ON SPECTRAL CHARACTERIZATION OF NONUNIFORM HYPERBOLICITY

MUNA ABU ALHALAWA AND DAVOR DRAGIČEVIĆ

Abstract. We give a complete functional theoretic characterization of tempered exponential dichotomies in terms of the invertibility of certain linear operators acting on a suitable Frechét space. In sharp contrast to previous results, we consider noninvertible linear cocycles acting on infinite-dimensional spaces. The principal advantage of our results is that they avoid the use of Lyapunov norms.

1. Introduction

The problem of characterizing hyperbolic behaviour of dynamical systems in terms of the spectral properties of certain linear operators has a long history that goes back to the pioneering works of Perron [29] and Li [22]. More precisely, Perron [29] established a complete characterization of the exponential stability of a linear differential equation

$$x' = A(t)x$$

in $\mathbb{R}^n$ in terms of the solvability (in x) of the nonlinear equation

$$x' = A(t)x + f(t), \tag{1}$$

where f and x belong to suitable function spaces. Similar results for the discrete time dynamics were obtained by Li. The condition that (1) has a (unique) solution x in some space Y_1 for any choice of f that belongs to some (possibly different) space Y_2 is commonly referred to as *admissibility* condition. Clearly, this requirement can be formulated in terms of the linear operator

$$(Lx)(t) = x' - A(t)x$$

acting between suitable function spaces.

The major contribution to this line of the research is due to Massera and Schäffer [24]. Indeed, in a constrast to the work of Perron, they have established complete characterization (in terms of admissibility)

2010 *Mathematics Subject Classification.* Primary: 37D99.

Key words and phrases. Tempered exponential dichotomies, Lyapunov exponents, Frechét spaces.

D.D. was supported in part by an Australian Research Council Discovery Project DP150100017 and by Croatian Science Foundation under the project IP-2014-09-2285.

of the notion of (uniform) exponential dichotomy which includes the notion of exponential stability as a particular case. More precisely, rather then considering only the dynamics that exhibits stable behaviour, they have considered the case of dynamics with the property that the phase space splits into two complementary directions, where in one direction dynamics exhibits stable behaviour while in the complementary direction it possesses an unstable (chaotic) behaviour. In addition, they have developed an axiomatic approach to the problem of constructing all possible pairs (Y_1, Y_2) of function spaces with the property that the corresponding admissibility condition is equivalent to the existence of exponential dichotomy.

To the best of our knowledge, the first results in this direction that deal with the infinite-dimensional dynamics are due to Dalec′kiĭ and Kreĭn [14]. For more recent results devoted to continuous and discrete evolution families, we refer to [1, 18, 19, 21, 27, 32, 35, 36, 37, 39, 41] for those dealing with uniform exponential behaviour and to [3, 6, 26, 34, 43] for those that consider various concepts of nonuniform exponential behaviour.

In the context of smooth dynamical systems, first results are due to Mather [25] who proved that a smooth diffeomorphisms f of a compact Riemannian manifold M is Anosov if and only if the operator Γ defined by

$$(\Gamma v)(x) = Df(f^{-1}(x))v(f^{-1}(x)),$$

on the space of all continuous vector fields v on M is hyperbolic (see [12] for related results in the case of flows). Subsequent related results consider the general case of linear cocycles (or the so-called linear skew product flows) acting on Banach spaces. We refer to [11, 13, 20, 31, 33, 38, 40] and references therein. We stress that all of those works consider only uniform hyperbolic behaviour.

In the paper [42] devoted to the roughness property of nonuniform hyperbolicity (the so-called tempered dichotomy) for linear cocycles on Banach spaces, the authors posed a questions on whether it is possible to give functional theoretic characterization of nonuniform behaviour. It turns out that the answer to this question is positive and such characterization was developed in [4] (see also [5]) and applied to the above mentioned roughness property of tempered dichotomies. However, the approach developed in [4] is far from satisfactory since the construction of appropriate spaces on which the Mather type of operator acts is given in terms of the so-called Lyapunov norms which transform nonuniform behaviour into the uniform. Thus, in order to use this charaterization to detect nonuniform behaviour, one would first need to construct appropriate Lyapunov norms. Hence, the results in [4] are unfortunately only of limited applicability.

In the present paper, we propose an alternative functional theoretic characterization of tempered exponential dichotomies. More precisely, we show that the existence of tempered exponential dichotomy (under the assumptions of the multiplicative ergodic theorem) is equivalent to the invertibility of Mather-type operators acting on a certain Frechét space. Although dealing with Frechét instead of Banach spaces is in principle harder, we feel that nevertheless our results have an advantage over those in [4]. The reason for this is that our Frechét space is build in terms of the original norm and not in terms of Lyapunov norms. We stress that our approach is close in spirit to that developed in [16] for Lyapunov regular trajectories in the finite-dimensional setting, associated to a smooth diffeomorphism.

In order to formulate an explicit result, let $(\Omega, \mathcal{F}, \mathbb{P}, \sigma)$ be an invertible and ergodic measure preserving dynamical system. Moreover, let $\mathcal{A}$ be a linear cocycle over this system which takes values in a family of compact and injective operators on some Banach space. We will construct a Frechét space Y as well as family of continuous operators $\mathbb{M}_\omega \colon Y \to Y$, $\omega \in \Omega$ such that the following result (which is a combination of Theorems 9 and 10) is valid.

