On Operating Envelope Protection Design for Nonlinear Discrete-time Systems

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Abstract: This paper is concerned with the operating envelope protection design for a broad class of nonlinear discrete-time systems based on a Robust Control Invariant (RCI) set framework. Existing techniques for a direct design of a RCI set for a general system suffer from the computational intractability connected with the complexity of a system model and model dimension. For a nonlinear system with mild assumptions on its model here we use a suitably defined linear model with additive uncertainty confined in a finite set and its corresponding maximum RCI set. Robust constraints satisfaction for the original system is ensured for a pre-defined set of disturbance trajectories. The approach is based on a superposition of a nominal and perturbed dynamics, whereas the coupling effects between respective dynamics are encompassed with the additive uncertainty. The additive uncertainty set is estimated by employing the trajectories of disturbance and a system description. The presented approach is scalable with respect to the system dimension. A simple example from the wind energy field is used to illustrate the proposed method.

Keywords: operating envelope protection, (maximum) robust control invariant set, disturbance trajectories, superposition, linear model, additive uncertainty, envelope model, estimation, wind turbine

1. INTRODUCTION

Robust Control Invariant (RCI) set framework has a great importance in the field of robust system control, whereas it can be used to design a control policy, such that the constraints are respected for all realisations of the model uncertainties in the future (Blanchini, 1999). While the methods for a RCI set computation given a Linear Time-Invariant (LTI) system model with additive uncertainty are widely accepted, the methods that are developed for a wider class of nonlinear or Linear Parameter-Varying (LPV) models suffer from computational intractability (Pluymers, 2006; Kerrigan, 2001).

The largest RCI set of a constrained system subject to additive uncertainty, denoted as the Maximum RCI (MRCI) set, is computed by intersecting the consecutive one-step robustly controllable sets within the bounds of the imposed system constraints (Blanchini, 1999; Blanchini and Miani, 2008). Typically used approaches in the MRCI set design for nonlinear and parameter-varying systems include Piece-Wise Affine (PWA) or LPV models. The method for one-step robustly controllable set computation of a given PWA model (Kerrigan, 2001) involves enumerating all possible transitions between the affine models. The model uncertainty is taken into account by computing the Minkowski set difference between the polyhedral partition representing targeted regions and the polytope of uncertainty in each iteration of the MRCI set computation algorithm which often discards the applicability of the approach for complex models (Vašak et al., 2007). Alamo et al. (2007) suggest employing the outer and inner convex set approximations for the computation of the one-step robustly controllable set that greatly reduce the computational and memory burden compared to the exact algorithm. Nevertheless, since the set approximation algorithms suffer from the curse of dimensionality, i.e. the approximation error may degrade dramatically with the rise of model dimensions, the scalability of the proposed method should be further investigated.

When applied to a LPV model of the system, it is somewhat easier to compute the one-step robustly controllable set. It includes computation of the one-step robustly controllable set for each of the extremal linear models, whereas the result is obtained by their intersection. One-step robustly controllable set issued for the LPV model is convex, thus allowing the application of algorithms with the lower computational complexity for its computation. Nevertheless, the MRCI set for a LPV model is solvable only for systems with the low number of states and extreme LTI models representing the LPV model (Pluymers, 2006).

The general framework of the robust control correction synthesis in a set theoretic approach may be found in Raković et al. (2009). In respective paper authors identify two approaches for the control input correction, specifically the *inf-sup* and *sup-inf* correction syntheses, whereas the applied approach depends whether the disturbance that acts on the system is known or unknown when the necessary control input correction is determined. The presented framework is validated on a discrete-time LTI model of a system. The robust control correction synthesis for LTI systems is also presented in Vašak and Perić (2008), Vašak and Perić (2010) within a multi-mass electrical drive application, with experimental results (Vašak et al., 2010) and efficient on-line implementation (Vašak et al., 2011). In Kvasnica et al. (2012) authors discuss the hierarchical design methodology in the control system layout that employs the low-level controller to ensure satisfaction of system constraints, control system stability and fault tolerance with respect to the technical failures in communication between the high-level controller and the actuators.

Design of the robust control correction system that is presented in this paper originates from the research conducted by Vašak et al. and complies with the general ideas presented in Raković et al. (2009) and Kvasnica et al. (2012). The contribution of our approach is the simplification of the general robust control correction synthesis which may be applied on a broad class of nonlinear systems subject to the predefined set of disturbance trajectories. The design method is based on a RCI set framework, whereas the employed model that is used for the RCI set computation has a specific structure of a linear discrete-time invariant system with an additive uncertainty, whereas it complies with the efficient methods for the robust control invariant set computation, e.g. Raković et al. (2007); Raković and Barić (2010). In order to derive a simplified approach for the protection system synthesis we introduce a new notion of the robust control invariance with respect to the provided set of disturbance trajectories. Existence of such a set is the leading assumption for the method.

The paper is organised as follows. Section 2 gives the outline of the notation and definitions that are important for the understanding of the presented material. Section 3 presents the RCI set design problem for a nonlinear discrete-time system given that the disturbance trajectories of the system are provided. The operating envelope protection design method is proposed in Section 4. Section 5 illustrates the proposed approach by an example and Section 6 summarises conclusions.

2. NOTATION AND DEFINITIONS

In the recurrence equations, superscript $+$ on top of a variable symbol is used to denote the state of respective variable at the next time instant, whereas its current state is briefly denoted with a variable symbol, e.g. x^+ and x denote the next and current state respectively. The state of a variable at the specific time instant is denoted with the time value in the subscript, e.g. x_t denotes the state at the (discrete) time instant t. Countable set is denoted with a calligraphic font, whereas the uncountable set is denoted with a blackboard bold font.

