ICCES 2017
International Conference on Computational &
Experimental Engineering and Science
26-30th June 2017
Funchal, Madeira Island, Portugal

Mixed MLPG Staggered Solution Procedure in Gradient Elasticity for Modeling of Heterogeneous Materials

Boris Jalušić[1], Jurica Sorić[1] and Tomislav Jarak[1]

Abstract A mixed MLPG collocation method is applied for modeling of deformation responses of heterogeneous materials in gradient elasticity. Herein, a heterogeneous material domain comprised of two isotropic homogeneous parts with different material elastic properties is considered. The solution for the entire domain is obtained by enforcing the corresponding boundary conditions along the interface of the homogeneous domains. For the approximation of the unknown field variables the Moving Least Squares (MLS) functions with interpolatory conditions are applied. The strain gradient elasticity based on the Aifantis theory with one microstructural parameter is utilized. The original fourth-order equilibrium equations of gradient elasticity are solved in a staggered manner as an uncoupled sequence of two sets of second-order differential equations. The proposed mixed meshless approach is tested and demonstrated by a representative numerical example.

Keywords: Mixed meshless approach, collocation method, staggered solution procedure, heterogeneous materials

1 Introduction

Nowadays, a large number of different meshless methods are utilized for the modeling of material deformation responses. This is due to their beneficial characteristics in comparison to standard mesh-based methods. The meshless numerical approaches are able to overcome problems such as element distortion and time-demanding mesh generation process. Nevertheless, the calculation of meshless approximation functions due to its high computational cost is still a major drawback. This deficiency can be alleviated to a certain extent by using the mixed Meshless Local Petrov-Galerkin (MLPG) Method paradigm [Atluri, Liu, Han (2006)]. In the present contribution, the MLPG formulation based on the mixed approach is adapted for the modeling of deformation responses of heterogeneous materials based on the strain gradient elasticity theory. A heterogeneous structure consists of two homogeneous materials which are discretized by grid points in which equilibrium equations are imposed. In addition, the strain gradient elasticity based on the Aifantis theory with only one microstructural parameter is considered. The gradient theory is used in order to more accurately capture the material behaviour near the interface between regions with different material properties and to remove jumps in the strain fields that can be observed when a classical theory of linear elasticity is used. The solution of fourth-order differential equations arising in non-classic theories requires a high-order of approximation functions [Askes, Aifantis (2011)]. Hence, using the Finite Element Method (FEM) for solving this type of problems is not a wise choice since standard formulations need to possess C^1 continuity, which leads to complicated shape functions with large number of nodal degrees of freedom, even if mixed elements are utilized [Amanatidou, Aravas (2002)]. Therefore, these FEM procedures should not be used due to their inefficiency related to high numerical costs [Askes, Aifantis (2011)]. On the other hand, the required C^1 continuity is obtainable in a simple and a straightforward manner, when the meshless methods are considered [Atluri (2004)]. In the proposed method, the fourth-order equilibrium equations of gradient elasticity are solved as an uncoupled sequence of two sets of the second-order differential equations [Askes, Morata (2008)], for the purpose of further decreasing the continuity requirement of the formulation. Hence, two different boundary value problems, local (classical) and non-local (gradient), are being solved, where the solution of the former problem is used as an input in the latter problem. In both boundary value problems, independent variables are approximated using meshless functions in such a way that each material is treated as a separate problem [Chen, Wang, Hu, Chi (2009)]. The global solution for the entire heterogeneous structure is acquired by enforcing appropriate boundary conditions along the interface of two homogeneous domains. The application of the staggered solution scheme [Askes, Morata (2008)], utilizing the mixed meshless approach, results in less complicated meshless formulation which only has the C^0 requirement on the approximation functions.

A collocation meshless method is used, which may be considered as a special case of the MLPG approach, where the Dirac delta function is used as the test function. Since the collocation method is employed, the strong form of equilibrium equations is employed and time-consuming numerical integration process is avoided. The MLS approximation functions [Atluri (2004)] with interpolatory properties (IMLS) are applied [Most, Bucher (2008)]. This enables simple imposition of essential boundary conditions as in FEM. Natural boundary conditions on outer edges are enforced via the direct collocation approach. In the local

[1] Faculty of Mechanical Engineering and Naval Architecture, University of Zagreb, Ivana Lučića 5, 10000 Zagreb, Croatia.

problem, the classical linear elastic boundary value problem for each homogeneous material is discretized by using the independent approximations of classical strains $\varepsilon_{ij}^{\mathrm{c}}$ and classical displacements u_i^{c}. In order to derive the final closed system of classical discretized equations with the classical displacements as only unknowns, the approximated classical strains are expressed in terms of classical displacements using appropriate kinematic relations. In the similar manner, for the discretization of the non-local boundary value problem, independent approximations of the gradient displacements u_i^{g} and the derivatives of gradient displacements $u_{i,k}^{\mathrm{g}}$ are utilized. Herein, to obtain the final solvable system of discretized gradient equations, the approximated derivatives are written in terms of gradient displacements at the collocation nodes. The mixed MLPG collocation method for the modeling of deformation responses of a heterogeneous material using gradient elasticity is presented and explained at large in Section 2. The proposed method is tested and analyzed by considering a problem of the clamped heterogeneous plate subjected to uniform displacement at the right end in Section 3. In Section 4 concluding remarks and further research guidelines are given.

