reliability analysis.
- All the presented methods can be easily implemented to the existing procedures and computer codes for the reliability analysis and neither of them requires additional structural response evaluation.

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REFERENCES

[1] P.L. Liu, A.D. Kiureghian, Multivariate distribution models with prescribed marginals and covariance's, Prob. Eng. Mech. Vol. 1, No. 2, (1986) 105-112.

[2] Živković, T., P., Eigenvalues of the real generalized eigenvalue perturbed by a low-rank perturbation, J. of Math. Chem., 9 (1992), 55-73.

[3] Madsen, S. Kren, N.C. Lind, Methods of Structural Safety, Prentice-Hill, Englewood Cliffs, New Jersey, 1986.

[4] Karamchandani, C.A. Cornell, Sensitivity estimation within first and second order reliability methods, Structural Safety, 11 (1992), 95-197.

[5] Mansour, A., L. Hovem, Probability-Based Ship Safety Analysis, J. of Ship Research, Vol. 38, No. 4., 1994, 329-339.

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APPENDIX A

The relation of ρ' and ρ is uniquely expressed as $\rho'_{km} = \rho_{km} F$, see Ref. [2] for commonly used two-parametric distributions.

The derivatives of the terms in the Nataf correlation matrix can be expressed as shown:

$$\frac{\partial \rho'_{km}}{\partial \rho_{km}} F_{km} + \rho_{km} \frac{\partial F_{km}}{\partial \rho_{km}}, \quad k,m = 1,2,...,n \,;$$

Two groups of two parametric distributions are considered

Group I	Group II
U-Uniform	LN-Lognormal
SE-Shifted exponential	GM-Gamma
SR-Shifted Rayleigh	T2L-Type-II Largest value
T1L-Type-I Largest value	T3S-Type-II smallest value
T2S-Type-II Smallest value	

There are five categories of formulae for F. In some cases, F depends also on the coefficient of variation δ. The values for $\frac{\partial F_{ij}}{\partial \rho_{ij}}$ are given for each of five categories.

(I Cat.) $F_{ij}=const.$ for x_j belonging to group 1 and x_i normal: $\frac{\partial F_{ij}}{\partial \rho_{ij}} = 0$.

(II Cat.) $F_{ij} = F(\delta_j)$ for x_j belonging to group 2 and x_i normal: $\frac{\partial F_{ij}}{\partial \rho_{ij}} = 0$

(III Cat.) $F_{ij} = F(\rho_{ij})$ for both x_i and x_j belonging to group 1:

$x_j^{x_i}$	U	SE	SR	T1L	T1S
U	-.094ρ				
SE	+.058ρ	-.367+.306ρ			
SR	-.016ρ	-.100+.042ρ	-.029ρ		
T1L	+.030ρ	+.154+.062ρ	+.045+.012ρ	+.069+.010ρ	
T1S	+.030ρ	+.154+.062ρ	+.045+.012ρ	+.069+.010ρ	-069+.010ρ

(IV Cat.) $F_{ij} = F(\rho_{ij}, \delta_j)$ for x_i belonging to group 1 and x_j belonging to group 2:

$x_j^{x_i}$	U	SE	SR	T1L	T1S
LN	+.020ρ	+.003+.050ρ-.437δ_j	+.001+.008ρ-.130δ_j	+.001+.008ρ-.197δ_j	-.001+.086ρ+.197δ_j
GM	+.004ρ	+.003+.028ρ-.296δ_j	+.001+.004ρ-.090δ_j	+.001+.006ρ-.132δ_j	-.001+.006ρ+.132δ_j
T2L	+.148ρ	-.152+.260ρ-.728δ_j	-.038+.056ρ-.229δ_j	-.060+.040ρ-.332δ_j	-.001+.040ρ+.332δ_j
T3L	-.010ρ	+.145+.020ρ-.467δ_j	+.042-.136δ_j	+.065+.006ρ-.211δ_j	-.065+.006ρ+.211δ_j

(V Cat.) $F = F(\rho_{ij}, \delta_i, \delta_j)$ for both x_i and x_j belonging to group 2:

For x_i and x_j lognormally distributed:

$x_j^{x_i}$	LN	GM	T2L	T3S
GM	$+.033+.004\rho-.104\delta_i-.119\delta_j$	$+.022+.002\rho-.077(\delta_i+\delta_j)$		
T2L	$+.082+.036\rho-.441\delta_i-.277\delta_j$	$+.056+.024\rho-.313\delta_i-.182\delta_j$	$+.054-.110\rho-.060\rho^2_j-$ $-.570(\delta_i+\delta_j)+.514\rho(\delta_i+\delta_j)-$ $-.371(\delta_i+\delta_j)$	
T3L	$+.052\rho$	$+.034+.006\delta_i-.111\delta_j$	$+.146+.026\rho+.005\delta_i-$ $-.481\delta_j$	$-.004-.002\rho-$ $-.005(\delta_i+\delta_j)$

For both x_i and x_j lognormally distributed:

$$\frac{\partial F(\rho, \delta_i, \delta_j)}{\partial \rho} = F\left[\frac{\delta_i \delta_j}{(1+\rho\delta_i\delta_j)\cdot \ln(1+\rho\delta_i\delta_j)} - \frac{1}{\rho}\right]$$

APPENDIX B

(a) The Cholesky decomposition is of the form $\mathbf{R}=\mathbf{L}\mathbf{L}^T$ or $\mathbf{R}^{-1}=\mathbf{M}^T\mathbf{M}$.

The elements $\lambda_{ij}, i = 1,2,..., n;\ j = 1,2,..., i$, of the lower-triangular matrix $\mathbf{L}$ are as follows:

$$\lambda_{ii} = \left(\rho_{ii} - \sum_{r=1}^{i-1} \lambda_{ir}^2\right)^{\frac{1}{2}} \tag{B-1}$$

$$\lambda_{ij} = \frac{1}{\lambda_{jj}}\left(\rho_{ij} - \sum_{r=1}^{j-1} \lambda_{ir}\lambda_{jr}\right)^{\frac{1}{2}} \quad \text{for } i>j; \quad \lambda_{ij}=0 \quad \text{for } i<j; \tag{B-2}$$

The elements $\mu_{ij},\ i = 1,2,...,n;\ j = 1,2,...,i$, of the matrix $\mathbf{M}=\mathbf{L}^{-1}$ can be determined as follows:

$$\mu_{ii} = \frac{1}{\lambda_{ii}} \tag{B-3}$$

$$\mu_{ij} = -\frac{1}{\lambda_{ii}} \sum_{r=1}^{i-1} \lambda_{ir}\mu_{rj} \quad \text{for } i>j; \quad (\mu_{ij}=0 \quad \text{for } i<j;) \tag{B-4}$$

(b) The matrix of derivatives $\left(\partial \mathbf{L} / \partial \rho_{km}\right)$ of the matrix $\mathbf{L}$ w.r.t. ρ_{km} can be determined in recursion as follows:

$$\frac{\partial \lambda_{ii}}{\partial \rho_{km}} = \frac{1}{\lambda_{ii}} \sum_{r=1}^{i-1} \lambda_{ir} \frac{\partial \lambda_{ir}}{\partial \rho_{km}} \tag{B-5}$$

$$\frac{\partial \lambda_{ij}}{\partial \rho_{km}} = \frac{1}{\lambda_{jj}}\left[\frac{\partial \rho_{ij}}{\partial \rho_{km}} - \frac{\rho_{ij}}{\lambda_{jj}}\frac{\partial \lambda_{jj}}{\partial \rho_{km}} - \sum_{r=1}^{j-1}\left(\lambda_{ir}\frac{\partial \lambda_{ir}}{\partial \rho_{km}} + \lambda_{ir}\frac{\partial \lambda_{jr}}{\partial \rho_{km}} - \frac{\lambda_{jr}\lambda_{ir}}{\lambda_{jj}}\frac{\partial \lambda_{jj}}{\partial \rho_{km}}\right)\right],\ i>j \tag{B-6}$$