Given two sets $\mathbb{U}$ and $\mathbb{V}$, such that $\mathbb{U} \subset \mathbb{R}^n$ and $\mathbb{V} \subset \mathbb{R}^n$, the Minkowski sum is defined by $\mathbb{U} \oplus \mathbb{V} \triangleq \{u + v \,|\, u \in \mathbb{U}, v \in \mathbb{V}\}$ and the Minkowski set difference is $\mathbb{U} \ominus \mathbb{V} \triangleq \{\delta \in \mathbb{R}^n \,|\, \delta + v \in \mathbb{U}, \forall v \in \mathbb{V}\}$. Given a vector x and a convex set $\mathbb{U}$ we briefly write $x \oplus \mathbb{U}$ instead of $\{x\} \oplus \mathbb{U}$. We use $||x||$ to denote the $L2$ norm of vector x. *Polytope* is a compact (closed and bounded) convex intersection of a finite number of half-spaces. LTI discrete-time system is *strictly stable* if all its poles are within the unit circle in the complex plane.

3. ROBUST CONTROL INVARIANCE WITH RESPECT TO A PRE-DEFINED SET OF DISTURBANCE TRAJECTORIES

Consider a nonlinear discrete-time system of the form

$$x^+ = f(x, u, d), \tag{1}$$

where $x \in \mathbb{R}^n$ is the current state of the system, $u \in \mathbb{R}^p$ is the current control input and $d \in \mathbb{R}^r$ is the disturbance acting on the system. The state update function f is assumed continuous and derivable from all sides in the considered domain. Disturbance is independent of the state and input and may change at time instants, whereas the uncertainty of disturbance estimation is constrained within a convex and compact set. The disturbance is assumed to be realised as one of the trajectories from a pre-defined set of disturbance trajectories $\mathcal{F} = \{\mathcal{F}_{d_1}, \mathcal{F}_{d_2}, \ldots, \mathcal{F}_{d_N}\}$. Although the disturbance realisation uncertainty may be also considered, it is omitted in this paper for simplicity of presentation. The output equation of the system is given with

$$y = h(x, u, d) + v, \tag{2}$$

where $y \in \mathbb{R}^m$ is the vector of measurements, v is the measurement noise and h is the system output function. The system state vector x and the disturbance d are assumed observable. The state update function f as well as the output function h are supposed to be known.

The system should be kept within the convex set of imposed constraints

$$(x, u) \in \mathbb{P}_x \times \mathbb{P}_u, \tag{3}$$

whereas the constraints defining the set $\mathbb{P}_x \times \mathbb{P}_u$ directly follow from the physical limitations and safety requirements of the system at hand. In terms of discussion from above, the maximum robust control invariant set of system (1) subject to the disturbance realisation scenarios in $\mathcal{F}$, denoted with $\mathbb{I}_x^{\mathcal{F}}$, implies that there exists the set of admissible control inputs $u \in \mathbb{U}^{\mathcal{F}}(x) \subseteq \mathbb{P}_u$ such that

$$\mathbb{I}_x^{\mathcal{F}} = \bigcap_{i=1}^{N} \{x_t \,|\, f(x_t, u_t, d_t), f(f(x_t, u_t, d_t), u_{t+1}, d_{t+1}) \ldots \ldots \in \mathbb{I}_x^{\mathcal{F}}, \{d_t, d_{t+1}, \ldots\} = \mathcal{F}_{d_i}\}, \tag{4}$$

where the disturbance realisations are generated according to the provided disturbance trajectory $\mathcal{F}_{d_i}$. The disturbance generation is conducted as follows. Assume that the initial time stamp of each disturbance trajectory is t. Then, each of the trajectories is expanded in the unbounded time interval according to the following scheme

$$\{\ldots \mathcal{F}_{d_i,t}, \ldots \mathcal{F}_{d_i,t}, \mathcal{F}_{d_i,t+1}, \mathcal{F}_{d_i,t+2}, \ldots \mathcal{F}_{d_i,t+n_{\mathcal{F}_i}-1}, \ldots \mathcal{F}_{d_i,t+n_{\mathcal{F}_i}-1}, \ldots\},$$

where $n_{\mathcal{F}_i}$ is the sequence length of the i-th disturbance trajectory. The condition in (4) then implies invariance with respect to each of the extended disturbance trajectories.

Note that $\mathbb{I}_x^{\mathcal{F}}$ is the over-approximation of the MRCI set corresponding to the system (1) with disturbance realisation d being confined in an arbitrary convex and compact set $\mathbb{D}$ that encompasses all possible disturbance realisations from $\mathcal{F}$. Introduced disturbance set-based approach will allow certain simplifications in the design of the operating envelope protection system.

The admissible control input set $\mathbb{U}^{\mathcal{F}}(x)$ in dependence of the system state x is usually represented with a set $\mathbb{I}_{xu}^{\mathcal{F}}$, whereas $\mathbb{I}_x^{\mathcal{F}} = \text{proj}_x\, \mathbb{I}_{xu}^{\mathcal{F}}$ and

$$\mathbb{U}^{\mathcal{F}}(x) = \{u \,|\, (x, u) \in \mathbb{I}_{xu}^{\mathcal{F}}\}. \tag{5}$$

Term proj_x denotes the projection operation on the x space.

