Limit theorems for random dynamical systems using the spectral method

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Piecewise expanding maps

Let $I = [0, 1]$ denote the unit interval equipped with Borel σ-algebra $\mathcal{B}$ and a Lebesgue measure m. We say that $T \colon I \to I$ is a **piecewise expanding map** if there exists a partition

$$0 = x_0 < x_1 < \ldots < x_{k-1} < x_k = 1$$

and $\alpha > 1$ such that:

1. restriction $T|_{(x_{i-1}, x_i)}$ is a C^1 function which can be extended to a C^1 function on $[x_{i-1}, x_i]$;
2. $|T'(x)| \geq \alpha$ for $x \in (x_{i-1}, x_i)$;
3. $g(x) = \frac{1}{|T'(x)|}$ is a function of bounded variation.

Deterministic setting

Let T be a piecewise expanding map and consider the associated **transfer operator** $\mathcal{L} \colon L^1(m) \to L^1(m)$ by

$$\mathcal{L}f(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T'(y)|}.$$

We note that $\mathcal{L}$ doesn't have good spectral properties as an operator on $L^1(m)$. However, it has as an operator on BV (space of functions of bounded variation). More precisely, $\mathcal{L} \colon BV \to BV$ is a **quasicompact operator**. This means that it can be written as

$$\mathcal{L} = \sum_{i=1}^{k} \lambda_i \Pi_i + N,$$

where λ_i are eigenvalues for $\mathcal{L}$, $|\lambda_i| = r(\mathcal{L}) = 1$, each Π_i is a projections onto an one-dimensional subspace of BV,

$\Pi_i N = N \Pi_i = 0$ and $r(N) < 1$. Some important consequences:

1. there exist an **absolutely continuous invariant measure** for T, i.e. 1 is an eigenvalue of $\mathcal{L}$ with a positive eigenvector;
2. under some additional assumptions acim is **unique** and **mixing**; we denote it by μ (from now on we assume that this is the case);
3. we have exponential **decay of correlation** and **limit laws** (central limit theorem, local central limit theorem, large deviations, almost sure invariance principle...)

Central limit theorem

Assume that $\phi \colon I \to \mathbb{R}$ bounded observable in BV such that $\int_{[0,1]} \phi \, d\mu = 0$. For each $n \in \mathbb{N}$, let

$$S_n = \sum_{k=0}^{n-1} \phi \circ T^k.$$

Theorem (Rousseau–Egele, 1983)

We have that $\lim_{n\to\infty} \int_{[0,1]} \frac{S_n^2}{n} = \sigma^2$, *where*

$$\sigma^2 = \int_{[0,1]} \phi^2 \, d\mu + 2\sum_{n=1}^{\infty} \int_{[0,1]} \phi(\phi \circ T^n) \, d\mu < \infty.$$

If $\sigma^2 > 0$, *then* $\frac{S_n}{\sqrt{n}}$ *converges in distribution to* $N(0, \sigma^2)$.

Large deviation principle

Theorem

If $\sigma^2 > 0$, then there exists $\delta > 0$ and a strictly convex, continuous and nonnegative function $c\colon (-\delta, \delta) \to \mathbb{R}$ which vanishes only at 0 such that

$$\lim_{n\to\infty} \frac{1}{n} \log \mu(S_n > n\varepsilon) = -c(\varepsilon), \quad \text{for } \varepsilon \in (0, \delta).$$

Ideas of the proofs

We define

$$\mathcal{L}^{\theta}(g) = \mathcal{L}(e^{\theta\phi} g), \quad \text{for } g \in BV \text{ and } \theta \in \mathbb{C}.$$

Since $\theta \mapsto \mathcal{L}^{\theta}$ is analytic, for θ sufficiently close to 0,

$$\mathcal{L}^{\theta} = \omega(\theta)\Pi(\theta) + N(\theta),$$

where $\Pi(\theta)$ is a projection of rank 1 and $r(N(\theta)) < |\omega(\theta)|$. CLT ($d\mu = f\,dm$): for $t \in \mathbb{R}$ we have that

$$\lim_{n\to\infty} \int_{[0,1]} e^{itS_n/\sqrt{n}}\, d\mu = \lim_{n\to\infty} \int_{[0,1]} (\mathcal{L}^{it/\sqrt{n}})^n (f)\, dm = \lim_{n\to\infty} \omega(it/\sqrt{n})^n$$

$$= e^{-t^2\sigma^2/2}.$$

LDP:

we first show that $\omega'(0) = 0$ and $\omega''(0) = \sigma^2$ and then that

$$\lim_{n\to\infty} \frac{1}{n} \log \int_{[0,1]} e^{\theta S_n} \, d\mu = \Lambda(\theta),$$

where $\Lambda(\theta) = \log \omega(\theta)$, for $\theta \in \mathbb{R}$ sufficiently close to 0.