Theorem 1. *The cocycle $\mathcal{A}$ admits a tempered dichotomy if and only if* $\mathrm{Id} - \mathbb{M}_\omega$ *is an invertible operator on Y for $\mathbb{P}$-a.e. $\omega \in \Omega$, where* Id *denotes the identity operator on Y.*

We hope that our results will be applicable to the study of nonuniformly hyperbolic dynamical systems. We emphasize that since the landmark works of Oseledets [28] and in particulary Pesin [30] this theory has become one of the central themes of the modern dynamical systems theory (see [8] for a detailed exposition). We refer to [7, 10, 23] and references therein for the discussions regarding the various extensions of this theory to the infinite-dimensional setting.

2. Preliminaries

In this section we recall some notions and collect previous auxiliary results that will be used in the following section.

2.1. **Linear cocycles and tempered exponential dichotomy.** Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assume that $\sigma \colon \Omega \to \Omega$ is an invertible, $\mathbb{P}$-preserving transformation which is ergodic. Furthermore, let X be a separable Banach space and denote by $B(X)$ the space of all bounded linear operators on X. Finally, let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

We say that a map $\mathcal{A} \colon \Omega \times \mathbb{N}_0 \to B(X)$ is a *linear cocycle* over σ if:

1. $\mathcal{A}(\omega, 0) = \mathrm{Id}$ for $\omega \in \Omega$;
2. $\mathcal{A}(\omega, n+m) = \mathcal{A}(\sigma^m(\omega), n)\mathcal{A}(\omega, m)$ for $\omega \in \Omega$ and $n, m \in \mathbb{N}_0$;
3. $$\int_\Omega \log^+ \|A(\omega)\| \, d\mathbb{P}(\omega) < \infty, \tag{2}$$
   where
   $$A(\omega) = \mathcal{A}(\omega, 1), \quad \text{for } \omega \in \Omega; \tag{3}$$

4. $\omega \mapsto A(\omega)x$ is a measurable map from Ω to X for each $x \in X$.

We recall that the map A given by (3) is called the *generator* of a cocycle $\mathcal{A}$.

We also introduce the notion of a tempered random variable. We say that a measurable map $K\colon \Omega \to (0,\infty)$ is a *tempered random variable* if

$$\lim_{n\to \pm\infty} \frac{1}{n} \log K(\sigma^n(\omega))=0, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

The following well-known result (see [2] for example) will be useful in our arguments.

Proposition 2. *Assume that $K\colon \Omega \to (0,\infty)$ is a tempered random variable. Then, for $\varepsilon>0$ there exists a measurable map $C\colon \Omega \to (0,\infty)$ such that:*

1. *for $\mathbb{P}$-a.e. $\omega \in \Omega$,*
$$K(\omega) \le C(\omega); \tag{4}$$
2. *for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every $n\in \mathbb{Z}$, we have that*
$$C(\sigma^n(\omega)) \le C(\omega)e^{\varepsilon |n|}. \tag{5}$$

Finally, we recall the notion of a tempered exponential dichotomy [4, 42]. We say that the cocycle $\mathcal{A}$ with generator A as in (3) admits a *tempered exponential dichotomy* if there exist $\lambda>0$, a tempered random variable $K\colon \Omega \to (0,\infty)$ and a family of projections $P(\omega) \in B(X)$, $\omega \in \Omega$ such that:

1. $\omega \mapsto P(\omega)x$ is a measurable map for each $x\in X$;
2. for $\mathbb{P}$-a.e. $\omega \in \Omega$,
$$A(\omega)P(\omega)=P(\sigma(\omega))A(\omega)$$
and the map
$$A(\omega)|\operatorname{Ker} P(\omega)\colon \operatorname{Ker} P(\omega) \to \operatorname{Ker} P(\sigma(\omega))$$
is invertible;
3. for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every $n\in \mathbb{N}$,
$$\|\mathcal{A}(\omega, n)P(\omega)\| \le K(\omega)e^{-\lambda n} \tag{6}$$
and
$$\|\mathcal{A}(\omega, -n)(\mathrm{Id}-P(\omega))\| \le K(\omega)e^{-\lambda n}, \tag{7}$$
where
$$\mathcal{A}(\omega, -n):=(\mathcal{A}(\sigma^{-n}(\omega), n)|\operatorname{Ker} P(\sigma^{-n}(\omega)))^{-1}$$
is a well-defined linear map from $\operatorname{Ker} P(\omega)$ to $\operatorname{Ker} P(\sigma^{-n}(\omega))$.

2.2. **Multiplicative ergodic theorem.** We now recall the version of the multiplicative ergodic theorem established by Lian and Lu [23]. For the sake of simplicity, we will restrict our attention to the case of compact cocycles.

Theorem 3. *Assume that $\mathcal{A}$ is a linear cocyle such that $A(\omega)$ is a compact and injective operator for $\mathbb{P}$-a.e. $\omega \in \Omega$, where $A(\omega)$ is given by* (3)*. Then either:*