4. METHOD FOR THE OPERATING ENVELOPE PROTECTION DESIGN

Suppose that the system (1) is controlled with a controller of a general structure, with u_c as its output signal. Given that the set of admissible control inputs $\mathbb{U}^{\mathcal{F}}(x)$ is computed, the control input correction that has to be applied in order that the system is steered within the imposed constraints is given with

$$\tilde{u} = u^* - u_c, \tag{6}$$

$$u^* = \min_u \quad ||u - u_c||, \quad \text{s.t. } u \in \mathbb{U}^{\mathcal{F}}(x). \tag{7}$$

In practice, the admissible control input set $\mathbb{U}^{\mathcal{F}}(x)$ can be hardly computed for a general class of nonlinear systems due to the complexity of $\mathbb{I}_x^{\mathcal{F}}$ computation. In that regard, we propose a computationally viable approach for the operating envelope protection design in this paper. The approach is based on computing the inner approximation of the admissible control input set $\hat{\mathbb{U}}^{\mathcal{F}}(x) \subseteq \mathbb{U}^{\mathcal{F}}(x)$ in a vicinity of nominal trajectories of the system (1). Nominal trajectories are result of the applied internal controller input u_r that is confined in the set $\mathbb{U}_r \subset \mathbb{P}_u$ whereas the system is exposed to the disturbance trajectories in $\mathcal{F}$. Set of admissible control inputs $\hat{\mathbb{U}}^{\mathcal{F}}(x)$ is in this paper approximated with $u_r \oplus \tilde{\mathbb{U}}(z)$, where the derived system state z represents a suitable extension of the system state-space, as will be discussed in the sequel. Internal controller is here represented with a black box model, while its structure is irrelevant for the design of the operating envelope protection system. The same applies to the controller block.

In terms of the previous discussion, the presented operating envelope protection is implemented with

$$u^* = \min_u \quad ||u - u_c||, \quad \text{s.t. } u \in \hat{\mathbb{U}}^{\mathcal{F}}(x) \equiv u_r \oplus \tilde{\mathbb{U}}(z), \tag{8}$$

whereas its integration in the control system is depicted in Fig. 1. Note that the variables $\hat{x}$, $\hat{z}$ and $\hat{d}$ are generally estimated from the available measurements y, inputs u^*, $\tilde{u}$ and as such are employed in the control system. Unlike the notation in Fig. 1, the hats of the estimated variables are omitted in the great part of this paper for simplicity, but the implications of the estimation uncertainty are noted later on.

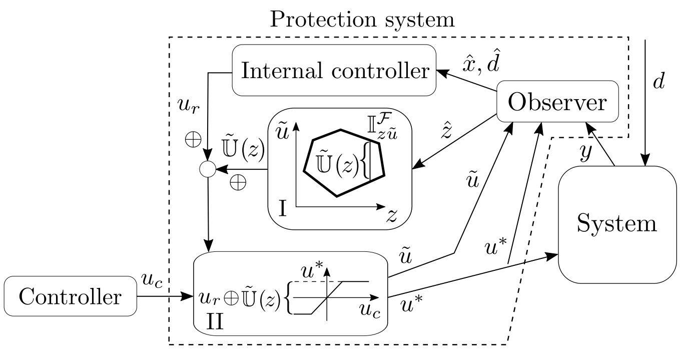

Fig. 1. Structural scheme of a RCI set-based operating envelope protection system; block 'I' determines the set of admissible control input perturbations $\tilde{\mathbb{U}}(z)$; block 'II' replaces the input u_c with the vector u^* from $u_r \oplus U(z)$ nearest to it (for 1D input case, this is a saturation function)

4.1 Linear time-invariant model with additive uncertainty

Similar to the approach used in the tube MPC design strategy, in Mayne et al. (2005), let us divide the system dynamics in two independent parts, the nominal dynamics around the steady-state operating point and the perturbed dynamics. For each disturbance trajectory $\mathcal{F}_{d_i}$ holds the following state update equation of the nominal dynamics in the coordinate system of the steady-state operating point

$$\bar{x}^+ = f(\bar{x} + x_0(d), u_r, d) - x_0(d^+), \tag{9}$$

where x_0 is the steady-state of the system (1) for the provided disturbance d. The shorter notation $x_r \equiv \bar{x} + x_0(d)$ will be also used in the remainder of this paper. The perturbed system dynamics models the behaviour of the system around the nominal trajectory under the influence of controls perturbed from u_r,

$$\tilde{x}^+ = A\tilde{x} + B\tilde{u} + \tilde{w}. \tag{10}$$

The actual state of the system (1), $x_0+\bar{x}+\tilde{x}$, is the result of the control input $u_r + \tilde{u}$, whereas under certain conditions the update of $\tilde{x}$ can be considered independent of the nominal system state $\bar{x}$, the nominal control input u_r and the disturbance d. Respective dependence, as well as the non-modelled nonlinearities are comprised in the additive uncertainty of the model with perturbations, denoted with $\tilde{w}$. Matrices A, B in (10) are used to approximate the dynamics of the system (1) in vicinity of the nominal trajectories. They are identified according to the procedure in Appendix A.

Proposition 1. The consistently estimated state-space model (10) is strictly stable provided that the system (1) is stabilised with the nominal control input u_r.