The derivative of the matrix **M** can be determined as $(\partial \mathbf{M}/\partial \rho_{km}) = -\mathbf{M}(\partial \mathbf{L}/\partial \rho_{km})\mathbf{M}$ or in recursion as follows:

$$\frac{\partial \mu_{ii}}{\partial \rho_{km}} = -\frac{1}{\lambda_{ii}^2}\frac{\partial \lambda_{ii}}{\partial \rho_{km}} \tag{B-7}$$

$$\frac{\partial \mu_{ij}}{\partial \rho_{km}} = -\frac{1}{\lambda_{ii}}\left[\sum_{r=j}^{i-1}\left(\mu_{rj}\frac{\partial \lambda_{ir}}{\partial \rho_{km}} + \lambda_{ir}\frac{\partial \mu_{rj}}{\partial \rho_{km}} - \frac{\lambda_{ir}\mu_{rj}}{\lambda_{ii}}\frac{\partial \lambda_{ii}}{\partial \rho_{km}}\right)\right] \tag{B-8}$$

The derivatives of the elements of the matrix $\mathbf{R}^{-1}$ can be obtained as follows:

$$\frac{\partial \rho_{km}^{-1}}{\partial \rho_{km}} = \sum_{r=k}^{n}\left(\mu_{ik}\frac{\partial \mu_{im}}{\partial \rho_{km}} + \mu_{im}\frac{\partial \mu_{ik}}{\partial \rho_{km}}\right) \tag{B-9}$$

(c) If derivatives are to be calculated w.r.t all or a great number of correlation coefficients, the procedure can be made efficient considering derivatives of the relation $\mathbf{R}=\mathbf{L}\mathbf{L}^T$ w.r.t ρ_{km}:

$$\mathbf{r}_{KM} = \frac{\partial \mathbf{L}}{\partial \rho_{km}}\mathbf{L}^T + \mathbf{L}\frac{\partial \mathbf{L}^T}{\partial \rho_{km}} \tag{B-10}$$

Taking into account symmetry of **R** or $\mathbf{r}_{KM}$ and lower triangular form of **L,** a system of $(n^2+n)/2$ equations in $(n^2+n)/2$ unknown terms of $\dfrac{\partial \mathbf{L}}{\partial \rho_{km}}$ can be formed. If they are stored row by row in a vector, the coefficient matrix of the system of equations has also a lower triangular form. Using relations (B-3) and (B-4), this matrix can be easily inverted. Each column of the inverse matrix represents all lower triangle terms of the derivative of **L** matrix w.r.t ρ_{km} stored in the same way. Only one row of coefficient matrix is needed at a time the calculation of the needed columns.

APPENDIX C

(a) The derivatives of the integral of the bivariate normal distribution:

$$\phi_2\left(\beta_i, \beta_j, z\right) = \frac{1}{2\pi\left(1-z^2\right)^{1/2}} \exp\left(\frac{1}{2} \frac{\beta_i^2 + \beta_j^2 - 2z\beta_i\beta_j}{\left(1-z^2\right)}\right) \tag{C-1}$$

w.r.t to correlation coefficient ρ_{km} can be expressed as:

$$\begin{aligned} \frac{\partial}{\partial \rho_{km}}\left[\int_0^{\gamma_{ij}} \phi_2\left(\beta_i, \beta_j; z\right) dz\right] &= \left(\beta_i \frac{\partial \beta_i}{\partial \rho_{km}} + \beta_j \frac{\partial \beta_j}{\partial \rho_{km}}\right) \int_0^{\gamma_{ij}} \frac{1}{1-z^2} \phi_2\left(\beta_i, \beta_j; z\right) dz + \\ &+ \left(\beta_j \frac{\partial \beta_i}{\partial \rho_{km}} + \beta_i \frac{\partial \beta_j}{\partial \rho_{km}}\right) \int_0^{\gamma_{ij}} \frac{z}{1-z^2} \phi_2\left(\beta_i, \beta_j; z\right) dz + \phi_2\left(\beta_i, \beta_j; \gamma_{ij}\right) \frac{\partial \gamma_{ij}}{\partial \rho_{km}} \end{aligned} \tag{C-2}$$

The values of the integrals in eqns. (C-2) can be obtained by numerical integration.