2 Mixed MLPG Method for Gradient Elasticity

The two-dimensional heterogeneous material which occupies the global computational domain Ω surrounded by the global outer boundary Γ is considered. The boundary Γ_{s} represents the interface between two subdomains, Ω^{+} and Ω^{-}, with different homogeneous material properties. Γ_{s} separates the global domain Ω in such a manner that $\Omega = \Omega^{+} \cup \Omega^{-}$ and $\Gamma = \Gamma^{+} \cup \Gamma^{-}$.

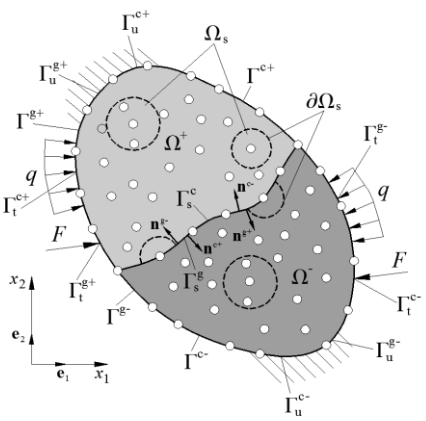

Figure 1: Two-dimensional heterogeneous material

Furthermore, since in the staggered procedures two different boundary value problems are solved one after the other, the global boundary Γ can be denoted as Γ^{c} or Γ^{g} to distinguish whether the classical or gradient boundary value problem is being solved. The same analogy applies to all other boundaries, where some kind of boundary condition is prescribed, e.g. the interface boundary Γ_{s} is in the classical boundary value problem denoted as $\Gamma_{\mathrm{s}}^{\mathrm{c}}$, while in the gradient one it is denoted $\Gamma_{\mathrm{s}}^{\mathrm{g}}$. Hence, the typical heterogeneous material being analyzed is now portrayed in Fig. 1. The governing equations for the presented example are the strong form 2D equilibrium equations which have to be satisfied within the global computational domain Ω divided into Ω^{+} and Ω^{-}. According to the staggered solution procedure described in [Askes, Morata (2008)], two sets of second-order partial differential equations can be utilized to describe the deformation of the heterogeneous material. These equations are here written for each homogeneous material separately. Thus, the first equation set representing the classical boundary value problem is equal to

$$\sigma_{ij,x_j}^{\mathrm{c}+} + b_i^{\mathrm{c}+} = 0, \quad \text{within } \Omega^{+}, \tag{1}$$

$$\sigma_{ij,x_j}^{\mathrm{c}-} + b_i^{\mathrm{c}-} = 0, \quad \text{within } \Omega^{-}, \tag{2}$$

While the second equation set for the non-local gradient problem utilize a microstructural parameter l and is expressed as

$$u_i^{\mathrm{g}+} - l^2 u_{i,mm}^{\mathrm{g}+} = u_i^{\mathrm{c}+}, \quad \text{within } \Omega^{+}, \tag{3}$$

$$u_i^{\mathrm{g}-} - l^2 u_{i,mm}^{\mathrm{g}-} = u_i^{\mathrm{c}-}, \quad \text{within } \Omega^{-}. \tag{4}$$

As evident, firstly the classical boundary value problem is solved, whose solution is then used as an input on the right hand side of the gradient equations. In this operator-split procedure, the classical and gradient boundary conditions need to be satisfied on the outer boundaries of the heterogeneous structure, depending on which problem is currently being solved. Hence, as in [Atluri, Liu, Han (2006)], the classical boundary conditions include the displacements u_i^{c} and tractions t_i^{c} equal to

$$u_i^{\mathrm{c}+} = \bar{u}_i^{\mathrm{c}+}, \quad \text{on } \Gamma_{\mathrm{u}}^{\mathrm{c}+}, \tag{5}$$

$$u_i^{\mathrm{c}-} = \bar{u}_i^{\mathrm{c}-}, \quad \text{on } \Gamma_{\mathrm{u}}^{\mathrm{c}-}, \tag{6}$$

$$t_i^{\mathrm{c}+} = \sigma_{ij}^{\mathrm{c}+} n_j^{\mathrm{c}+} = \bar{t}_i^{\mathrm{c}+}, \quad \text{on } \Gamma_{\mathrm{t}}^{\mathrm{c}+}, \tag{7}$$

$$t_i^{\mathrm{c}-} = \sigma_{ij}^{\mathrm{c}-} n_j^{\mathrm{c}-} = \bar{t}_i^{\mathrm{c}-}, \quad \text{on } \Gamma_{\mathrm{t}}^{\mathrm{c}-}, \tag{8}$$

while the gradient boundary conditions can be the displacements u_i^{g} and second-order normal derivatives of displacements R_i^{g} [Polizzotto (2003)], where κ_{ijk} denotes the third-order tensor comprised of second derivatives of displacements u_i^{g}