1. *There is a finite sequence of numbers*
$$\lambda_1 > \lambda_2 > \cdots > \lambda_k > \lambda_\infty = -\infty$$
and a decomposition
$$X = E_1(\omega) \oplus \cdots \oplus E_k(\omega) \oplus E_\infty(\omega)$$
such that for $\mathbb{P}$-a.e. $\omega \in \Omega$,
$$A(\omega)E_i(\omega) = E_i(\sigma(\omega)), i = 1, \dots, k \quad \textit{and} \quad A(\omega)E_\infty(\omega) \subset E_\infty(\sigma(\omega)),$$
$$\lim_{|n| \to \infty} \frac{1}{n} \log\|\mathcal{A}(\omega, n)x\| = \lambda_i \quad \textit{for } x \in E_i(\omega) \setminus \{0\} \textit{ and } i \in \{1, \dots k\}$$
and
$$\lim_{n \to \infty} \frac{1}{n} \log\|\mathcal{A}(\omega, n)x\| = \lambda_\infty \quad \textit{for } x \in E_\infty(\omega).$$
Moreover, each $E_i(\omega), i = 1, \dots, k$ is a finite-dimensional subspace of X.
2. *There exists an infinite sequence of numbers*
$$\lambda_1 > \lambda_2 > \cdots > \lambda_k > \ldots > \lambda_\infty = -\infty$$
and for each $k \in \mathbb{N}$ a decomposition
$$X = E_1(\omega) \oplus \cdots \oplus E_k(\omega) \oplus F_k(\omega)$$
such that for $\mathbb{P}$-a.e. $\omega \in \Omega$,
$$A(\omega)E_i(\omega) = E_i(\sigma(\omega)), \; i = 1, \dots, k \quad \textit{and} \quad A(\omega)F_k(\omega) \subset F_k(\sigma(\omega)),$$
$$\lim_{|n| \to \infty} \frac{1}{n} \log\|\mathcal{A}(\omega, n)x\| = \lambda_i, \quad \textit{for } x \in E_i(\omega) \setminus \{0\} \textit{ and } i = 1, \dots, k$$
and
$$\limsup_{n \to \infty} \frac{1}{n} \log\|\mathcal{A}(\omega, n)x\| \le \lambda_{i+1}, \quad \textit{for } x \in F_k(\omega).$$
Moreover, each $E_i(\omega), i \neq \infty$ is a finite-dimensional subspace of X.

We note that the numbers λ_i are called *Lyapunov exponents* of the cocycle $\mathcal{A}$. In addition, subspaces $E_i(\omega)$ are called *Oseledets subspaces.* Now we are in position to state sufficient conditions for the existence of tempered exponential dichotomy. The following result was established by Lian and Lu [23].

Theorem 4. *Assume that $\mathcal{A}$ is a linear cocycle satisfying assumptions of Theorem 3. If all Lyapunov exponents of $\mathcal{A}$ are nonzero, then $\mathcal{A}$ admits a tempered exponential dichotomy.*

2.3. **Frechét space.** We now introduce our Frechét space that will play a central role in our arguments. Set

$$Y=\left\{\mathbf{x}=(x_n)_{n\in\mathbb{Z}}\subset X:\limsup_{|n|\to\infty}\frac{1}{|n|}\log\|x_n\|\le 0\right\}.$$

It is easy to verify that Y is a vector space. Furthermore, for each $k\in\mathbb{N}$ set

$$Y_k=\left\{\mathbf{x}=(x_n)_{n\in\mathbb{Z}}\subset X:\|\mathbf{x}\|_k<\infty\right\},$$

where

$$\|\mathbf{x}\|_k=\sup_{n\in\mathbb{Z}}(\|x_n\|e^{-|n|/k}).$$

It is straightforward to show that:

- $(Y_k,\|\cdot\|_k)$ is a Banach space for each $k\in\mathbb{N}$;
- $$Y=\bigcap_{k\in\mathbb{N}}Y_k; \tag{8}$$
- for each $k\in\mathbb{N}$
$$Y_{k+1}\subset Y_k; \tag{9}$$
- $\|\mathbf{x}\|_k\le\|\mathbf{x}\|_{k+1}$ for every $\mathbf{x}\in Y$ and $k\in\mathbb{N}$.

It follows from the above properties that we can equip Y with the structure of the *graded Frechét space* by saying that the sequence $(\mathbf{x}^l)_{l\in\mathbb{N}}\subset Y$ converges to $\mathbf{x}\in Y$ if and only if $(\mathbf{x}^l)_{l\in\mathbb{N}}$ converges to $\mathbf{x}$ in Y_k for each $k\in\mathbb{N}$.

2.4. **Mather-type operator.** The other crucial ingredient in our characterization is the construction of appropriate linear operators. Let $\mathcal{A}$ be a linear cocycle and consider its generator A given by (3). For $\omega\in\Omega$, we define

$$(\mathbb{M}_\omega\mathbf{x})_n=A(\sigma^{n-1}(\omega))x_{n-1},\quad \text{for } n\in\mathbb{Z} \text{ and } \mathbf{x}=(x_n)_{n\in\mathbb{Z}}\subset X.$$

Obviously, $\mathbb{M}_\omega$ is a linear map. Let us now establish several auxiliary results.

Lemma 1. *For $\mathbb{P}$-a.e. $\omega\in\Omega$,*

$$\mathbb{M}_\omega\colon Y_{k+1}\to Y_k$$

is a well-defined and bounded linear operator for each $k\in\mathbb{N}$.

Proof. It follows from (2) that there exist $a>0$ and a tempered random variable $K\colon\Omega\to(0,\infty)$ such that

$$\|\mathcal{A}(\omega,n)\|\le K(\omega)e^{an}, \tag{10}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every $n \in \mathbb{N}$. Fix now $k \in \mathbb{N}$ and choose $\varepsilon > 0$ such that $\varepsilon < \frac{1}{k} - \frac{1}{k+1}$. Furthermore, let C be given by Proposition 2 and let $\Omega'_k \subset \Omega$ be such that $\mathbb{P}(\Omega'_k) = 1$ and that (4), (5) and (10) hold for each $\omega \in \Omega'_k$. It follows that

$$\begin{aligned}
e^{-|n|/k}\|(\mathbb{M}_\omega \mathbf{x})_n\| &= e^{-|n|/k}\|A(\sigma^{n-1}(\omega))x_{n-1}\| \\
&\le e^{-|n|/k}K(\sigma^{n-1}(\omega))e^a\|x_{n-1}\| \\
&\le e^{-|n|/k}C(\sigma^{n-1}(\omega))e^a\|x_{n-1}\| \\
&\le C(\omega)e^{a-|n|/k+\varepsilon|n-1|}\|x_{n-1}\| \\
&\le C(\omega)e^{a+1/k}e^{(\varepsilon-1/k)|n-1|}\|x_{n-1}\| \\
&\le C(\omega)e^{a+1/k}e^{-|n-1|/(k+1)}\|x_{n-1}\| \\
&\le C(\omega)e^{a+1/k}\|\mathbf{x}\|_{k+1},
\end{aligned}$$