In contrast to the tube MPC approach, one cannot ensure the satisfaction of the imposed constraints for the nominal system state $x_r = x_0 + \bar{x}$ that is a consequence of the nominal control input u_r. Therefore, let us divide the states of the system in two subsystems, whereas the nominal trajectories of the first subsystem states, indexed with the set $\mathcal{A}$, do not compromise the imposed system constraints and the nominal trajectories of the second subsystem states, indexed with the set $\mathcal{B}$, exceed the limits of the imposed system constraints for at least one nominal trajectory. The system constraints (3) will remain satisfied as far as

$$\begin{bmatrix} \tilde{x}_{\mathcal{A}} \\ \bar{x}_{\mathcal{B}} + \tilde{x}_{\mathcal{B}} \end{bmatrix} \in \mathbb{P}_x \ominus (\mathbb{X}_{r_{\mathcal{A}}} \times \mathbb{X}_{0_{\mathcal{B}}}) \tag{11}$$

and

$$\tilde{u} \in \mathbb{P}_u \ominus \mathbb{U}_r, \tag{12}$$

where $\mathbb{X}_{r_{\mathcal{A}}}$ is the smallest convex set that contains the nominal trajectories of the subsystem $\mathcal{A}$ state vector, $x_{r_{\mathcal{A}}}$. Similarly, $\mathbb{X}_{0_{\mathcal{B}}}$ denotes the smallest convex set that contains the steady-state operating points of the subsystem $\mathcal{B}$, $x_{0_{\mathcal{B}}}$. For the problem to be well-posed, sets $\mathbb{P}_x \ominus (\mathbb{X}_{r_{\mathcal{A}}} \times \mathbb{X}_{0_{\mathcal{B}}})$ and $\mathbb{P}_u \ominus \mathbb{U}_r$ ought to be full-dimensional. Note that the states of the nominal subsystem $\mathcal{A}$ are not considered in the state vector (11) since they comply with the imposed system constraints, resulting with the reduced number of the model state dimension. In a specific case the state vector of the proposed LTI model can comprise the states of the subsystem $\mathcal{B}$ exclusively.

According to the discussion, system (1) is represented with the LTI state-space model comprising the states of system perturbation and the nominal dynamics of the subsystem $\mathcal{B}$ in the coordinate system of the steady-state operating point,

$$z^+ = \begin{bmatrix} \tilde{x}^+ \\ \bar{x}_{\mathcal{B}}^+ \end{bmatrix} = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & S \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \bar{x}_{\mathcal{B}} \end{bmatrix} + \begin{bmatrix} B \\ \mathbf{0} \end{bmatrix} \tilde{u} + \begin{bmatrix} \tilde{w} \\ \bar{w}_{\mathcal{B}} \end{bmatrix}, \quad (13)$$

where z is the state vector of the model. The nominal dynamics of the subsystem $\mathcal{B}$ around the operating point is represented with the envelope model. The envelope model is estimated to include each of the nominal subsystem $\bar{x}_{\mathcal{B}}$ trajectories resulting from the given set of disturbance trajectories $\mathcal{F}$. Matrix S of the envelope model is used to model the shape of the envelope propagation and the additive uncertainty $\bar{w}_{\mathcal{B}}$ to define its scale.

The existence of the robust control invariant set of the LTI model with the additive uncertainty (13) subject to the constraints (11) and (12), denoted with $\mathbb{I}_{z\tilde{u}}^{\mathcal{F}}$, ensures that the imposed constraints of the system supplied with the nominal control input u_r and the control input corrections $\tilde{u}$ remain respected. The computation of the (maximum) RCI set for model (13) is discussed in Subsection 4.4.

4.2 Envelope of nominal trajectories in coordinate system of steady-state operating point

By consequently employing all disturbance trajectories from the set $\mathcal{F}$, the set of nominal system trajectories is obtained if the system is supplied with the control input u_r. The nominal trajectories of the subsystem $\mathcal{B}$ in the coordinate system of the disturbance-dependent operating point are then derived, which are used to estimate the corresponding envelope model defined with the propagation matrix S and the propagation uncertainty set $\bar{\mathbb{W}}_{\mathcal{B}}$ (13),

$$\bar{x}_{\mathcal{B}}^+ = S\bar{x}_{\mathcal{B}} + \bar{w}_{\mathcal{B}},\ \bar{w}_{\mathcal{B}} \in \bar{\mathbb{W}}_{\mathcal{B}}. \quad (14)$$

The envelope model is estimated from a collected set of the nominal state pairs $\{(\bar{x}_{\mathcal{B}_j}, \bar{x}_{\mathcal{B}_j}^+)\}$, $\forall j \in \mathcal{J}$, where $\mathcal{J}$ is the corresponding set of indices, such that

$$\bar{x}_{\mathcal{B}_j}^+ \in S\bar{x}_{\mathcal{B}_j} \oplus \bar{\mathbb{W}}_{\mathcal{B}},\ \forall j \in \mathcal{J}, \quad (15)$$

whereas the propagation uncertainty is minimised while imposing the bounds to the steady-state set of the envelope. The corresponding optimisation problem is presented in the following proposition.

Proposition 2. The envelope model is estimated by solving the following optimisation problem

$$\begin{aligned} \min_{L,Y} \quad & -\operatorname{logdet}(L) \\ \text{s.t.} \quad & \begin{bmatrix} I & L\bar{x}_{\mathcal{B}_j}^+ - Y\bar{x}_{\mathcal{B}_j} \\ \left(L\bar{x}_{\mathcal{B}_j}^+ - Y\bar{x}_{\mathcal{B}_j}\right)^\top & 1 \end{bmatrix} \succeq 0, \quad (16) \\ & (L - Y) - L_s \succeq 0,\ L \succ 0,\ Y \succ 0,\ \forall j \in \mathcal{J}, \end{aligned}$$

where $(L^\top L)^{-1}$ is a diagonal shape matrix of the ellipsoid that over-approximates the uncertainty set $\bar{\mathbb{W}}_{\mathcal{B}}$ and $Y = LS$ is also a diagonal matrix. The largest steady-state ellipsoidal set of the envelope model is described by the shape matrix $(L_s^\top \cdot L_s)^{-1}$, where L_s is a suitably selected symmetric matrix, such that the over-approximation error is sufficiently small.