$$u_i^{\mathrm{g}+} = \bar{u}_i^{\mathrm{g}+}, \quad \text{on } \Gamma_{\mathrm{u}}^{\mathrm{g}+}, \tag{9}$$

$$u_i^{\mathrm{g}-} = \bar{u}_i^{\mathrm{g}-}, \quad \text{on } \Gamma_{\mathrm{u}}^{\mathrm{g}-}, \tag{10}$$

$$R_i^{\mathrm{g}+} = n_j^{\mathrm{g}+} n_k^{\mathrm{g}+} \kappa_{ijk}^{\mathrm{g}+} = \frac{\partial^2 u_i^{\mathrm{g}+}}{\partial n^{\mathrm{g}+2}} = \bar{R}_i^{\mathrm{g}+}, \quad \text{on } \Gamma_{\mathrm{t}}^{\mathrm{g}+}, \tag{11}$$

$$R_i^{\mathrm{g}-} = n_j^{\mathrm{g}-} n_k^{\mathrm{g}-} \kappa_{ijk}^{\mathrm{g}-} = \frac{\partial^2 u_i^{\mathrm{g}-}}{\partial n^{\mathrm{g}-2}} = \bar{R}_i^{\mathrm{g}-}, \quad \text{on } \Gamma_{\mathrm{t}}^{\mathrm{g}-}. \tag{12}$$

Furthermore, to acquire the solution for the entire structure, the conditions on the interface boundaries, $\Gamma_{\mathrm{s}}^{\mathrm{c}}$ and $\Gamma_{\mathrm{s}}^{\mathrm{g}}$, need to be enforced, for both the classical and the gradient problem. According to [Askes, Morata (2008)], if the classical elasticity problem is solved, these boundary conditions are the continuity of displacements and reciprocity of tractions

$$u_i^{\mathrm{c}+} - u_i^{\mathrm{c}-} = 0, \quad \text{on } \Gamma_{\mathrm{s}}^{\mathrm{c}}, \tag{13}$$

$$\sigma_{ij}^{\mathrm{c}+} n_j^{\mathrm{c}+} + \sigma_{ij}^{\mathrm{c}-} n_j^{\mathrm{c}-} = 0, \quad \text{on } \Gamma_{\mathrm{s}}^{\mathrm{c}}. \tag{14}$$

On the other hand, if the gradient problem is considered, the interface boundary conditions include the continuity of displacements and reciprocity of first-order normal derivatives of displacements

$$u_i^{\mathrm{g}+} - u_i^{\mathrm{g}-} = 0, \quad \text{on } \Gamma_{\mathrm{s}}^{\mathrm{g}}, \tag{15}$$

$$\frac{\partial u_i^{\mathrm{g}+}}{\partial n^{\mathrm{g}+}} + \frac{\partial u_i^{\mathrm{g}-}}{\partial n^{\mathrm{g}-}} = 0, \quad \text{on } \Gamma_{\mathrm{s}}^{\mathrm{g}}. \tag{16}$$

The two-dimensional heterogeneous continuum Ω is discretized by two set of nodes $I = 1,2,...,N$ and $M = 1,2,...,P$, where N and P indicate the total number of nodes within Ω^+ and Ω^-, respectively. Herein, the same sets and position of the nodes are used for the discretization of both the classical and the gradient boundary value problem. Now, for each considered discretization node, the MLPG concept [Atluri (2004)] is applied, wherein the Dirac delta test function is chosen as the weight function in local weak forms, and the local approximation domains are defined around each node in order to compute the connectivity between the nodes. For the nodes positioned on the interface boundaries, the approximation domains are truncated in such a manner that the discretization nodes from one homogeneous material influence only the nodes belonging to that material. For the discretization of both boundary value problems, the mixed collocation procedure [Atluri, Liu, Han (2006)] is utilized. All unknown field variables are approximated separately within subdomains Ω^+ and Ω^-, where the same approximation functions are employed for all field components. For the shape function construction, the well-known MLS approximation scheme [Atluri (2004)] is employed. The interpolatory properties of the MLS approximation function are achieved by utilizing the weight function according to [Most, Bucher (2008)]. Since the discretization of the classical boundary value problem using the mixed MLPG approach is well documented in the scientific literature, the description of the obtained equations for the classical problem is here skipped and the reader is referred to [Jalušić, Sorić, Jarak (2017)], where this approach is described in depth. In this contribution, the main focus is shifted to the discretization of the gradient boundary value problem and the corresponding boundary conditions. Here, the displacement and derivatives of displacements are unknown field variables. Thus, for the nodes within the material Ω^+, and nodes positioned on the boundaries $\Gamma_{\mathrm{u}}^{\mathrm{g}+}$, $\Gamma_{\mathrm{t}}^{\mathrm{g}+}$, and $\Gamma_{\mathrm{s}}^{\mathrm{g}+}$ these approximations are written as

$$u_i^{\mathrm{g}+(h)}(\mathbf{x}) = \sum_{J=1}^{N_{\Omega_\mathrm{s}}} \phi_J^+(\mathbf{x}) (\hat{u}_i^{\mathrm{g}+})_J, \quad \text{within } \Omega^+, \tag{17}$$