for each $n \in \mathbb{Z}$, $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_{k+1}$ and $\omega \in \Omega'_k$. We conclude that

$$\|\mathbb{M}_\omega \mathbf{x}\|_k \le C(\omega)e^{a+1/k}\|\mathbf{x}\|_{k+1}, \tag{11}$$

for every $\mathbf{x} \in Y_{k+1}$ and $\omega \in \Omega'_k$. Set $\Omega' = \cap_{k=1}^\infty \Omega'_k$. Then, $\mathbb{P}(\Omega') = 1$ and it follows readily from (11) that $\mathbb{M}_\omega \colon Y_{k+1} \to Y_k$ is a well-defined and bounded operator for each $\omega \in \Omega'$ and $k \in \mathbb{N}$. □

As a direct consequence of Lemma 1, we obtain the following result.

Proposition 5. *The operator $\mathbb{M}_\omega \colon Y \to Y$ is a well-defined and continuous operator for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

Proof. By Lemma 1, there exists a full-measure set $\Omega' \subset \Omega$ such that $\mathbb{M}_\omega \colon Y_{k+1} \to Y_k$ is a well-defined and bounded linear operator for each $k \in \mathbb{N}$ and $\omega \in \Omega'$.

We begin by noting that $\mathbb{M}_\omega Y \subset Y$ for each $\omega \in \Omega'$. Indeed, for any $\mathbf{x} \in Y$ we have that $\mathbf{x} \in Y_{k+1}$ for each $k \in \mathbb{N}$ (see (8)). Hence, Lemma 1 implies that $\mathbb{M}_\omega \mathbf{x} \in Y_k$ for every $k \in \mathbb{N}$. We conclude that $\mathbb{M}_\omega \mathbf{x} \in Y$.

Let us now establish the continuity of $\mathbb{M}_\omega$. Take a sequence $(\mathbf{x}^l)_l$ that converges to $\mathbf{x}$ in Y. This implies that $(\mathbf{x}^l)_l$ converges to $\mathbf{x}$ in Y_{k+1} for each $k \in \mathbb{N}$. By (11), we have that $(\mathbb{M}_\omega \mathbf{x}^l)_l$ converges to $\mathbb{M}_\omega \mathbf{x}$ in Y_k for every $k \in \mathbb{N}$ and therefore also in Y. The proof is completed. □

3. Main results

In this section we obtain the main results of this paper, i.e. we establish the complete characterization of tempered exponential dichotomies in terms of the invertibility of operators $\mathrm{Id} - \mathbb{M}_\omega$ on the space Y. Before we establish several auxiliary lemmas, we will introduce some additional notation. Assume that the cocycle $\mathcal{A}$ admits a tempered dichotomy and let $P(\omega)$, $\omega \in \Omega$ be the associated family of projections. For $\omega \in \Omega$, we define linear operators Γ_ω^i, $i=1,2$ by

$$(\Gamma_\omega^1 \mathbf{x})_n = \sum_{m=0}^\infty \mathcal{A}(\sigma^{n-m}(\omega), m) P(\sigma^{n-m}(\omega)) x_{n-m} \tag{12}$$

and

$$(\Gamma_\omega^2 \mathbf{x})_n = \sum_{m=1}^\infty \mathcal{A}(\sigma^{n+m}(\omega), -m)(\mathrm{Id} - P(\sigma^{n+m}(\omega))) x_{n+m}, \tag{13}$$

where $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$.

Lemma 2. *Assume that the cocycle $\mathcal{A}$ admits a tempered dichotomy. Then, $\Gamma_\omega^1 \colon Y_{k+1} \to Y_k$ given by (12) is a well-defined and bounded linear operator for $\mathbb{P}$-a.e. $\omega \in \Omega$ and sufficiently large $k \in \mathbb{N}$.*

Proof. Take an arbitrary $k \in \mathbb{N}$ satisfying $1/k < \lambda$ and choose $\varepsilon \in (0, \frac{1}{k} - \frac{1}{k+1})$. Furthermore, let C be given by Proposition 2 (with respect to K as in the notion of tempered dichotomy). It follows from (4), (5) and (6) that

$$\begin{aligned} e^{-|n|/k} \|(\Gamma_\omega^1 \mathbf{x})_n\| &\le e^{-|n|/k} \sum_{m=0}^\infty K(\sigma^{n-m}(\omega)) e^{-\lambda m} \|x_{n-m}\| \\ &\le e^{-|n|/k} \sum_{m=0}^\infty C(\sigma^{n-m}(\omega)) e^{-\lambda m} \|x_{n-m}\| \\ &\le C(\omega) e^{-|n|/k} \sum_{m=0}^\infty e^{\varepsilon|n-m|} e^{-\lambda m} \|x_{n-m}\| \\ &\le C(\omega) \sum_{m=0}^\infty e^{(\varepsilon - 1/k)|n-m|} e^{(1/k - \lambda)m} \|x_{n-m}\| \\ &\le C(\omega) \sum_{m=0}^\infty e^{-|n-m|/(k+1)} e^{(1/k-\lambda)m} \|x_{n-m}\| \\ &\le \frac{C(\omega)}{1 - e^{1/k - \lambda}} \|\mathbf{x}\|_{k+1}, \end{aligned}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every $n \in \mathbb{Z}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_{k+1}$. Hence,

$$\|\Gamma_\omega^1 \mathbf{x}\|_k \le \frac{C(\omega)}{1 - e^{1/k-\lambda}} \|\mathbf{x}\|_{k+1}, \tag{14}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_{k+1}$. The proof of the lemma is completed. □

The following result follows from Lemma 2 in a same way as Proposition 5 follows from Lemma 1.

