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Characteristic scales of bounded L^2 sequences

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Abstract

Practical applications of semiclassical measures are tightly connected with a so called oscillatory property, prevailing leakage of information related to high frequencies. In this paper we propose a complementary, concentratory property which prevents loss of information related to low frequencies. We demonstrate that semiclassical measures attain the best performance level if both the properties are satisfied simultaneously, and address a question if this is possible to achieve for an arbitrary bounded L^2 sequence, providing a negative answer. Comparison of H-measures with semiclassical ones is presented, showing precedence of the latter for problems exhibiting just a single frequency scale. Finally, we present some (strong) compactness results based on the above properties.

Keywords: semiclassical measures, H-measures, oscillatory sequence, concentratory sequence, characteristic scale, Kolmogorov-Riesz compactness theorem

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1. Introduction

In the studies of partial differential equations one often needs to deal with sequences converging weakly, but not strongly in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$. Some possible causes of such situations are governed by oscillations. For example, sequence $u_n(\mathbf{x}) := e^{2\pi i \frac{\mathsf{k}}{\varepsilon_n}\cdot\mathbf{x}}$, where $\mathsf{k} \in \mathbf{Z}^d \setminus \{\mathsf{0}\}$ and $\varepsilon_n \to 0^+$, is the simplest example of oscillations known under the name *plane wave*, and it converges weakly to zero in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$, but it does not converge strongly.

Among various methods and tools suitable for exploring such sequences, microlocal defect functionals proved to be quite successful, e.g. H-measures [5, 12], semiclassical measures [6, 10], H-distributions [3], etc.

The first microlocal defect functional was introduced independently by Luc Tartar [12] and Patrick Gérard [5] around 1990, and is called *H-measure* or *microlocal defect measure*. It is the Radon measure on cospherical bundle $\Omega\times\mathrm{S}^{d-1}$ over open subset $\Omega\subseteq\mathbf{R}^d$ and the definition in the case of local spaces can be provided by the following theorem (cf. [5, Theorem 1] or [12, Theorem 1.1]):

Theorem 1. (existence of H-measures) *For a weakly converging sequence* $\mathsf{u}_n \rightharpoonup \mathsf{0}$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$, *there exists a subsequence* $(\mathsf{u}_{n'})$ *and an* $r\times r$ *hermitian non-negative matrix Radon measure* $\boldsymbol{\mu}_H$ *on* $\Omega\times\mathrm{S}^{d-1}$ *such that for any* $\varphi_1,\varphi_2\in\mathrm{C}_c(\Omega)$ *and* $\psi\in\mathrm{C}(\mathrm{S}^{d-1})$ *one has:*

$$\lim_{n'}\int_{\mathbf{R}^d}\Big(\widehat{\varphi_1\mathsf{u}_{n'}}(\boldsymbol{\xi})\otimes\widehat{\varphi_2\mathsf{u}_{n'}}(\boldsymbol{\xi})\Big)\psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right)d\boldsymbol{\xi} = \langle\boldsymbol{\mu}_H,(\varphi_1\bar{\varphi}_2)\boxtimes\psi\rangle$$
$$= \int_{\Omega\times\mathrm{S}^{d-1}}\varphi_1(\mathbf{x})\bar{\varphi}_2(\mathbf{x})\psi(\boldsymbol{\xi})\,d\bar{\boldsymbol{\mu}}_H(\mathbf{x},\boldsymbol{\xi})\,.$$

The above measure $\boldsymbol{\mu}_H$ *is called* the H-measure *associated to the (sub)sequence* $(\mathsf{u}_{n'})$. ∎

When the *whole* sequence admits the H-measure (i.e. the definition is valid without passing to a subsequence), we say that the sequence is *pure*.

Notation. Let us explain the notation used in the previous theorem, which shall be kept throughout the paper.

By S^{d-1} we denote the $(d-1)$-dimensional unit sphere (used here only in the Fourier space), while open $\Omega\subseteq\mathbf{R}^d$ stands for the physical space.

By $\otimes$ we denote the tensor product of vectors on $\mathbf{C}^r$, defined by $(\mathsf{a}\otimes\mathsf{b})\mathsf{v} = (\mathsf{v}\cdot\mathsf{b})\mathsf{a}$, where $\cdot$ stands for the (complex) scalar product ($\mathsf{a}\cdot\mathsf{b} := \sum_{i=1}^r a_i\bar{b}_i$), resulting in $[\mathsf{a}\otimes\mathsf{b}]_{ij} = a_i\bar{b}_j$, while $\boxtimes$ denotes the tensor product of functions in different variables. By $\langle\cdot,\cdot\rangle$ we denote any sesquilinear dual product, which we take to be antilinear in the first variable, and linear in the second.

The Fourier transform we define as $\hat{\mathsf{u}}(\boldsymbol{\xi}) := \int_{\mathbf{R}^d} e^{-2\pi i\boldsymbol{\xi}\cdot\mathbf{x}}\mathsf{u}(\mathbf{x})\,d\mathbf{x}$, and its inverse as $(\mathsf{u})^\vee(\boldsymbol{\xi}) := \int_{\mathbf{R}^d} e^{2\pi i\boldsymbol{\xi}\cdot\mathbf{x}}\mathsf{u}(\mathbf{x})\,d\mathbf{x}$. In order to have the Fourier transform well-defined on Ω we identify functions defined on Ω with their extensions by zero to the whole $\mathbf{R}^d$.