$$(\nabla u_i^{\mathrm{g}+})^{(h)}(\mathbf{x}) = \sum_{J=1}^{N_{\Omega_\mathrm{s}}} \phi_J^+(\mathbf{x}) (\hat{u}_{\mathrm{G}i}^{\mathrm{g}+})_J, \quad \text{within } \Omega^+, \tag{18}$$

where ϕ_J^+ represents the nodal value of two-dimensional shape function for node J, N_{Ω_s} stands for the number of nodes within the approximation domain Ω_s, while $\left(\hat{u}_i^{\mathrm{g}+}\right)_J$ and $\left(\hat{u}_{\mathrm{G}i}^{\mathrm{g}+}\right)_J$ denote the nodal values of the displacement and derivatives of displacement components. Now, firstly the governing equations of the gradient problem, (3) and (4), are rewritten in their matrix form at the discretization nodes in the domains Ω^+ and Ω^-

$$\mathbf{u}_I^{\mathrm{g}+} - l^2 \left[\nabla^{+\mathrm{T}} \cdot \left(\nabla^+ \mathbf{u}_I^{\mathrm{g}+} \right) \right] = \mathbf{u}_I^{\mathrm{c}+}, \tag{19}$$

$$\mathbf{u}_M^{\mathrm{g}-} - l^2 \left[\nabla^{-\mathrm{T}} \cdot \left(\nabla^- \mathbf{u}_M^{\mathrm{g}-} \right) \right] = \mathbf{u}_M^{\mathrm{c}-}, \tag{20}$$

where $\nabla^2 = \nabla^{\mathrm{T}} \cdot (\nabla)$ denotes the Laplacian operators written in matrix form. Hence, the operators ∇^+ and ∇^- are equal to

$$\nabla^{+\mathrm{T}} = \begin{bmatrix} \frac{\partial(\ )^+}{\partial x_1}(\mathbf{x}_I) & 0 & \frac{\partial(\ )^+}{\partial x_2}(\mathbf{x}_I) & 0 \\ 0 & \frac{\partial(\ )^+}{\partial x_1}(\mathbf{x}_I) & 0 & \frac{\partial(\ )^+}{\partial x_2}(\mathbf{x}_I) \end{bmatrix}, \tag{21}$$

$$\nabla^{-\mathrm{T}} = \begin{bmatrix} \frac{\partial(\ )^-}{\partial x_1}(\mathbf{x}_M) & 0 & \frac{\partial(\ )^-}{\partial x_2}(\mathbf{x}_M) & 0 \\ 0 & \frac{\partial(\ )^-}{\partial x_1}(\mathbf{x}_M) & 0 & \frac{\partial(\ )^-}{\partial x_2}(\mathbf{x}_M) \end{bmatrix}. \tag{22}$$

The governing equations (19) and (20) are now simultaneously discretized by the approximations (17) and (18) resulting in

$$\sum_{J=1}^{N_{\Omega_s}} \phi_J^+ \mathbf{u}_J^{\mathrm{g}+} - l^2[\nabla^{+\mathrm{T}} \cdot (\sum_{K=1}^{N_{\Omega_s}} \phi_K^+ \mathbf{u}_{\mathrm{G}K}^{\mathrm{g}+})] = \mathbf{u}_I^{\mathrm{c}+}, \tag{23}$$

$$\sum_{J=1}^{N_{\Omega_s}} \phi_J^- \mathbf{u}_J^{\mathrm{g}-} - l^2[\nabla^{-\mathrm{T}} \cdot (\sum_{K=1}^{N_{\Omega_s}} \phi_K^- \mathbf{u}_{\mathrm{G}K}^{\mathrm{g}-})] = \mathbf{u}_M^{\mathrm{c}-}. \tag{24}$$

In the above equations, $\mathbf{u}_{\mathrm{G}}^{\mathrm{g}+}$ and $\mathbf{u}_{\mathrm{G}}^{\mathrm{g}-}$ denote the vectors of unknown derivatives of displacement defined by

$$[\mathbf{u}_{\mathrm{G}}^{\mathrm{g}+}]^{\mathrm{T}} = [\frac{\partial \hat{u}_1^{\mathrm{g}+}}{\partial x_1} \quad \frac{\partial \hat{u}_2^{\mathrm{g}+}}{\partial x_1} \quad \frac{\partial \hat{u}_1^{\mathrm{g}+}}{\partial x_2} \quad \frac{\partial \hat{u}_2^{\mathrm{g}+}}{\partial x_2}], \tag{25}$$

$$[\mathbf{u}_{\mathrm{G}}^{\mathrm{g}-}]^{\mathrm{T}} = [\frac{\partial \hat{u}_1^{\mathrm{g}-}}{\partial x_1} \quad \frac{\partial \hat{u}_2^{\mathrm{g}-}}{\partial x_1} \quad \frac{\partial \hat{u}_1^{\mathrm{g}-}}{\partial x_2} \quad \frac{\partial \hat{u}_2^{\mathrm{g}-}}{\partial x_2}]. \tag{26}$$