Once that the problem (16) is solved, the exact polytopic uncertainty set $\bar{\mathbb{W}}_{\mathcal{B}}$ is computed from the given set of pairs $\{(\bar{x}_{\mathcal{B}_j}, \bar{x}_{\mathcal{B}_j}^+)\}$, $\forall j \in \mathcal{J}$. The uncertainty set of the nominal dynamics $\bar{\mathbb{W}}_{\mathcal{B}}$ may be represented with the over-approximation (Le Guernic, 2009) of the exact uncertainty to reduce the computational complexity of the (M)RCI set computation.

Remark 1. One may additionally reduce the over-approximation error of the envelope by introducing the nominal state observation error $\bar{o}_{\mathcal{B}}$,

$$\bar{y}_{\mathcal{B}} = \bar{x}_{\mathcal{B}} + \bar{o}_{\mathcal{B}}, \quad (17)$$

where the observation error $\bar{o}_{\mathcal{B}}$ resides in the suitably determined compact and convex set $\bar{\mathbb{O}}_{\mathcal{B}}$ with origin in its interior. The envelope model constraint then becomes

$$\bar{y}_{\mathcal{B}_j}^+ \in \bar{x}_{\mathcal{B}_j}^+ \oplus \bar{\mathbb{O}}_{\mathcal{B}},\ \forall j \in \mathcal{J}. \quad (18)$$

Given that the nominal state observation error is introduced, the model state vector and corresponding constraints for the MRCI set derivation should be reformulated as

$$\begin{bmatrix} \tilde{x}_{\mathcal{A}} \\ \bar{y}_{\mathcal{B}} + \tilde{x}_{\mathcal{B}} \end{bmatrix} \in \mathbb{P}_x \ominus \left(\mathbb{X}_{r_{\mathcal{A}}} \times \left(\mathbb{X}_{0_{\mathcal{B}}} \oplus \bar{\mathbb{O}}_{\mathcal{B}}\right)\right). \quad (19)$$

4.3 Uncertainty of perturbed dynamics

The applied control input corrections will steer the system away from the nominal trajectory. Since the considered system is nonlinear, the uncertainty of the perturbed dynamics depends on the current system state, control input and disturbance. The uncertainty set of the perturbed dynamics is therefore estimated in the reachable sets of the system subject to the disturbance trajectories in $\mathcal{F}$, whereas the control input is confined in $u_r \oplus (\mathbb{P}_u \ominus \mathbb{U}_r)$ and state constraints provided with $\mathbb{P}_x$. The reachability analysis should account for the estimation uncertainties of the system state and the disturbance, while the output of the internal controller u_r depends on respective estimates. Accurate reachability can be performed for the low-dimensional systems given that they can be approximated arbitrarily well by a linearised model with the additive uncertainty at each time instant, whereas for reachability of the higher-dimensional system one should consider certain approximations (Le Guernic, 2009; Dang and Maler, 1998; Girard, 2005).

The estimation of the perturbed dynamics uncertainty is conducted by employing the pre-defined set of disturbance trajectories in the reachability analysis, whereas the samples of perturbed dynamics uncertainty $\{\tilde{w}_k\}$, $\forall k \in \mathcal{K}$, where $\mathcal{K}$ is the corresponding set of indices, are estimated at each time instant of each disturbance trajectory $\mathcal{F}_{d_i}$ according to

$$\begin{aligned} \arg\min_{\tilde{w}_k, \bar{w}_{\mathcal{B}}} \quad & \tilde{w}_k^\top \tilde{w}_k \\ \text{s.t.} \quad & f(x, u_r + \tilde{u}, d) = A\tilde{x} + B\tilde{u} + \ldots \\ & \begin{bmatrix} f_{\mathcal{A}}(x_r, u_r, d) \\ S\bar{x}_{\mathcal{B}} + \bar{w}_{\mathcal{B}} + x_{0_{\mathcal{B}}} \end{bmatrix} + \tilde{w}_k,\ \bar{w}_{\mathcal{B}} \in \bar{\mathbb{W}}_{\mathcal{B}}, \end{aligned} \quad (20)$$

where $f_{\mathcal{A}}$ denotes the indexing of subsystem dimensions $\mathcal{A}$, the state x in (20) is in the time instant t of the reachability analysis sampled from the set of reachable system states $\mathbb{X}_t$, the control input correction $\tilde{u}$ is sampled from $\mathbb{P}_u \ominus \mathbb{U}_r$, disturbance d is realised according to the disturbance trajectory set $\mathcal{F}_{d_i}$, $\tilde{x}$ equals $x - x_0 - \bar{x}$, whereas $\bar{x}$ and x_0 are determined from the state of the nominal system trajectory and disturbance realisation d respectively. Observed samples of the uncertainty $\{\tilde{w}_k\}$, $\forall k \in \mathcal{K}$, at time instant t are used to estimate the convex hull of the perturbed dynamics uncertainty $\tilde{\mathbb{W}}_t$.

Estimated uncertainty sets of the perturbed dynamics in time instants are used to determine the equivalent fixed uncertainty set of the perturbed dynamics $\tilde{\mathbb{W}}_\infty$. The set $\tilde{\mathbb{W}}_\infty$ is derived according to the following principles. Starting from the time instant t_0, the one-step state update equation is given with $A\tilde{x}_{t_0} + B\tilde{u}_{t_0} + \tilde{w}_{t_0}$ and

$$\tilde{w}_{t_0} \in \tilde{\mathbb{W}}_{t_0} \subseteq \tilde{\mathbb{W}}_\infty. \quad (21)$$