Random Lasota-Yorke maps

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and assume that $\sigma \colon \Omega \to \Omega$ is invertible transformation that preserves $\mathbb{P}$. Furthermore, assume that $\mathbb{P}$ is ergodic. We now take the collection T_ω, $\omega \in \Omega$ of piecewise expanding maps. By $\mathcal{L}_\omega$ we denote the transfer operator associated to T_ω. For $\omega \in \Omega$ and $n \in \mathbb{N}$, set

$$T_\omega^n = T_{\sigma^{n-1}\omega} \circ \ldots \circ T_{\sigma\omega} \circ T_\omega$$

and

$$\mathcal{L}_\omega^n = \mathcal{L}_{\sigma^{n-1}\omega} \circ \ldots \circ \mathcal{L}_{\sigma\omega} \circ \mathcal{L}_\omega.$$

The associated skew-product transformation $\tau \colon \Omega \times I \to \Omega \times I$ is given by $\tau(\omega, x) = (\sigma\omega, T_\omega x)$.

We assume that:

1. there exists $K > 0$ such that $\|\mathcal{L}_\omega\| \le K$ for $\mathbb{P}$-a.e. $\omega \in \Omega$;
2. there exists $N \in \mathbb{N}$ and measurable $\alpha^N, \beta^N \colon \Omega \to (0, \infty)$ with $\int_\Omega \log \alpha^N(\omega)\, d\mathbb{P}(\omega) < 0$ such that for any $f \in BV$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\|\mathcal{L}_\omega^N f\|_{BV} \le \alpha^N(\omega)\|f\|_{BV} + \beta^N(\omega)\|f\|_1;$$

3. there exist $D, \lambda > 0$ such that $\|\mathcal{L}_\omega^n f\|_{BV} \le De^{-\lambda n}\|f\|_{BV}$ for $f \in BV$, $\int f\, dm = 0$, $n \in \mathbb{N}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$;
4. there exists $N \in \mathbb{N}$ such that for any $a > 0$ and sufficiently large $n \in \mathbb{N}$, there is $c > 0$ such that $\operatorname{essinf} \mathcal{L}_\omega^{nN} f \ge c\|f\|_1$, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

   $f \in C_a := \{f \in BV : f \ge 0 \text{ and } var(f) \le a\|f\|_1\}$.

Then, there exists a **unique acim** (w.r.t. $\mathbb{P} \times m$) μ for τ such that $\pi_* \mu = \mathbb{P}$, where $\pi \colon \Omega \times I \to \Omega$ is a projection. We can regard μ as a collection of **fiber measures** μ_ω, $\omega \in \Omega$ on I:

$$\int_{\Omega \times I} \phi(\omega, x)\, d\mu = \int_\Omega \int_I \phi(\omega, x)\, d\mu_\omega(x)\, d\mathbb{P}(\omega).$$

We consider **observables** $\phi \colon \Omega \times I \to \mathbb{R}$ such that

$$\text{esssup}_{(\omega,x)} |\phi(\omega, x)| < \infty \quad \text{and} \quad \text{esssup}_\omega \, var(\phi(\omega, \cdot)) < \infty.$$

Moreover, we assume that

$$\int_{[0,1]} \phi(\omega, \cdot)\, d\mu_\omega = 0, \quad \omega \in \Omega.$$

We form Birkhoff sums

$$S_n(\omega, x) = \sum_{i=0}^{n-1} (\phi \circ \tau^i)(\omega, x) = \sum_{i=0}^{n-1} \phi(\sigma^i \omega, T_\omega^i x).$$

We are interested in the **quenched** type of limit theorems i.e. those that give an information about the asymptotic behaviour of Birkhoff sums w.r.t. to μ_ω for "typical" ω.

Previous work:

1. Kifer, 1998: quenched limit theorems but not with spectral method (main example: random subshifts of finite type);
2. Aimino-Nicol-Vaienti, 2014: spectral method but the base space is assumed to be a Bernoulli shift (piecewise expanding maps);
3. Ayyer-Liverani-Stenlund, 2008: same as above but for random toral automorphisms.