As obvious, the equations (23) and (24) represent an unsolvable system since the global number of nodal unknowns is larger than the number of equations. Thus, the system of equations is here closed simply by enforcing the compatibility at each node between the approximated nodal derivatives of displacements, $\mathbf{u}_{\mathrm{G}}^{\mathrm{g}+(h)}(\mathbf{x}_K) \approx \mathbf{u}_{\mathrm{G}K}^{\mathrm{g}+}$ and $\mathbf{u}_{\mathrm{G}}^{\mathrm{g}-(h)}(\mathbf{x}_K) \approx \mathbf{u}_{\mathrm{G}K}^{\mathrm{g}-}$, and the nodal displacements $\hat{\mathbf{u}}_J^{\mathrm{g}+}$ and $\hat{\mathbf{u}}_J^{\mathrm{g}-}$, respectively. Hence, the compatibility equations written using kinematic differential operators, $\mathbf{D}_{\mathrm{K}}^{\mathrm{g}+}$ and $\mathbf{D}_{\mathrm{K}}^{\mathrm{g}-}$, are

$$\hat{\mathbf{u}}_{\mathrm{G}}^{\mathrm{g}+} = \mathbf{D}_{\mathrm{K}}^{\mathrm{g}+} \hat{\mathbf{u}}^{\mathrm{g}+}, \tag{27}$$

$$\hat{\mathbf{u}}_{\mathrm{G}}^{\mathrm{g}-} = \mathbf{D}_{\mathrm{K}}^{\mathrm{g}-} \hat{\mathbf{u}}^{\mathrm{g}-}. \tag{28}$$

Equations (27) and (28) are now again written at every discretization node and discretized by (17), which yields

$$\mathbf{u}_{\mathrm{G}K}^{\mathrm{g}+} = \sum_{J=1}^{N_{\Omega_s}} \mathbf{D}_{\mathrm{K}}^{\mathrm{g}+} \phi_J^+(\mathbf{x}_K) \mathbf{u}_J^{\mathrm{g}+} = \sum_{J=1}^{N_{\Omega_s}} \mathbf{G}_{KJ}^+ \mathbf{u}_J^{\mathrm{g}+}, \tag{29}$$

$$\mathbf{u}_{\mathrm{G}K}^{\mathrm{g}-} = \sum_{J=1}^{N_{\Omega_s}} \mathbf{D}_{\mathrm{K}}^{\mathrm{g}-} \phi_J^-(\mathbf{x}_K) \mathbf{u}_J^{\mathrm{g}-} = \sum_{J=1}^{N_{\Omega_s}} \mathbf{G}_{KJ}^- \mathbf{u}_J^{\mathrm{g}-}, \tag{30}$$

where $\mathbf{G}_{KJ}^+ = \mathbf{G}_J^+(\mathbf{x}_K)$ and $\mathbf{G}_{KJ}^- = \mathbf{G}_J^-(\mathbf{x}_K)$ indicate the matrices consisting of the first-order derivatives of shape functions, written analogously to operators in (21) and (22). Inserting the discretized compatibility relations (29) and (30), into the discretized governing equations (23) and (24), a solvable system of linear algebraic equations with only the nodal displacements as unknowns is attained

$$\mathbf{K}_{IJ}^{\mathrm{g}+} \hat{\mathbf{u}}_J^{\mathrm{g}+} = \mathbf{F}_I^{\mathrm{g}+}, \quad \text{within } \Omega^+, \tag{31}$$

$$\mathbf{K}_{MJ}^{\mathrm{g}-} \hat{\mathbf{u}}_J^{\mathrm{g}-} = \mathbf{F}_M^{\mathrm{g}-}, \quad \text{within } \Omega^-, \tag{32}$$

where the gradient nodal coefficient matrices $\mathbf{K}_{IJ}^{\mathrm{g}+}$ and $\mathbf{K}_{MJ}^{\mathrm{g}-}$ are equal to

$$\mathbf{K}_{IJ}^{\mathrm{g}+} = \sum_{J=1}^{N_{\Omega_s}} \mathbf{S}_{IJ}^+ - l^2[\sum_{K=1}^{N_{\Omega_s}} \mathbf{G}_{IK}^{\mathrm{g}+\,\mathrm{T}} \sum_{J=1}^{N_{\Omega_s}} \mathbf{G}_{KJ}^{\mathrm{g}+}], \tag{33}$$

$$\mathbf{K}_{MJ}^{\mathrm{g}-} = \sum_{J=1}^{N_{\Omega_s}} \mathbf{S}_{MJ}^- - l^2[\sum_{K=1}^{N_{\Omega_s}} \mathbf{G}_{MK}^{\mathrm{g}-\,\mathrm{T}} \sum_{J=1}^{N_{\Omega_s}} \mathbf{G}_{KJ}^{\mathrm{g}-}]. \tag{34}$$

Herein, the matrices $\mathbf{S}_{IJ}^+$ and $\mathbf{S}_{MJ}^-$ are the diagonal matrices comprising of nodal shape function values

$$\mathbf{S}_{IJ}^+ = \begin{bmatrix} \phi_J^+(\mathbf{x}_I) & 0 \\ 0 & \phi_J^+(\mathbf{x}_I) \end{bmatrix}, \tag{35}$$

$$\mathbf{S}_{MJ}^- = \begin{bmatrix} \phi_J^-(\mathbf{x}_M) & 0 \\ 0 & \phi_J^-(\mathbf{x}_M) \end{bmatrix}. \tag{36}$$