The next step is given with $A^2\tilde{x}_{t_0} + AB\tilde{u}_{t_0} + B\tilde{u}_{t_0+1} + A\tilde{w}_{t_0} + \tilde{w}_{t_0+1}$, thus

$$\tilde{w}_{t_0} \in \tilde{\mathbb{W}}_{t_0}, \tilde{w}_{t_0+1} \in \tilde{\mathbb{W}}_{t_0+1},\ A\tilde{\mathbb{W}}_{t_0} \oplus \tilde{\mathbb{W}}_{t_0+1} \subseteq A\tilde{\mathbb{W}}_\infty \oplus \tilde{\mathbb{W}}_\infty. \quad (22)$$

Therefore, given the sequence of the uncertainty sets $\{\tilde{\mathbb{W}}_l\}_{l=t_0}^{t_0+i_0}$, estimate of the fixed uncertainty set $\tilde{\mathbb{W}}_\infty$ is for the first i_0 time instants lower-bounded with

$$\begin{aligned} &\tilde{\mathbb{W}}_{t_0} \subseteq \tilde{\mathbb{W}}_\infty, \\ &\tilde{\mathbb{W}}_{t_0+1} \oplus A\tilde{\mathbb{W}}_{t_0} \subseteq \tilde{\mathbb{W}}_\infty \oplus A\tilde{\mathbb{W}}_\infty, \\ &\vdots \\ &\tilde{\mathbb{W}}_{t_0+i_0-1} \oplus A\tilde{\mathbb{W}}_{t_0+i_0-2} \oplus \ldots \oplus A^{i_0-1}\tilde{\mathbb{W}}_{t_0} \subseteq \\ &\qquad \tilde{\mathbb{W}}_\infty \oplus A\tilde{\mathbb{W}}_\infty \oplus \ldots \oplus A^{i_0-1}\tilde{\mathbb{W}}_\infty. \end{aligned} \quad (23)$$

Note that estimating the smallest fixed uncertainty set $\tilde{\mathbb{W}}_\infty$ that complies with (23) may be computationally intractable for large i_0. Therefore, irrespective of the parameter i_0 value, the over-approximating fixed uncertainty set $\tilde{\mathbb{W}}_\infty$ may be estimated by considering a certain number of newest sets in the sequence and by performing the estimate $\tilde{\mathbb{W}}_\infty$ iteratively. E.g. if three newest uncertainty sets are considered to update the fixed uncertainty set at time instant t, the following set inequality should apply

$$\tilde{W}_t \oplus A\tilde{W}_{t-1} \oplus A^2\tilde{W}_{t-2} \subseteq \tilde{\mathbb{W}}_\infty \oplus A\tilde{\mathbb{W}}_\infty \oplus A^2\tilde{\mathbb{W}}_\infty. \quad (24)$$

Once the uncertainty set $\tilde{\mathbb{W}}_\infty$ is determined, the overall uncertainty of the LTI model equals to

$$\mathbb{W} = \tilde{\mathbb{W}}_\infty \times \bar{\mathbb{W}}_{\mathcal{B}}. \quad (25)$$

4.4 (Maximum) RCI set computation

The design of an operating envelope protection based on a RCI set framework reduces to the MRCI set computation of the LTI model (13) with constraints (11), (12) and the additive uncertainty set (25). In order to ensure that the set of admissible control input corrections is full-dimensional, that is $\mathbb{U}_r \subset \mathbb{P}_u$, hard constraints should be imposed to the output of the internal controller. The state estimation error may be included in the RCI set design procedure, as discussed in Vašak and Perić (2010), Hure et al. (2016).

One should keep in mind that the bigger is the set of admissible control actions $\mathbb{P}_u \ominus \mathbb{U}_r$ the larger is the uncertainty set $\tilde{\mathbb{W}}_\infty$ of the perturbation model (10). Depending on the system and the applied control law for u_r, the respective uncertainty may rise significantly when enlarging the admissible set of control input corrections above some threshold, which may result with an empty MRCI set. In that regard, the size measure of the resulting MRCI set can be a performance measure for the selection of $\mathbb{U}_r$.

For a linear model with linear constraints, the maximum robust control invariant set is an intersection of halfspaces and can be computed in an iterative manner by employing the algorithm which can be found in Blanchini (1999), Blanchini and Miani (2008). There is no guarantee that the respective algorithm will terminate in finite time, hence it it usually run with an upper bound on the number of iterations or with a lower bound on the size measure of difference between constructed sets in consecutive iterations. One can find the arbitrarily accurate polyhedral robust control invariant under-approximation of the maximum robust control invariant set in a finite number of steps by employing the approach in Blanchini (1991). Set of algorithms for the MRCI set computation may be found in the MPT3 toolbox (Kvasnica et al., 2015). Otherwise, for the higher-dimensional models one can e.g. employ the parameterised RCI set computation algorithm, as discussed in Raković and Barić (2010), or use the piecewise ellipsoidal sets (Kurzhanskiy and Varaiya, 2006) to represent the MRCI set under-approximation.

4.5 Deployment of designed RCI set

The set of admissible control inputs is given with $u_r \oplus \tilde{\mathbb{U}}(z)$, where u_r is the output of the internal controller and $\tilde{\mathbb{U}}(z)$ is the set of admissible control input corrections with respect to the model state z. The state of the model should be estimated from the available measurements of the system. Note that the system state x is represented with a sum of the operating point x_0, the nominal system state $\bar{x}$ around the operating point and the perturbation state $\tilde{x}$. The approach to estimate the feasible model state from the one that is provided by the certain state estimation algorithm is proposed in this subsection.