Related work on **sequential dynamics**: Bakhtin, Conze-Raugi, Conze-Le Borgne-Roger, Nandori-Szasz-Varju.

MET, Froyland/Lloyd/Quas, 2013.

Assume that $(\Omega, \mathcal{F}, \mathbb{P}, \sigma)$ is an ergodic m.p.s. where Ω is a Borel subset of a separable, complete metric space. Furthermore, let B be a Banach space and $\mathcal{L} = \mathcal{L}_\omega$, $\omega \in \Omega$ a family of bounded linear operators on B such that the map $\omega \mapsto \mathcal{L}_\omega$ is Borel-measurable. Then, for a.e. $\omega \in \Omega$, the following limits exist

$$\Lambda(\mathcal{L}) := \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^n\| \text{ and } \kappa(\mathcal{L}) := \lim_{n\to\infty} \frac{1}{n} \log ic(\mathcal{L}_\omega^n),$$

where $ic(\mathcal{L}_\omega^n) = \inf\{r > 0 :$
$\mathcal{L}_\omega^n(B(0,1))$ can be covered with finitely many balls of radius $r\}$.

MET2

If $\kappa(\mathcal{L}) < \Lambda(\mathcal{L})$, then there exists $1 \le l \le \infty$ and a sequence of Lyapunov exponents

$$\Lambda(\mathcal{L}) = \lambda_1 > \lambda_2 > \ldots > \lambda_l > \kappa(\mathcal{L}) \quad (\text{if } 1 \le l < \infty)$$

or

$$\Lambda(\mathcal{L}) = \lambda_1 > \lambda_2 > \ldots \quad \text{and} \quad \lim_{n\to\infty} \lambda_n = \kappa(\mathcal{L}) \quad (\text{if } l = \infty);$$

and for $\mathbb{P}$-almost every $\omega \in \Omega$ there exists a unique splitting (called the *Oseledets splitting*) of B into closed subspaces

$$B = V(\omega) \oplus \bigoplus_{j=1}^{l} Y_j(\omega),$$

MET3

depending measurably on ω and such that:

1. For each $1 \leq j \leq l$, $\dim Y_j(\omega) < \infty$, Y_j is equivariant i.e. $\mathcal{L}_\omega Y_j(\omega) = Y_j(\sigma\omega)$ and for every $y \in Y_j(\omega) \setminus \{0\}$,
$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^n y\| = \lambda_j.$$
2. V is equivariant i.e. $\mathcal{L}_\omega V(\omega) \subseteq V(\sigma\omega)$ and for every $v \in V(\omega)$,
$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^n v\| \leq \kappa(\mathcal{L}).$$

In order to be able to apply MET for our cocycle of transfer operators, we will require that: Ω is a Borel subset of a separable, complete metric space and that

$$\text{the map} \quad \omega \to T_\omega \quad \text{has a countable range}$$

We have $\dim Y_1(\omega) = 1$ and $Y_1(\omega) = span\{v_\omega^0\}$, where $d\mu_\omega = v_\omega^0 dm$.

We also form a twisted cocycle. More precisely, for $\omega \in \Omega$ and $\theta \in \mathbb{C}$, we define

$$\mathcal{L}_\omega^\theta(h) = \mathcal{L}_\omega(e^{\theta\phi(\omega,\cdot)}h), \quad h \in BV.$$

Theorem

For $\theta \in \mathbb{R}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \int_{[0,1]} e^{\theta S_n(\omega,\cdot)}\, d\mu_\omega = \Lambda(\theta),$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ where $\Lambda(\theta)$ is a top Lyapunov exponent of the cocycle $\mathcal{L}_\omega^\theta$, $\omega \in \Omega$.

Regularity of Λ

Key points ($d\mu_\omega = v_\omega^0\, dm$):

1. we construct the top space as $v_\omega^0 + \mathcal{W}^\theta(\omega, \cdot)$ where $\mathcal{W}^\theta$ is a (unique) solution of $F(\theta, \mathcal{W}) = 0$, where

$$F(\theta, \mathcal{W}) = \frac{\mathcal{L}^\theta_{\sigma^{-1}\omega}(v^0_{\sigma^{-1}\omega} + \mathcal{W}(\sigma^{-1}\omega, \cdot))}{\int (\mathcal{L}^\theta_{\sigma^{-1}\omega}(v^0_{\sigma^{-1}\omega} + \mathcal{W}(\sigma^{-1}\omega, \cdot)))\, dm} - \mathcal{W}(\omega, \cdot) - v^0_\omega,$$

   where $\mathcal{W} \in \mathcal{S}$ and

$$\mathcal{S} := \{\mathcal{W} \colon \Omega \times I \to \mathbb{C} : \mathcal{W}(\omega, \cdot) \in BV,\ \text{esssup}_\omega \|W(\omega, \cdot)\|_{BV} < \infty\}.$$