Proposition 6. *Assume that the cocycle $\mathcal{A}$ admits a tempered dichotomy. Then, the operator $\Gamma_\omega^1 \colon Y \to Y$ given by* (12) *is a well-defined and continuous operator for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

Proof. By Lemma 2, there exists a full-measure set $\Omega' \subset \Omega$ such that $\Gamma_\omega^1 \colon Y_{k+1} \to Y_k$ is a well-defined and bounded linear operator for each $k \in \mathbb{N}$ sufficiently large and $\omega \in \Omega'$.

We begin by noting that $\Gamma_\omega^1 Y \subset Y$ for each $\omega \in \Omega'$. Indeed, for any $\mathbf{x} \in Y$ we have that $\mathbf{x} \in Y_{k+1}$ for each $k \in \mathbb{N}$ (see (8)). Hence, Lemma 2 implies that $\Gamma_\omega^1 \mathbf{x} \in Y_k$ for every $k \in \mathbb{N}$ sufficiently large. Hence, (8) and (9) imply that $\Gamma_\omega^1 \mathbf{x} \in Y$.

We now prove that Γ_ω^1 is continuous. Take a sequence $(\mathbf{x}^l)_l$ that converges to $\mathbf{x}$ in Y. This implies that $(\mathbf{x}^l)_l$ converges to $\mathbf{x}$ in Y_{k+1} for each $k \in \mathbb{N}$. By (14), we have that $(\Gamma_\omega^1 \mathbf{x}^l)_l$ converges to $\Gamma_\omega^1 \mathbf{x}$ in Y_k for every $k \in \mathbb{N}$ sufficiently large. This implies that $(\Gamma_\omega^1 \mathbf{x}^l)_l$ converges to $\Gamma_\omega^1 \mathbf{x}$ in Y. ☐

Lemma 3. *Assume that the cocycle $\mathcal{A}$ admits a tempered dichotomy. Then, the operator $\Gamma_\omega^2 \colon Y_{k+1} \to Y_k$ given by* (13) *is a well-defined and bounded for $\mathbb{P}$-a.e. $\omega \in \Omega$ and sufficiently large $k \in \mathbb{N}$.*

Proof. Using the same notation as in the proof of Lemma 2, it follows from (4), (5) and (6) that

$$\begin{aligned}
e^{-|n|/k} \|(\Gamma_\omega^2 \mathbf{x})_n\| &\le e^{-|n|/k} \sum_{m=1}^{\infty} K(\sigma^{n+m}(\omega)) e^{-\lambda m} \|x_{n+m}\| \\
&\le e^{-|n|/k} \sum_{m=1}^{\infty} C(\sigma^{n+m}(\omega)) e^{-\lambda m} \|x_{n+m}\| \\
&\le C(\omega) e^{-|n|/k} \sum_{m=1}^{\infty} e^{\varepsilon |n+m|} e^{-\lambda m} \|x_{n+m}\| \\
&\le C(\omega) \sum_{m=1}^{\infty} e^{-|n+m|/(k+1)} e^{(1/k-\lambda)m} \|x_{n+m}\| \\
&\le C(\omega) \frac{e^{1/k-\lambda}}{1-e^{1/k-\lambda}} \|\mathbf{x}\|_{k+1},
\end{aligned}$$

for μ-a.e. $\omega \in \Omega$ and every $n \in \mathbb{Z}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_{k+1}$. Hence,

$$\|\Gamma_\omega^2 \mathbf{x}\|_k \le C(\omega) \frac{e^{1/k-\lambda}}{1-e^{1/k-\lambda}} \|\mathbf{x}\|_{k+1}, \tag{15}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and every $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_{k+1}$. This immediately yields the conclusion of the lemma. ☐

Lemma 3 implies the following result whose proof is analogous to the proof of Proposition 6.

Proposition 7. *Assume that the cocycle $\mathcal{A}$ admits a tempered dichotomy.Then, the operator $\Gamma_\omega^2 \colon Y \to Y$ given by* (13) *is a well-defined and continuous operator for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

Proposition 8. *Assume that the cocycle $\mathcal{A}$ admits a tempered dichotomy. Then, we have the following:*

1. *the operator $\Gamma_\omega := \Gamma_\omega^1 - \Gamma_\omega^2 \colon Y_{k+1} \to Y_k$ is a well-defined and bounded for $\mathbb{P}$-a.e. $\omega \in \Omega$ and sufficiently large $k \in \mathbb{N}$, where Γ_ω^1 and Γ_ω^2 are given by* (12) *and* (13) *respectively;*
2. *the operator $\Gamma_\omega \colon Y \to Y$ is a well-defined and continuous operator for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

Proof. The first assertion follows from Lemma 2 and Lemma 3, while the second follows from Proposition 6 and Proposition 7. □

The connection between Γ_ω and $\mathbb{M}_\omega$ is given by the following result.