Let us suppose that the delayed control input vector u_{t-1}, vector of measurements y_t and its variance is at disposition for the system state estimation at time instant t. By performing the iteration of the nonlinear state estimation, where the state vector is defined with

$$q = \begin{bmatrix} x \\ x_{\mathcal{B}_0} \\ \tilde{x} \end{bmatrix}, \quad (26)$$

corresponding state estimate $\hat{q}$ and covariance matrix P are estimated. Respective estimates can be derived by employing the method for nonlinear estimation, e.g. the unscented Kalman filter (Wan and Van Der Merwe, 2000) or generally established extended Kalman filter (Jazwinski, 2007). Once obtained estimates $\hat{q}$ and P are used to find the feasible state q^* with the maximum probability,

$$\begin{aligned} q^* = \min_q \quad & (q-\hat{q})^\top P^{-1}(q-\hat{q}) \\ \text{s.t.} \quad & \begin{bmatrix} \tilde{x} \\ \bar{x}_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} \tilde{x} \\ x_{\mathcal{B}} - x_{\mathcal{B}_0} - \tilde{x}_{\mathcal{B}} \end{bmatrix} \in \mathbb{I}_z. \end{aligned} \quad (27)$$

Model state estimate $\hat{z}$ is uniquely determined with q^*. The set of admissible control inputs $\tilde{\mathbb{U}}(\hat{z})$ is easily obtained given the state of the model $\hat{z}$. The system is supplied with the control input that is determined as a projection of the controller input u_c on the set of the admissible control inputs, resulting with u^* as depicted in Fig. 1. The simple control correction algorithms for the lower-dimensional control inputs can be found in Vašak and Perić (2010) and Hure et al. (2016).

5. ILLUSTRATIVE EXAMPLE

5.1 Simplified wind turbine model

Presented method for the RCI set design is evaluated on a simplified model of a direct-drive wind turbine system operating above the nominal wind speed (Burton et al., 2001; Hure et al., 2016), given with the following equations

$$T_\beta \dot{\beta} + \beta = \beta_{\text{ref}}, \tag{28a}$$

$$J_t \dot{\omega} = T_a(\beta, \omega, v_w) - T_{g,\text{nom}}, \tag{28b}$$

$$y = [\beta,\ \ \omega]^\top, \tag{28c}$$

where β is the pitch angle of the blades, β_{ref} is the pitch angle reference and T_β is the time constant of the blade pitching dynamics; J_t is the inertia of the turbine, ω is the rotational speed of the turbine, with $T_a(\beta, \omega, v_w)$ is given the nonlinear static function of the turbine aerodynamics, v_w is the rotor-effective wind speed and $T_{g,\text{nom}}$ is the rated generator torque; y is the vector of measurements. Imposed system constraints are given with

$$\begin{aligned} -2 \le \beta \le 35\ [°], \qquad -11 \le \dot{\beta} \le 11\ [°/\text{s}], \\ 2.304 \le \omega \le 3.35\ [\text{rad/s}],\ \ -2 \le \beta_{\text{ref}} \le 35\ [°]. \end{aligned} \tag{29}$$

The set of disturbance trajectories $\mathcal{F}$ is composed of the turbulent wind speed data collected at the real wind farm site.

The internal controller that is considered for a RCI set design has a form of the static feed-forward control law with disturbance as its only input variable, $\bar{\beta}_{\text{ref}} = f_{f,0}(v_w)$, whereas $u_r \equiv \bar{\beta}_{ref}$ and $d \equiv v_w$. The nominal system dynamics is constrained with respect to

$$\begin{aligned} 0 \le \bar{\beta} \le 32\ [°],\ -8 \le \dot{\bar{\beta}} \le 8\ [°/s], \\ 0 \le \bar{\beta}_{\text{ref}} \le 32\ [°]. \end{aligned} \tag{30}$$

The wind speed estimation error $|\hat{v}_w - v_w| < 1$ [m/s] is included in the design of the protection system. The wind speed estimation error is in the simulation emulated with the band-limited white noise signal.

5.2 Problem formulation

RCI set is designed for the LTI model of the wind turbine with the additive uncertainty, obtained according to the presented method,

$$z^+ = \begin{bmatrix} 0.779 & 0 & 0 \\ -0.0042 & 0.973 & 0 \\ 0 & 0 & 0.916 \end{bmatrix} z + \begin{bmatrix} 0.221 \\ -0.0005 \\ 0 \end{bmatrix} \tilde{u} + w,$$

$$[0,\ -0.0053,\ -0.0258]^\top \le w \le [0,\ 0.0086,\ 0.0471]^\top, \tag{31}$$

where the state vector is defined with $z = [\tilde{\beta}, \tilde{\omega}, \bar{\omega}]^\top$, $\tilde{x} \equiv [\tilde{\beta}, \tilde{\omega}]^\top$ and $\bar{x}_{\mathcal{B}} \equiv \bar{\omega}$, whereas for the control input correction applies $\tilde{u} \equiv \tilde{\beta}_{\text{ref}}$. The observation error $-0.015 \le o_{\mathcal{B}} \le 0.015$[rad/s] and the steady-state envelope parameter $L_s = 1.455$ were used for the estimation of the envelope model. The imposed constraints of the model are given with

$$\begin{aligned} -2 \le \tilde{\beta} \le 3\ [°],\ -3 \le \dot{\tilde{\beta}} \le 3\ [°/s], \\ -0.5086 \le \tilde{\omega} + \bar{\omega} \le 0.5086\ [\text{rad/s}]. \end{aligned} \tag{32}$$

5.3 Results

The MRCI set is determined for the model and uncertainty set in (31) with constraints (32) and the result is depicted in Fig. 2, obtained by employing the MPT3 toolbox (Herceg et al., 2013). The computed MRCI set is used to form the admissible control input set $u_r \oplus \tilde{\mathbb{U}}(z)$ of the wind turbine control system.