2. for θ close to 0, the top Oseledets space of the twisted cocycle $\mathcal{L}^\theta_\omega$ is one-dimensional;
3. $\Lambda(\theta) = \int \log \left| \int e^{\theta\phi(\omega, \cdot)}(v^0_\omega + \mathcal{W}^\theta(\omega, \cdot))\, dm \right| d\mathbb{P}(\omega)$.

Also, $\Lambda'(0) = 0$ and $\Lambda''(0) = \Sigma^2$, where Σ^2 is a variance.

Theorem (Large deviation principle)

Assume that $\Sigma^2 > 0$. Then, there exists $\varepsilon_0 > 0$ and a function $c\colon (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ which is nonnegative, continuous, strictly convex, vanishing only at 0 and such that

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\omega(S_n(\omega,\cdot) > n\varepsilon) = -c(\varepsilon), \quad \text{for } 0 < \varepsilon < \varepsilon_0 \text{ and a.e. } \omega.$$

We can also obtain CLT.

Theorem (Central limit theorem)

If $\Sigma^2 > 0$, we have that

$$\lim_{n\to\infty} \int g(S_n(\omega,\cdot)/\sqrt{n})\, d\mu_\omega = \int g\, dN(0,\Sigma^2),$$

for g continuous and bounded and a.e. $\omega \in \Omega$.

Idea of the proof

We need to show that

$$\lim_{n\to\infty}\int e^{it\frac{S_n(\omega,\cdot)}{\sqrt{n}}}\,d\mu_\omega = e^{-\frac{t^2\Sigma^2}{2}}, \quad \text{for a.e. } \omega\in\Omega.$$

This follows by proving that:

1.
$$\lim_{n\to\infty}\int e^{it\frac{S_n(\omega,\cdot)}{\sqrt{n}}}\,d\mu_\omega = \lim_{n\to\infty}\prod_{j=0}^{n-1}\lambda_{\sigma^j\omega}^{\frac{it}{\sqrt{n}}},$$

   where

   $\lambda_\omega^\theta = \int \mathcal{L}_{\sigma^{-1}\omega}^\theta(v_{\sigma^{-1}\omega}^0 + \mathcal{W}^\theta(\sigma^{-1}\omega,\cdot))\,dm =: H(\theta,\mathcal{W}^\theta)(\omega);$

2. by Taylor expansion of $\theta\to H(\theta,\mathcal{W}^\theta)$ around 0:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\lambda_{\sigma^j\omega}^{\frac{it}{\sqrt{n}}} = -\frac{t^2\Sigma^2}{2}.$$

LCLT

Theorem

Assume that $\Lambda(it) < 0$ *for* $t \in \mathbb{R} \setminus \{0\}$. *Then, for* $\mathbb{P}$*-.a.e.* $\omega \in \Omega$ *and every bounded interval* $J \subset \mathbb{R}$, *we have*

$$\lim_{n \to \infty} \sup_{s \in \mathbb{R}} \left| \Sigma \sqrt{n} \mu_\omega(s + S_n \phi(\omega, \cdot) \in J) - \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2n\Sigma^2}} \right| = 0.$$

Theorem

The following is equivalent:

1. $\Lambda(it) < 0$ *for* $t \in \mathbb{R} \setminus \{0\}$;
2. *the equation* $e^{it\phi(\omega,\cdot)} \mathcal{L}_\omega^* \psi_{\sigma\omega} = \gamma_\omega^{it} \psi_\omega$, *where* $\gamma_\omega^{it} \in S^1$, $\psi_\omega \in BV^*$ *has measurable solutions only for* $t = 0$ *(when* $\gamma_\omega^0 = 1$ *and* $\psi_\omega = m$*).*

Further developments

Our results include **piecewise expanding maps in higher dimension:**

D. Dragičević, G. Froyland, C. González-Tokman and S. Vaienti: *A spectral approach for quenched limit theorems for random expanding dynamical systems*, Communications in Mathematical Physics, in press.

Work in progress: random composition of hyperbolic diffeomorphisms.