The gradient nodal force vectors $\mathbf{F}_I^{\mathrm{g}+}$ and $\mathbf{F}_M^{\mathrm{g}-}$ in (31) and (32) are composed of the known values of classical displacements. As obvious, by utilizing the staggered procedure and the presented mixed meshless strategy, the coefficient matrices $\mathbf{K}_{IJ}^{\mathrm{g}+}$ and $\mathbf{K}_{MJ}^{\mathrm{g}-}$ are assembled using only the first-order derivatives of shape functions. All approximation functions in this contribution possess the interpolation property at the nodes. Consequently, the essential boundary conditions are enforced straightforwardly, analogously to the procedure in FEM. Therefore, by discretizing the displacement boundary conditions (9) and (10) with the approximation (17), we obtain

$$\bar{\mathbf{u}}_I^{\mathrm{g}+} = \sum_{J=1}^{N_{\Omega_s}} \phi_J^+ \hat{\mathbf{u}}_J^{\mathrm{g}+}, \quad \text{on } \Gamma_{\mathrm{u}}^{\mathrm{g}+}, \tag{37}$$

$$\bar{\mathbf{u}}_I^{\mathrm{g}-} = \sum_{J=1}^{N_{\Omega_s}} \phi_J \hat{\mathbf{u}}_J^{\mathrm{g}-}, \quad \text{on } \Gamma_{\mathrm{u}}^{\mathrm{g}-}, \tag{38}$$

The natural boundary conditions (11) and (12) on the boundaries $\Gamma_{\mathrm{t}}^{\mathrm{g}+}$ and $\Gamma_{\mathrm{t}}^{\mathrm{g}-}$ are imposed using the direct collocation approach. Here, in order to derive the discretized equation of the natural boundary conditions dependent only on the nodal values of unknown displacements, the compatibility between second-order and first-order derivatives of displacements at the collocation nodes is imposed. Hence, for the heterogeneous structure this compatibility can be written using differential operators, $\mathbf{D}_{\mathrm{K}}^{\mathrm{sg}+}$ and $\mathbf{D}_{\mathrm{K}}^{\mathrm{sg}-}$, equal to

$$\hat{\mathbf{u}}_{\mathrm{SG}}^{\mathrm{g}+} = \mathbf{D}_{\mathrm{K}}^{\mathrm{sg}+} \hat{\mathbf{u}}_{\mathrm{G}}^{\mathrm{g}+}, \tag{39}$$

$$\hat{\mathbf{u}}_{\mathrm{SG}}^{\mathrm{g}-} = \mathbf{D}_{\mathrm{K}}^{\mathrm{sg}-} \hat{\mathbf{u}}_{\mathrm{G}}^{\mathrm{g}-}, \tag{40}$$

where $\hat{\mathbf{u}}_{\mathrm{SG}}^{\mathrm{g}+}$ and $\hat{\mathbf{u}}_{\mathrm{SG}}^{\mathrm{g}-}$ denote the vectors of unknown nodal second-order derivatives of displacements. Now, by employing the equations (39) and (40), and the compatibility between the first-order derivatives and the displacements defined by (27) and (28), along with the displacement approximation (17) we obtain the following discretized expressions for gradient natural boundary conditions

$$\bar{\mathbf{R}}_I^{\mathrm{g}+} = \mathbf{N}_I^{\mathrm{SG}+} \sum_{K=1}^{N_{\Omega_\mathrm{s}}} \mathbf{H}_{IK}^{+} \sum_{J=1}^{N_{\Omega_\mathrm{s}}} \mathbf{G}_{KJ}^{+} \hat{\mathbf{u}}_J^{\mathrm{g}+}, \quad \text{on } \Gamma_\mathrm{t}^{\mathrm{g}+}, \tag{41}$$

$$\bar{\mathbf{R}}_M^{\mathrm{g}-} = \mathbf{N}_M^{\mathrm{SG}-} \sum_{K=1}^{N_{\Omega_\mathrm{s}}} \mathbf{H}_{MK}^{-} \sum_{J=1}^{N_{\Omega_\mathrm{s}}} \mathbf{G}_{KJ}^{-} \hat{\mathbf{u}}_J^{\mathrm{g}-}, \quad \text{on } \Gamma_\mathrm{t}^{\mathrm{g}-}. \tag{42}$$