Lemma 4. *Assume that the cocycle $\mathcal{A}$ admits a tempered dichotomy. Then,*

$$(\Gamma_\omega(\mathrm{Id} - \mathbb{M}_\omega))\mathbf{x} = \mathbf{x},$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$, k sufficiently large and every $\mathbf{x} \in Y_{k+1}$. In particular,

$$(\Gamma_\omega(\mathrm{Id} - \mathbb{M}_\omega))\mathbf{x} = \mathbf{x} \quad \textit{for } \mathbf{x} \in Y.$$

Proof. Observe that (recall that A is given by (3))

$$\begin{aligned}
&((\Gamma_\omega^1(\mathrm{Id} - \mathbb{M}_\omega))\mathbf{x})_n \\
&= \sum_{m=0}^{\infty} \mathcal{A}(\sigma^{n-m}(\omega), m) P(\sigma^{n-m}(\omega)) x_{n-m} \\
&\quad - \sum_{m=0}^{\infty} \mathcal{A}(\sigma^{n-m}(\omega), m) P(\sigma^{n-m}(\omega)) A(\sigma^{n-m-1}(\omega)) x_{n-m-1} \\
&= \sum_{m=0}^{\infty} \mathcal{A}(\sigma^{n-m}(\omega), m) P(\sigma^{n-m}(\omega)) x_{n-m} \\
&\quad - \sum_{m=0}^{\infty} \mathcal{A}(\sigma^{n-m-1}(\omega), m+1) P(\sigma^{n-m-1}(\omega)) x_{n-m-1} \\
&= P(\sigma^n(\omega)) x_n.
\end{aligned}$$

Similarly, we have that

$$\begin{aligned}
&((\Gamma_\omega^2(\mathrm{Id} - \mathbb{M}_\omega))\mathbf{x})_n \\
&= \sum_{m=1}^{\infty} \mathcal{A}(\sigma^{n+m}(\omega), -m)(\mathrm{Id} - P(\sigma^{n+m}(\omega))) x_{n+m} \\
&\quad - \sum_{m=1}^{\infty} \mathcal{A}(\sigma^{n+m}(\omega), -m)(\mathrm{Id} - P(\sigma^{n+m}(\omega))) A(\sigma^{n+m-1}(\omega)) x_{n+m-1}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{m=1}^{\infty} \mathcal{A}(\sigma^{n+m}(\omega), -m)(\mathrm{Id} - P(\sigma^{n+m}(\omega)))x_{n+m} \\
&\quad - \sum_{m=1}^{\infty} \mathcal{A}(\sigma^{n+m-1}(\omega), -(m-1))(\mathrm{Id} - P(\sigma^{n+m-1}(\omega)))x_{n+m-1} \\
&= -(\mathrm{Id} - P(\sigma^n(\omega)))x_n.
\end{aligned}$$

Hence,

$$\begin{aligned}
((\Gamma_\omega(\mathrm{Id} - \mathbb{M}_\omega))\mathbf{x})_n &= ((\Gamma_\omega^1(\mathrm{Id} - \mathbb{M}_\omega))\mathbf{x})_n - ((\Gamma_\omega^2(\mathrm{Id} - \mathbb{M}_\omega))\mathbf{x})_n \\
&= P(\sigma^n(\omega))x_n + (\mathrm{Id} - P(\sigma^n(\omega)))x_n \\
&= x_n,
\end{aligned}$$

which yields the first statement of the lemma. The second statement is a direct consequence of the first. □

Theorem 9. *Assume that the cocycle $\mathcal{A}$ admits a tempered dichotomy. Then,* $\mathrm{Id} - \mathbb{M}_\omega$ *is an invertible continuous operator on Y for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

Proof. By Lemma 4, we have that

$$\Gamma_\omega(\mathrm{Id} - \mathbb{M}_\omega) = \mathrm{Id} \quad \text{on } Y,$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. Let us now prove that

$$(\mathrm{Id} - \mathbb{M}_\omega)\Gamma_\omega = \mathrm{Id} \quad \text{on } Y, \tag{16}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$. Observe that (A is again as in (3))

$$\begin{aligned}
&(\Gamma_\omega^1\mathbf{x} - \mathbb{M}_\omega\Gamma_\omega^1\mathbf{x})_n \\
&= \sum_{m=0}^{\infty} \mathcal{A}(\sigma^{n-m}(\omega), m)P(\sigma^{n-m}(\omega))x_{n-m} \\
&\quad - A(\sigma^{n-1}(\omega)) \sum_{m=0}^{\infty} \mathcal{A}(\sigma^{n-m-1}(\omega), m)P(\sigma^{n-m-1}(\omega))x_{n-m-1} \\
&= \sum_{m=0}^{\infty} \mathcal{A}(\sigma^{n-m}(\omega), m)P(\sigma^{n-m}(\omega))x_{n-m} \\
&\quad - \sum_{m=0}^{\infty} \mathcal{A}(\sigma^{n-m-1}(\omega), m+1)P(\sigma^{n-m-1}(\omega))x_{n-m-1} \\
&= P(\sigma^n(\omega))x_n,
\end{aligned}$$

and thus

$$(\Gamma_\omega^1\mathbf{x} - \mathbb{M}_\omega\Gamma_\omega^1\mathbf{x})_n = P(\sigma^n(\omega))x_n, \tag{17}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$, every $n \in \mathbb{Z}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y$. Similarly,

$$\begin{aligned}
&(-\Gamma_\omega^2 \mathbf{x} + \mathbb{M}_\omega \Gamma_\omega^2 \mathbf{x})_n \\
&= -\sum_{m=1}^\infty \mathcal{A}(\sigma^{n+m}(\omega), -m)(\mathrm{Id} - P(\sigma^{n+m}(\omega)))x_{n+m} \\
&\quad + A(\sigma^{n-1}(\omega)) \sum_{m=1}^\infty \mathcal{A}(\sigma^{n+m-1}(\omega), -m)(\mathrm{Id} - P(\sigma^{n+m-1}(\omega)))x_{n+m-1} \\
&= -\sum_{m=1}^\infty \mathcal{A}(\sigma^{n+m}(\omega), -m)(\mathrm{Id} - P(\sigma^{n+m}(\omega)))x_{n+m} \\
&\quad + \sum_{m=1}^\infty \mathcal{A}(\sigma^{n+m-1}(\omega), -(m-1))(\mathrm{Id} - P(\sigma^{n+m-1}(\omega)))x_{n+m-1}, \\
&= (\mathrm{Id} - P(\sigma^n(\omega)))x_n,
\end{aligned}$$

and therefore

$$(-\Gamma_\omega^2 \mathbf{x} + \mathbb{M}_\omega \Gamma_\omega^2 \mathbf{x})_n = (\mathrm{Id} - P(\sigma^n(\omega)))x_n, \tag{18}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$, every $n \in \mathbb{Z}$ and $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y$. Finally, we observe that (17) and (18) readily imply (16). □

We now establish the converse of Theorem 9.

