Nonuniform spectrum on Banach spaces

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Preliminaries

Let $X = (X, \|\cdot\|)$ be a Banach space and let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of bounded linear operators on X. We consider the associated *linear nonautonomous difference equation* given by

$$x_{m+1} = A_m x_m, \quad m \in \mathbb{Z}. \tag{1}$$

Furthermore, let $\mathcal{A}(m,n)$, $m \geq n$ be a *linear cocycle* defined by

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n, & m > n, \\ \mathrm{Id} & m = n. \end{cases}$$

If $(x_m)_{m\in\mathbb{Z}} \subset X$ solves (1), then $x_m = \mathcal{A}(m,n)x_n$ for $m \geq n$.

(Non)uniform exponential dichotomy

We say that (1) admits a *uniform exponential dichotomy* if:

1. there exist projections $P_n \colon X \to X$ for each $n \in \mathbb{Z}$ satisfying

$$A_n P_n = P_{n+1} A_n \quad \text{for } n \in \mathbb{Z} \tag{2}$$

   and each map $A_n|\ker P_n \colon \ker P_n \to \ker P_{n+1}$ is invertible;

2. there exist $\lambda, D > 0$ such that

$$\|\mathcal{A}(m,n)P_n\| \le De^{-\lambda(m-n)} \quad \text{for } m \ge n$$

   and

$$\|\mathcal{A}(m,n)Q_n\| \le De^{-\lambda(n-m)} \quad \text{for } m \le n,$$

   where $Q_n = \mathrm{Id} - P_n$ and $\mathcal{A}(m,n) = (\mathcal{A}(n,m)|\ker P_m)^{-1}$ for $m < n$.

(Non)uniform exponential dichotomy

We say that (1) admits a *nonuniform exponential dichotomy* if:

1. there exist projections $P_n \colon X \to X$ for each $n \in \mathbb{Z}$ satisfying (2) and each map $A_n|\ker P_n \colon \ker P_n \to \ker P_{n+1}$ is invertible;
2. there exist a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|\mathcal{A}(m,n)P_n\| \le De^{-\lambda(m-n)+\varepsilon|n|} \quad \text{for } m \ge n$$

and

$$\|\mathcal{A}(m,n)Q_n\| \le De^{-\lambda(n-m)+\varepsilon|n|} \quad \text{for } m \le n,$$

where $Q_n = \mathrm{Id} - P_n$ and $\mathcal{A}(m,n) = (\mathcal{A}(n,m)|\ker P_m)^{-1}$ for $m < n$.

Proposition

Assume that the equation (1) *admits a nonuniform exponential dichotomy. For each* $n \in \mathbb{Z}$, *we have*

$$\operatorname{Im} P_n = \left\{ v \in X : \sup_{m \ge n} \|\mathcal{A}(m,n)v\| < +\infty \right\}$$

and $\operatorname{Im} Q_n$ *consists of all* $v \in X$ *for which there exists a sequence* $(x_m)_{m \le n} \subset X$ *such that* $x_n = v$, $x_m = A_{m-1}x_{m-1}$ *for* $m \le n$ *and* $\sup_{m \le n} \|x_m\| < +\infty$.

Ubiquity of nonuniform behaviour

Let $(\Omega, \mathcal{B}, \mu)$ be a probability space and let $f\colon \Omega \to \Omega$ be an invertible transformation that preserves μ, i.e. $\mu(B) = \mu(f^{-1}(B))$ for $B \in \mathcal{B}$. Furthermore, assume that μ is ergodic, i.e. for every $B \in \mathcal{B}$ such that $f^{-1}(B) = B$ we have that $\mu(B) \in \{0, 1\}$. Let M_d denote the space of all matrices of order d and consider a measurable map $A\colon X \to M_d$. We consider

$$A^{(n)}(\omega) = A(f^{n-1}(\omega)) \cdots A(f(\omega)) \cdot A(\omega), \quad \text{for } n \in \mathbb{N} \text{ and } \omega \in \Omega.$$

Ubiquity of nonuniform behaviour

Theorem (Froyland-Lloyd-Quas, 2010)

Assume that

$$\log^+\|A(\cdot)\| \in L^1(X, \mathcal{B}, \mu).$$

Then, there exist numbers $\infty > \lambda_1 > \ldots > \lambda_k \geq -\infty$ *and for* μ*-a.e.* $\omega \in \Omega$ *an decomposition*

$$\mathbb{R}^d = E_1(\omega) \oplus \ldots \oplus E_k(\omega)$$

such that $A(\omega)E_i(\omega) \subset E_i(f(\omega))$ *and*

$$\lim_{n\to\infty} \frac{1}{n} \log\|A^{(n)}(\omega)v\| = \lambda_i, \text{ for } v \in E_i(\omega) \setminus \{0\} \text{ and } i = 1, \ldots, k.$$

Ubiquity of nonuniform behaviour

Theorem (D.-Froyland, 2016)

Assume that $\lambda_i \neq 0$ for each $i \in \{1, \dots, k\}$. Then, for μ-a.e. $\omega \in \Omega$ the sequence $(A_n)_{n\in\mathbb{Z}}$ given by $A_n = A(f^n(\omega))$, $n \in \mathbb{Z}$ admits a nonuniform dichotomy.