In the above equations, the matrices $\mathbf{H}_{IK}^{+}$ and $\mathbf{H}_{MK}^{-}$ connect the second- and first-order derivatives of displacements via the first-derivatives of shape functions,

$$\mathbf{H}_{FK}^{+\,\mathrm{T}} = \begin{bmatrix} \frac{\partial \phi_K^+}{\partial x_1}(\mathbf{x}_I) & 0 & \frac{\partial \phi_K^+}{\partial x_2}(\mathbf{x}_I) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial \phi_K^+}{\partial x_1}(\mathbf{x}_I) & 0 & \frac{\partial \phi_K^+}{\partial x_2}(\mathbf{x}_I) \\ 0 & \frac{\partial \phi_K^+}{\partial x_2}(\mathbf{x}_I) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial \phi_K^+}{\partial x_2}(\mathbf{x}_I) & 0 \end{bmatrix}, \tag{43}$$

$$\mathbf{H}_{FK}^{-\,\mathrm{T}} = \begin{bmatrix} \frac{\partial \phi_K^-}{\partial x_1}(\mathbf{x}_M) & 0 & \frac{\partial \phi_K^-}{\partial x_2}(\mathbf{x}_M) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial \phi_K^-}{\partial x_1}(\mathbf{x}_M) & 0 & \frac{\partial \phi_K^-}{\partial x_2}(\mathbf{x}_M) \\ 0 & \frac{\partial \phi_K^-}{\partial x_2}(\mathbf{x}_M) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial \phi_K^-}{\partial x_2}(\mathbf{x}_M) & 0 \end{bmatrix}, \tag{44}$$

while the matrices $\mathbf{G}_{KJ}^{+}$ and $\mathbf{G}_{KJ}^{-}$ are analogous to the ones defined by (29) and (30). These equations are now inserted into the global coefficient matrix in the rows corresponding to the current node positioned on $\Gamma_\mathrm{t}^{\mathrm{g}+}$ and $\Gamma_\mathrm{t}^{\mathrm{g}-}$, respectively. For the nodes on the boundary $\Gamma_\mathrm{s}^{\mathrm{g}}$, the interface conditions (15) and (16) are discretized by using approximations (17) and (18), while also utilizing the discretized compatibility conditions, (29) and (30), in the reciprocity of natural boundary conditions. Hence, the final form of the discretized interface conditions of this procedure states

$$\sum_{J=1}^{N_{\Omega_\mathrm{s}}} \phi_J^{+} \hat{\mathbf{u}}_J^{\mathrm{g}+} = \sum_{J=1}^{N_{\Omega_\mathrm{s}}} \phi_J^{-} \hat{\mathbf{u}}_J^{\mathrm{g}-}, \quad \text{on } \Gamma_\mathrm{s}^{+}, \tag{45}$$

$$\mathbf{N}_I^{\mathrm{G}+} \sum_{J=1}^{N_{\Omega_\mathrm{s}}} \mathbf{G}_{IJ}^{+} \hat{\mathbf{u}}_J^{\mathrm{g}+} = -\mathbf{N}_M^{\mathrm{G}-} \sum_{J=1}^{N_{\Omega_\mathrm{s}}} \mathbf{G}_{MJ}^{-} \hat{\mathbf{u}}_J^{\mathrm{g}-}, \quad \text{on } \Gamma_\mathrm{s}^{-}, \tag{46}$$

where $\mathbf{N}_I^{\mathrm{G}+}$ and $\mathbf{N}_M^{\mathrm{G}-}$ denote the matrices composed of the unit normal vectors associated to the first-order derivatives of displacements.

3 Numerical Example

3.1 Plate under uniform displacement

A heterogeneous plate is utilized in order to test the ability of the proposed method to remove discontinuities from the strain field. The material properties of the left part of the plate are taken as $E^{+} = 1000$ and $\nu^{+} = 0.25$, while the material data of the right side are $E^{-} = 10000$ and $\nu^{-} = 0.3$. The geometry of each homogeneous subdomain is defined by the length $L = 3$ and the height $H = 3$. The left side of the plate is fixed, while the unit displacement is imposed on the right side. The geometry and the boundary conditions are defined and depicted in Fig. 2 and Fig. 3.

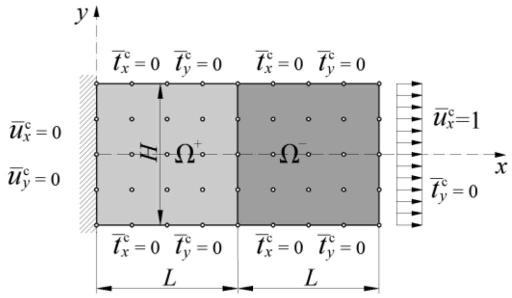

Figure 2: Plate with classical boundary conditions

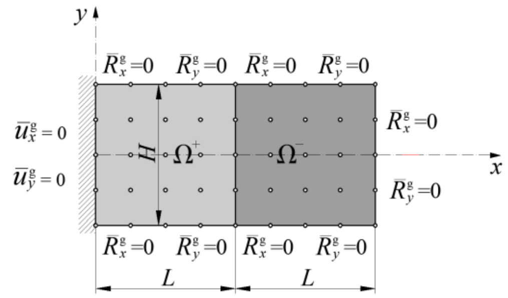

Figure 3: Plate with gradient boundary conditions

For the verification of the presented mixed collocation approach, the distributions of the strain components $\varepsilon_x^{\mathrm{g}}$ and $\varepsilon_{xy}^{\mathrm{g}}$ along the line $y = 0.9$ are portrayed in Fig. 4 and Fig. 5 for two different values of the microstructural parameter l. The plate is discretized by the uniform nodal distributions, in both x and y directions using 242 nodes, where h_s defines the horizontal and vertical distance between nodes. The second-order IMLS functions are applied for the solution of the problem with the size of the approximation domain equal to $r_\mathrm{s} = 2.4 h_\mathrm{s}$. As evident, from the distributions of the strain components, the use of the microstructural parameter larger than zero causes the change in the strain field at and around the interface of the homogeneous domains.