Obtained simulation results of the wind turbine rotational speed in case that the RCI set-based protection is employed and otherwise is given in Fig. 3a. Comparison of the controller output u_c, internal controller output u_r and the corrected control output u^* (refer to Fig. 1) of the control system with employed protection is shown in Fig. 3b. Control input u_c is the output of the wind turbine gain-scheduled PI controller.

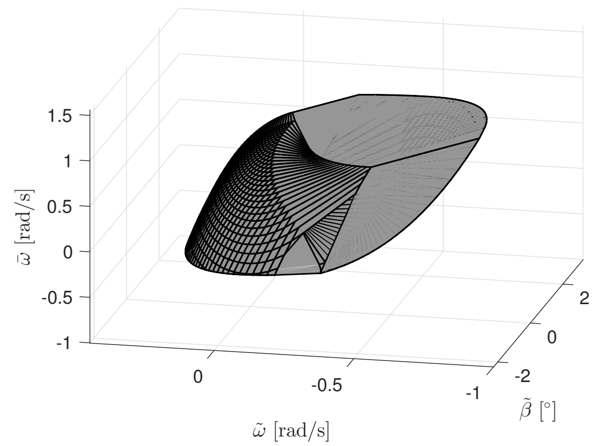

Fig. 2. The MRCI set of the wind turbine LTI model

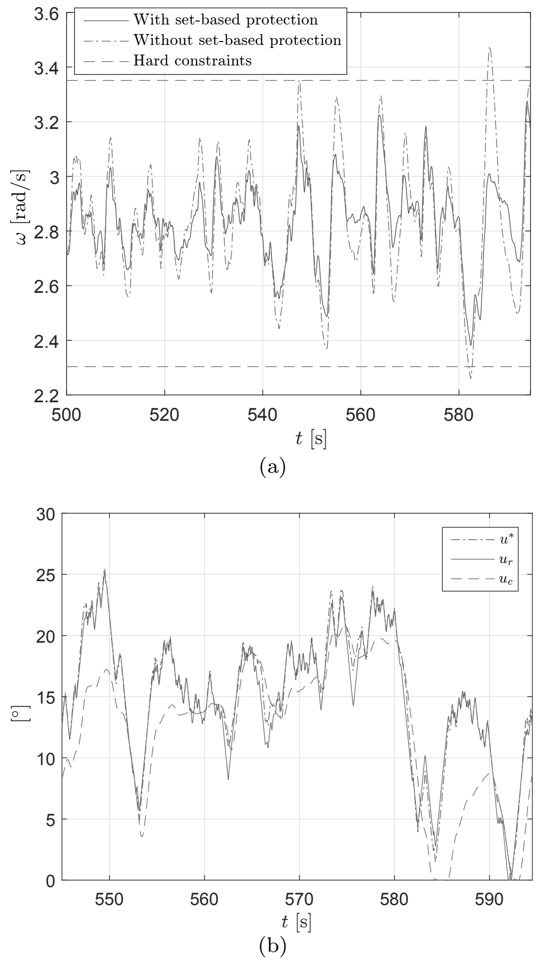

Fig. 3. (a) Selected section of the rotational turbine speed response when the protection system is used and otherwise ; (b) Comparison of the controller output u_c, the internal controller output u_r and the corrected control input u^* when the set-based protection system is employed

6. CONCLUSIONS

The RCI set-based operating envelope protection design for a nonlinear discrete-time system subject to the set of disturbance trajectories is discussed in this paper. The presented approach is based on a LTI model with an additive uncertainty representing the system dynamics around the nominal trajectory and an envelope model of the nominal trajectory. Respective models are estimated by employing the collected scenarios of disturbance realisations during the operation of the system.

Existing algorithms for the RCI set design of LTI systems subject to additive uncertainty enable the application of the presented method to a wide range of systems. The resulting RCI set can be used for the real-time implementation of the operating envelope protection with high safety requirements. The procedures that are presented in the RCI set design process may be further refined in order to get even larger invariant set of the system. In that regard, suggestions for the internal controller synthesis and selection of associated constraints may be given.

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Appendix A. IDENTIFICATION OF PERTURBED SYSTEM DYNAMICS

Given that the perturbation dynamics may be arbitrarily well approximated with the linearised model in the close vicinity of the nominal trajectory, the following prediction equation may be posed

$$\tilde{x}^{+} = A\tilde{x} + B\tilde{u} + e_i, \tag{A.1}$$

where

$$e_i = [A_i - A, B_i - B] \begin{bmatrix} \tilde{x}^\top & \tilde{u}^\top \end{bmatrix}^\top \tag{A.2}$$

is the prediction error given the linearisation matrices A_i, B_i of the system around the nominal trajectories.

If the quadratic prediction error function is to be minimised, the corresponding performance function is given with

$$e_i^\top e_i = \left|\left| [A_i - A, B_i - B] \begin{bmatrix} \tilde{x}^\top & \tilde{u}^\top \end{bmatrix}^\top \right|\right|^2 \tag{A.3}$$

Computationally viable approach for the minimisation of the worst-case prediction error (A.3) can be derived by employing the induced 2-norm (Meyer, 2000) as a quantity measure. By employing the induced 2-norm one can minimise the maximum norm of the prediction error on a boundary of the nominal trajectory neighbourhood that is defined with a unit sphere,

$$\min_{A,B} \max_{i, ||[\tilde{x}^\top\ \tilde{u}^\top]||_2 = 1} \left|\left| [A_i - A, B_i - B] \begin{bmatrix} \tilde{x}^\top & \tilde{u}^\top \end{bmatrix}^\top \right|\right|. \tag{A.4}$$

Good initial guess for the matrices A, B may be found in the finite set of linearised matrices $\{(A_i, B_i)\}$ that are obtained by linearisation of the system around the nominal trajectories.