Similar statements can be obtained in an infinite-dimensional setting where the versions of MET have been obtained by: *Ruelle, Mañé, Thieullen, Lian and Lu, Froyland, Lloyd and Quas, González-Tokman and Quas, Blumenthal.*

Nonuniform spectrum

Let us consider equation (1) and define Σ to be the set of all $\lambda \in \mathbb{R}$ with the property that the equation

$$x_{m+1} = e^{-\lambda} A_m x_m, \quad m \in \mathbb{Z}$$

doesn't admit a nonuniform exponential dichotomy. We say that Σ is a *nonuniform spectrum* associated to (1). Our main goals:

1. describe all possible structures of Σ;
2. discuss relationship between Σ and Lyapunov exponents associated to (1).

These questions where first studied by *Sacker and Sell* for the case of uniform spectrum.

Structure of the spectrum

Let $D \subset X$ be the closed unit ball centered at 0. Given a linear operator $A\colon X \to X$, we denote by $|A|_\alpha$ the infimum of all $r > 0$ with the property that $A(D)$ has a finite cover by balls of radius at most r. It is easy to verify that $|A|_\alpha \le \|A\|$ for A bounded. Moreover, if A is compact, then $|A|_\alpha = 0$. We will assume that

$$l_\alpha := \limsup_{n\to+\infty} \frac{1}{n} \log|\mathcal{A}(n,0)|_\alpha < \infty.$$

For each $a \in \mathbb{R}$ and $n \in \mathbb{Z}$, let

$$S_a(n) = \left\{ v \in X : \sup_{m\ge n} \left(e^{-a(m-n)} \|\mathcal{A}(m,n)v\| \right) < +\infty \right\}$$

and

let $U_a(n)$ be the set of all vectors $v \in X$ for which there exists a sequence $(x_m)_{m \le n} \subset X$ such that $x_n = v$, $x_m = A_{m-1}x_{m-1}$ for $m \le n$ and

$$\sup_{m \le n} \left(e^{-a(m-n)} \|x_m\| \right) < +\infty.$$

We note that if $a < b$, then

$$S_a(n) \subset S_b(n) \quad \text{and} \quad U_b(n) \subset U_a(n)$$

for $n \in \mathbb{Z}$. In addition, if $a \in \mathbb{R} \setminus \Sigma$, then

$$X = S_a(n) \oplus U_a(n) \quad \text{for } n \in \mathbb{Z}$$

and the projections P_n and Q_n associated to the equation $x_{m+1} = e^{-a} A_m x_m$ satisfy

$$\operatorname{Im} P_n = S_a(n) \quad \text{and} \quad \operatorname{Im} Q_n = U_a(n).$$

One can show that:

1. for $a \in (l_\alpha, \infty) \setminus \Sigma$ we have that $\dim U_a(n) < \infty$;
2. $\Sigma \cap (l_\alpha, \infty)$ is closed in (l_α, ∞). Moreover, for each $a \in (l_\alpha, \infty) \setminus \Sigma$ we have $S_a(n) = S_b(n)$ and $U_a(n) = U_b(n)$ for all $n \in \mathbb{Z}$ and all b in some open neighborhood of a;
3. for $a_1, a_2 \in (l_\alpha, \infty) \setminus \Sigma$ with $a_1 < a_2$ we have that $[a_1, a_2] \cap \Sigma \neq \emptyset$ if and only if $\dim U_{a_1} > \dim U_{a_2}$;
4. for $c \in (l_\alpha, \infty) \setminus \Sigma$, the set $\Sigma \cap [c, +\infty)$ is the union of finitely many closed intervals.