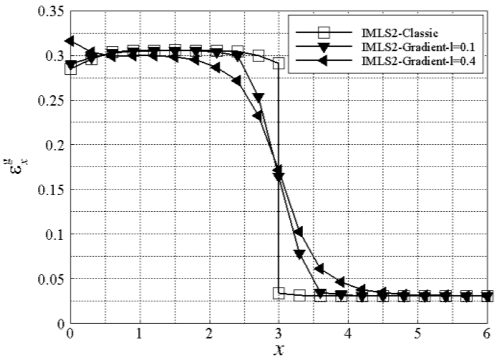

Figure 4: Distribution of strain ε_x^g for $y = 0.9$

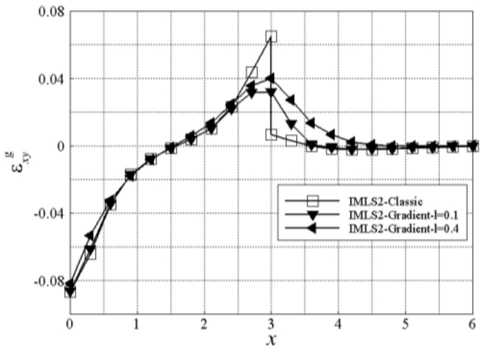

Figure 5: Distribution of strain ε_{xy}^g for $y = 0.9$

For $l \neq 0$ no discontinuity in the strain field is observed at the interface boundary. Accordingly, it can be concluded that the method is suitable for smoothing the strain field.

4 Conclusion

The mixed collocation method based on the Meshless Local Petrov-Galerkin (MLPG) concept has been proposed and applied for the modeling of deformation responses of heterogeneous materials based on gradient elasticity. The problem is solved in a staggered manner using the Aifantis strain gradient theory with only one unknown microstructural parameter, whereby firstly the boundary value problem of classical elasticity is solved, whose solution is then used as the input for the corresponding gradient boundary value problem. Both problems are described by the second-order equations, instead of the original fourth-order differential equations. By employing the mixed MLPG concept, the necessary derivative order of approximation functions is further reduced in the equations. Given that a collocation method is used, there is no need for numerical integration. Thus, the application of the staggered solution scheme and the mixed meshless approach results in an accurate and stable numerical formulation, where only the first-order derivatives of shape functions need to be calculated. The gradient theory is used here in order to more accurately capture the material behaviour near the interface between regions with different material properties and to remove jumps in the strain fields that can be observed when a classical theory of linear elasticity is used. This enables more physical description of the transition of the strain distributions between various homogeneous material regions inside heterogeneous structures. In further research, the described meshless computational strategy will be extended to the modeling of damage initiation in the zones where the strain localization is present, and considered for the use in meshless multiscale computation algorithms.

Acknowledgement: This work has been fully supported by Croatian Science Foundation under the project 2516.

References:

Amanatidou, E.; Aravas, N. (2002): Mixed Finite Element Formulations of Strain-gradient Elasticity Problems. *Computer Methods in Applied Mechanics and Engineering*, vol. 191, pp. 1723-1751.

Askes, H.; Aifantis, E. C. (2011): Gradient Elasticity in Statics and Dynamics: An overview of formulations, length scale identification procedures, finite element implementations and new results. *International Journal of Solids and Structures*, vol. 48, pp. 1962-1990.

Askes, H.; Morata, I.; Aifantis, E. C. (2008): Finite Element Analysis with Staggered Gradient Elasticity. *Computers & Structures*, vol. 86, pp. 1266-1279.

Atluri, S. N. (2004): *The Meshless Method (MLPG) for Domain & BIE Discretization*. Tech. Science Press, Forsyth, USA.

Atluri, S. N.; Liu, H. T.; Han, Z. D. (2006): Meshless Local Petrov-Galerkin (MLPG) Mixed Collocation Method for Elasticity Problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 14, no. 3, pp. 141-152.

Chen, J.-S.; Wang, L.; Hu, H.-Y.; Chi, S.-W. (2009): Subdomain radial basis collocation method for heterogeneous media. *International Journal for Numerical Methods in Engineering*, vol. 80, pp. 163-190.

Jalušić, B.; Sorić, J.; Jarak, T. (2017): Mixed Meshless Local Petrov-Galerkin Collocation Method for Modeling of Material Discontinuity. *Computational Mechanics*, vol. 59, pp. 1-19.

Most, T.; Bucher, C. (2008): New Concepts for Moving Least Squares: An Interpolating Nonsingular Weighting Function and Weighted Nodal Least Squares. *Engineering Analysis with Boundary Elements*, vol. 32, no. 6, pp. 461-470.

Polizzotto, C. (2003): Gradient Elasticity and Nonstandard Boundary Conditions. *International Journal of Solids and Structures*, vol. 40, no. 26, pp. 7399-7423.