The C_{ij} term in eqn. (64) is obtained by substitution of eqn. (C-2) in the derivative of the eqn. (63) and the collecting the appropriate terms w.r.t $\partial\beta_i / \partial\rho_{km}$ as shown:

$$C_{ij} = -\phi\left(\beta_i\right)\Phi\left(-\beta_j\right) - \beta_i \int_0^{\gamma_{ij}} \frac{\phi_2\left(\beta_i, \beta_j, z\right)}{1-z^2} dz + \beta_j \int_0^{\gamma_{ij}} \frac{z\phi_2\left(\beta_i, \beta_j, z\right)}{1-z^2} dz \tag{C-3}$$

The term C_{ij} is obtained by collecting the terms w.r.t $\partial\beta_j / \partial\rho_{km}$.

(b) The H_{ijkm} terms in eqn (64) for the derivatives of the mode correlation coefficients γ_{ij} w.r.t. correlation coefficients ρ_{km} are obtained via derivatives of the cosine of the angle between the normalized gradient vectors in the design points $\mathbf{U}_i^*$ and $\mathbf{U}_j^*$.

For gradient vectors the transformation between U and Y coordinates reads $\alpha' = \mathbf{A}^T \omega$ and the corresponding length of vector α' reads: $d = \left(\omega^T \mathbf{R}' \omega\right)^{1/2}$. The cosine of the angle is given by the expression:

$$\gamma_{ij} = \alpha_i^T \alpha_j = \frac{\alpha'^T_i}{d_i} \frac{\alpha'_j}{d_j} = \frac{\omega_i^T \mathbf{R}' \omega_j}{\left(\omega_i^T \mathbf{R}' \omega_i\right)^{1/2} \left(\omega_j^T \mathbf{R}' \omega_j\right)^{1/2}} \tag{C-4}$$

For given ω and γ, is the function of elements of $\mathbf{R}'$. If ω is calculated for $\mathbf{R}' = \mathbf{R}'_0$ and $\alpha' = \alpha'_i$ it can be expressed as:

$$\omega = \mathbf{A}_0^{-T} \frac{\mathbf{U}^*}{\beta} = \frac{\overline{\mathbf{Y}}^*}{\beta} \tag{C-5}$$

Substituting (C-5) into (C-4), taking derivative of γ_{ij} w.r.t ρ_{km} and noting that $\partial \mathbf{R}' / \partial \rho'_{km} = \mathbf{r}_{KM}$ the following expression is obtained:

$$H_{ijkm} = \frac{\partial \gamma_{ij}}{\partial \rho_{km}} = e_{km} \left\{ \frac{\left(\bar{y}_k\right)_i \left(\bar{y}_m\right)_j + \left(\bar{y}_k\right)_j \left(\bar{y}_m\right)_i}{\beta_i \beta_j} - \gamma_{ij} \left[\frac{\left(\bar{y}_k\right)_i \left(\bar{y}_m\right)_i}{\beta_i} + \frac{\left(\bar{y}_k\right)_j \left(\bar{y}_m\right)_j}{\beta_j} \right] \right\} \quad \text{(C-6)}$$

(c) If the mode correlation coefficient γ_{ij} is expressed via design point coordinates $\mathbf{U}_i^*$ and $\mathbf{U}_j^*$ of two modes "i" and "j" as

$$\bar{\gamma}_{ij} = \frac{Y_i^{*T} \mathbf{R}^{-1} Y_i^*}{\beta_i \beta_j} = \frac{\sum_{k=i}^{n} u_{ik}^* u_{jk}^*}{\beta_i \beta_j}, \quad \text{(C-7)}$$

the derivatives of the r.h.s can be expressed as:

$$\frac{1}{\beta_i \beta_j} \sum_{s=1}^{n} \left(\frac{\partial u_{is}^*}{\partial \rho_{km}} u_{js}^* + \frac{\partial u_{js}^*}{\partial \rho_{km}} u_{is}^* \right) - \gamma_{ij} \left(\frac{1}{\beta_i} \frac{\partial \beta_i}{\partial \rho_{km}} + \frac{1}{\beta_j} \frac{\partial \beta_j}{\partial \rho_{km}} \right) \quad \text{(C-8)}$$

The expression for $\bar{\gamma}_{ij}$ gives the same numerical result, but of the opposite sign than eqn. (C-6), since it represents the rate of the change of angle between position vectors $\mathbf{U}_i^*$ and $\mathbf{U}_j^*$ and not the normals α_i and α_j of the tangent hyperplanes to the failure surface needed in the calculation of γ_{ij}.