Theorem

Set $\Sigma_\alpha := \Sigma \cap (l_\alpha, \infty)$. *One of the following alternatives holds:*

- $\Sigma_\alpha = \emptyset$;
- $\Sigma_\alpha = (l_\alpha, \infty)$;
- $\Sigma_\alpha = I_1 \cup \bigcup_{n=2}^{k} [a_n, b_n]$, *where* $I_1 = [a_1, b_1]$ *or* $I_1 = [a_1, +\infty)$, *for some numbers*

$$b_1 \geq a_1 > b_2 \geq a_2 > \cdots > b_k \geq a_k > l_\alpha \tag{3}$$

  for some integer $k \in \mathbb{N}$;
- $\Sigma_\alpha = I_1 \cup \bigcup_{n=2}^{k-1} [a_n, b_n] \cup (l_\alpha, b_k]$, *where* $I_1 = [a_1, b_1]$ *or* $I_1 = [a_1, \infty)$, *for some numbers* a_n *and* b_n *as in* (3) *for some integer* $k \in \mathbb{N}$ *(when* $k = 1$ *we have* $\Sigma = (l_\alpha, b_1]$*)*;

Theorem (continued)

- $\Sigma_\alpha = I_1 \cup \bigcup_{n=2}^{\infty}[a_n, b_n]$, *where* $I_1 = [a_1, b_1]$ *or* $I_1 = [a_1, +\infty)$, *for some numbers*

$$b_1 \geq a_1 > b_2 \geq a_2 > \cdots > l_\alpha \tag{4}$$

  with $\lim_{n\to+\infty} a_n = l_\alpha$;
- $\Sigma_\alpha = I_1 \cup \bigcup_{n=2}^{\infty}[a_n, b_n] \cup (l_\alpha, b_\infty]$, *where* $I_1 = [a_1, b_1]$ *or* $I_1 = [a_1, +\infty)$, *for some numbers* a_n *and* b_n *as in* (4) *with* $b_\infty := \lim_{n\to+\infty} a_n > l_\alpha$.

Example

Take $X = l^2$ and numbers a_n and b_n as in (4) such that $\lim_{n\to+\infty} a_n = -\infty$. Consider (1) with A_n given by

$$A_n x = (A_n^1 x_1, A_n^2 x_2, A_n^3 x_3, \ldots),$$

where

$$A_n^j = \begin{cases} e^{b_j + \sqrt{n+1}\cos(n+1) - \sqrt{n}\cos n}, & n \geq 0, \\ e^{a_j + \sqrt{|n+1|}\cos(n+1) - \sqrt{|n|}\cos n}, & n < 0 \end{cases}$$

for $j \in \mathbb{N}$. One can show that A_n is a compact operator for each $n \in \mathbb{Z}$ and that

$$\Sigma = \bigcup_{n=1}^{\infty} [a_n, b_n].$$

Nonuniform spectrum and asymptotic behaviour

Consider the case when $\Sigma_\alpha = \bigcup_{n=1}^{k} [a_n, b_n]$, for some numbers a_n and b_n as in (3). Take any $c_i \in (b_{i+1}, a_i)$ for each $i = 1, \ldots, k-1$, $c_0 > b_1$ and $c_k \in (l_\alpha, a_k)$. Set

$$E_0(n) = U_{c_0}(n), \quad E_{k+1}(n) = S_{c_k}(n) \quad \text{and} \quad E_i(n) = S_{c_{i-1}}(n) \cap U_{c_i}(n),$$

for $i \in \{1, \ldots, k\}$. Then, we have that

- $X = \oplus_{i=1}^{k+1} E_i(n)$ for $n \in \mathbb{Z}$;
- $\lim_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}(m,n)v\| = \infty$ for $v \in E_0(n) \setminus \{0\}$;
- $\liminf_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}(m,n)v\|$, $\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}(m,n)v\|$ belong to $[a_i, b_i]$ for $v \in E_i(n) \setminus \{0\}$ and $i \in \{1, \ldots, k\}$;
- $\limsup_{m\to\infty} \frac{1}{m} \log\|\mathcal{A}(m,n)v\| \le l_\alpha$ for $v \in E_{k+1}(n) \setminus \{0\}$.

Consider now a nonlinear dynamics

$$x_{k+1} = A_k x_k + f_k(x_k) \tag{5}$$

For a class of nonlinear perturbations $(f_k)_k$ we have that for all solutions $(x_k)_k$ of (5) such that

$$a_k \le \liminf_{n\to\infty} \frac{1}{n} \log\|x_n\| \le \limsup_{n\to\infty} \frac{1}{n} \log\|x_n\| \le b_1,$$

there exists $i \in \{1, \ldots, k\}$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \log\|x_n\|, \ \limsup_{n\to\infty} \frac{1}{n} \log\|x_n\| \in [a_i, b_i].$$

Most interesting in the case of so-called Lyapunov regular sequences $(A_n)_n$.

Further developments

There are analogous results in the case of *continuous time*, i.e. for evolution families $T(t, s)$. J. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs 25, American Mathematical Society, Providence, RI, 1988.

D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics 840, Springer-Verlag, Berlin-New York, 1981.