Theorem 10. *Let $\mathcal{A}$ be a linear cocycle with generator A as in* (3) *such that $A(\omega)$ is injective and compact operator for $\mathbb{P}$-a.e. $\omega \in \Omega$. Furthermore, assume that $\mathrm{Id} - \mathbb{M}_\omega$ is an invertible operator on Y for $\mathbb{P}$-a.e. $\omega \in \Omega$. Then, the cocycle $\mathcal{A}$ admits a tempered dichotomy.*

Proof. In a view of Theorem 4, it is sufficient to show that all Lyapunov exponents of $\mathcal{A}$ nonzero. Assume the opposite, i.e. that zero is a Lyapunov exponent of $\mathcal{A}$ and let $E_0(\omega)$ denote the corresponding Oseledets subspace. Then, Theorem 3 implies that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log\|\mathcal{A}(\omega, n)v\| = 0, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and } 0 \neq v \in E_0(\omega). \tag{19}$$

Fix $0 \neq v \in E_0(\omega)$ and consider a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \subset X$ defined by

$$x_n = \mathcal{A}(\omega, n)v, \quad n \in \mathbb{Z}.$$

By (19), $\mathbf{x} \in Y$. It is easy to verify that $(\mathrm{Id} - \mathbb{M}_\omega)\mathbf{x} = \mathbf{x}$. Since $\mathbf{x} \neq 0$, we obtain the contradiction with the invertibility of $\mathrm{Id} - \mathbb{M}_\omega$. □

We note that Theorem 1 follows directly from Theorems 9 and 10.

Remark 1. *We remark that Theorems 3 and 4 are in fact valid under weaker assumption that the cocycle $\mathcal{A}$ is* quasicompact *and that $A(\omega)$ given by* (3) *is injective for $\mathbb{P}$-a.e. $\omega \in \Omega$. Consequently, all the results of this paper are also valid in this setting. We have decided to present our results for the particular case of compact cocycles in order to focus* *on the novelties of the present paper and to avoid introducing background on quasicompact cocycles (which can be found in* [9, 17, 23]*). We note that the notion of a quasicompact cocycle is not hard to introduce but is quite challenging to verify in practice that the cocycle possesses this property.*

Furthermore, in principle, the assumption on the injectivity of operators $A(\omega)$ could be eliminated. Indeed, the most recent versions of the multiplicative ergodic theorem (see [9, 17]*) require that the cocycle $\mathcal{A}$ is quasicompact and there are no requirements on the injectivity of operators $A(\omega)$. Regarding Theorem 4, it is known that in the finite-dimensional setting it holds without injectivity assumptions (see* [15, Theorem 2]*). We believe that those ideas, when combined with the tools developed in* [10]*, could be extended to the general case of quasi-compact cocycles on Banach spaces. However, we refrain from doing so since it would require many technical arguments that would completely overshadow main results of our paper.*

Acknowledgements

We would like to thank the referee for carefully reading our manuscript and for his/her valuable comments.

References

1. B. Aulbach and N. Van Minh, *The concept of spectral dichotomy for linear difference equations II*, J. Differ. Equ. Appl. **2** (1996), 251–262.
2. L. Arnold, *Random dynamical systems*, Springer-Verlag (1998).
3. L. Barreira, D. Dragičević and C. Valls, *Nonuniform hyperbolicity and admissibility*, Adv. Nonlinear Stud. **14** (2014), 791–811.
4. L. Barreira, D. Dragičević and C. Valls, *Tempered exponential dichotomies: admissibility and stability under perturbations*, Dyn. Syst. **31** (2016), 525–545.
5. L. Barreira, D. Dragičević and C. Valls, *Characterization of nonuniform exponential trichotomies for flows*, J. Math. Anal. Appl. **434** (2016), 376–400.
6. L. Barreira, D. Dragičević and C. Valls, *Admissibility on the half line for evolution families*, J. Anal. Math. **132** (2017), 157–176.
7. L. Barreira, D. Dragičević and C. Valls, *Nonuniform spectrum on Banach spaces*, Adv. Math. **371** (2017), 547–591.
8. L. Barreira and Ya. Pesin, *Nonuniform Hyperbolicity*, Encyclopedia of Mathematics and its Application 115, Cambridge University Press, 2007.
9. A. Blumenthal, *A volume-based approach to the multiplicative ergodic theorem on Banach spaces*, Discrete Contin. Dyn. Syst. **36** (2016), 2377–2403.
10. A. Blumenthal and L. S. Young, *Entropy, volume growth and SRB measures for Banach space mappings*, Invent. Math. **207** (2017), 833–893.
11. C. Chicone and Yu. Latushkin, *Evolution Semigroups in Dynamical Systems and Differential Equations*, Mathematical Surveys and Monographs 70, Amer. Math. Soc., 1999.
12. C. Chicone and R. Swanson, *Spectral theory for linearizations of dynamical svstems*, J. Differential Equations **40** (1981), 155–167.

13. S. N. Chow and H. Leiva, *Existence and Roughness of the Exponential Dichotomy for Skew-Product Semiflow in Banach Spaces*, J. Differential Equations **120** (1995), 429–477.
14. Ju. Dalec′kiĭ and M. Kreĭn, *Stability of Solutions of Differential Equations in Banach Space*, Translations of Mathematical Monographs 43, Amer. Math. Soc., 1974.
15. D. Dragičević and G. Froyland, *Hölder continuity of Oseledets splittings for semi-invertible operator cocycles*, Ergodic Theory Dynam. Systems, in press.
16. D. Dragičević and S. Slijepčević, *Characterization of hyperbolicity and generalized shadowing lemma*, Dyn. Syst. **26** (2011), 483–502.
17. C. González-Tokman and A. Quas, *A semi-invertible operator Oseledets theorem*, Ergodic Theory Dynam. Systems **34** (2014), 1230–1272.
18. N. Huy, *Exponential dichotomy of evolution equations and admissibility of function spaces on a half-line*, J. Funct. Anal. **235** (2006), 330–354.
19. Y. Latushkin, T. Randolph and R. Schnaubelt, *Exponential dichotomy and mild solution of nonautonomous equations in Banach spaces*, J. Dynam. Differential Equations **10** (1998), 489–510.
20. Y. Latushkin and R. Schnaubelt, *volution semigroups, translation algebra and exponential dichotomy of cocycles*, J. Differential Equations **159** (1999), 321–369.
21. Y. Latushkin and Y. Tomilov, *Fredholm differential operators with unbounded coefficients*, J. Differential Equations **208** (2005), 388–429.
22. T. Li, *Die Stabilittsfrage bei Differenzengleichungen*, Acta Math. **63** (1934), 99–41.
23. Z. Lian and K. Lu, *Lyapunov exponents and invariant manifolds for random dynamical systems in a Banach space*, Mem. Amer. Math. Soc. **206** (2010), no. 967.
24. J. Massera and J. Schäffer, *Linear Differential Equations and Function Spaces*, Pure and Applied Mathematics 21, Academic Press, 1966.
25. J. Mather, *Characterization of Anosov diffeomorphisms*, Indag. Math. **30** (1968), 479–483.
26. M. Megan, A. L. Sasu and B. Sasu, *On nonuniform exponential dichotomy of evolution operators in Banach spaces*, Integral Equations Operator Theory **44** (2002), 71–78.
27. M. Megan, A. L. Sasu and B. Sasu, *Discrete admissibility and exponential dichotomy for evolution families*, Discrete Contin. Dynam. Syst. **9** (2003), 383–398.
28. V. Oseledets, *A multiplicative ergodic theorem. Liapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc. **19** (1968), 197–221.
29. O. Perron, *Die Stabilitätsfrage bei Differentialgleichungen*, Math. Z. **32** (1930), 703–728.
30. Ya. Pesin, *Families of invariant manifolds corresponding to nonzero characteristic exponents*, Math. USSR-Izv. **10** (1976), 1261–1305.
31. C. Preda, *$(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X)$-admissibility and exponential dichotomy for cocycles*, J. Differential Equations **249** (2010), 578–598.
32. C. Preda and O. R. Onofei, *Discrete Schäffer spaces and exponential dichotomy for evolution families*, Monatsh. Math., in press.
33. C. Preda, P. Preda and A. Craciunescu, *Criterions for detecting the existence of the exponential dichotomies in the asymptotic behavior of the solutions of variational equations*, J. Funct. Anal. **258** (2010), 729–757.
34. P. Preda and M. Megan, *Nonuniform dichotomy of evolutionary processes in Banach spaces*, Bull. Austral. Math. Soc. **27** (1983), 31–52.

35. P. Preda, A. Pogan and C. Preda, (L^p, L^q)*-admissibility and exponential dichotomy of evolutionary processes on the half-line*, Integral Equations Operator Theory **49** (2004), 405–418.
36. P. Preda, A. Pogan and C. Preda, *Schäffer spaces and exponential dichotomy for evolutionary processes*, J. Differential Equations **230** (2006), 378–391.
37. A. L. Sasu and B. Sasu, *Exponential dichotomy on the real line and admissibility of function spaces*, Integral Equations Operator Theory **54** (2006), 113–130.
38. A. L. Sasu and B. Sasu, *Input-output conditions for the asymptotic behavior of linear skew-product flows and applications*, Comm. Pure Appl. Anal. **5** (2006), 551–569
39. A. L. Sasu and B. Sasu, *On the dichotomic behavior of discrete dynamical systems on the half-line*, Discrete Contin. Dynam. Syst. **33** (2013), 3057–3084.
40. A. L. Sasu and B. Sasu, *Admissibility and exponential trichotomy of dynamical systems described by skew-product flows*, J. Differential Equations **260** (2016), 1656–1689.
41. N. Van Minh, F. Räbiger and R. Schnaubelt, *Exponential stability, exponential expansiveness, and exponential dichotomy of evolution equations on the half-line*, Integral Equations Operator Theory **32** (1998), 332–353.
42. L. Zhou, K. Lu and W. Zhang, *Roughness of tempered dichotomies for infinite-dimensional random difference equations*, J. Differential Equations **254** (2013), 4024–4046.
43. L. Zhou, K. Lu and W. Zhang, *Equivalences between nonuniform exponential dichotomy and admissibility*, J. Differential Equations **262** (2017), 682–747.

Birzeit University, Birzeit, Palestine
E-mail address: mabualhalawa@birzeit.edu

School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052, Australia
E-mail address: d.dragicevic@unsw.edu.au

Department of Mathematics, University of Rijeka, Croatia
E-mail address: ddragicevic@math.uniri.hr