By $\mathrm{K}(\mathbf{x},r)$ and $\mathrm{K}[\mathbf{x},r]$ we denote open and closed balls around $\mathbf{x}$ of radius r.

Throughout the paper, when there is no fear of ambiguity, we pass to a subsequence without relabelling it. ∎

H-measures quantify the deflection from strong $\mathrm{L}^2_{\mathrm{loc}}$ precompactness in the sense that a trivial H-measure implies the strong convergence in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{C}^r)$ of the corresponding sequence, and vice versa. This property is a basis of standard compactness results obtained by means of H-measures (eg. [11, Theorem 2], [9, Theorem 7]).

However, they turn not to be the right object if we want to distinguish sequences with different characteristic lengths (e.g. different frequencies). Indeed, the H-measure associated to $u_n(\mathbf{x}) := e^{2\pi i\frac{\mathsf{k}}{\varepsilon_n}\cdot\mathbf{x}}$ (which is pure) is $\lambda\boxtimes\delta_{\frac{\mathsf{k}}{|\mathsf{k}|}}$, where λ is the Lebesgue measure on the physical space $\mathbf{R}^d$ (i.e. in $\mathbf{x}$), while $\delta_{\frac{\mathsf{k}}{|\mathsf{k}|}}$ is the Dirac mass in the unit vector $\boldsymbol{\xi} = \frac{\mathsf{k}}{|\mathsf{k}|}$. Thus, for any choice of $\varepsilon_n\to0^+$ we have the same H-measure.

The new object suitable for problems with characteristic lengths was introduced by Gérard [6] under the name *semiclassical measure*, while later an alternative construction using the Wigner transform was presented by Pierre-Louis Lions and Thierry Paul [10], denoting the object as *Wigner measure*. Here we present the existence result in a simpler, but equivalent form to the original Gérard's definition [6], without introducing the notion of (semiclassical) pseudodifferential operators (cf. [13, Chapter 32]).

Theorem 2. (existence of semiclassical measures) *If* (u_n) *is a bounded sequence in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$, *and* (ω_n) *a sequence of positive numbers such that* $\omega_n \to 0^+$, *then there exists a subsequence* $(\mathsf{u}_{n'})$ *and an* $r\times r$ *hermitian non-negative matrix Radon measure* $\boldsymbol{\mu}_{sc}^{(\omega_{n'})}$ *on* $\Omega\times\mathbf{R}^d$ *such that for any* $\varphi_1,\varphi_2\in \mathrm{C}^\infty_c(\Omega)$ *and* $\psi\in\mathcal{S}(\mathbf{R}^d)$ *one has:*

$$\lim_{n'}\int\limits_{\mathbf{R}^d}\Big(\widehat{\varphi_1\mathsf{u}_{n'}}(\boldsymbol{\xi})\otimes\widehat{\varphi_2\mathsf{u}_{n'}}(\boldsymbol{\xi})\Big)\psi(\omega_{n'}\boldsymbol{\xi})\,d\boldsymbol{\xi} = \langle\boldsymbol{\mu}_{sc}^{(\omega_{n'})},(\varphi_1\bar\varphi_2)\boxtimes\psi\rangle$$

$$= \int\limits_{\Omega\times\mathbf{R}^d}\varphi_1(\mathbf{x})\bar\varphi_2(\mathbf{x})\psi(\boldsymbol{\xi})\,d\bar{\boldsymbol{\mu}}_{sc}^{(\omega_{n'})}(\mathbf{x},\boldsymbol{\xi})\,.$$

The above measure $\boldsymbol{\mu}_{sc}^{(\omega_{n'})}$ *is called* the semiclassical measure with (semiclassical) scale $(\omega_{n'})$ *associated to the (sub)sequence* $(\mathsf{u}_{n'})$. ∎

When there is no fear of ambiguity, we assume that we have already passed to a subsequence determining a semiclassical measure, and reduce the notation to $\boldsymbol{\mu}_{sc}=\boldsymbol{\mu}_{sc}^{(\omega_n)}$.

When the *whole* sequence admits a semiclassical measure with scale (ω_n) (i.e. the definition is valid without passing to a subsequence), we say that the sequence is (ω_n)-*pure*.

If $\mathsf{u}_n \rightharpoonup \mathsf{u}$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ then

$$\boldsymbol{\mu}_{sc}^{(\omega_n)} = (\mathsf{u}\otimes\mathsf{u})\lambda\boxtimes\delta_{\mathsf{0}}+\boldsymbol{\nu}_{sc}^{(\omega_n)}\,, \tag{1}$$

where $\boldsymbol{\nu}_{sc}^{(\omega_n)}$ is the semiclassical measure, with the same scale as $\boldsymbol{\mu}_{sc}^{(\omega_n)}$, associated to the sequence $(\mathsf{u}_n-\mathsf{u})$.

In the above theorem we have used a notion of the boundedness in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ which is meant with respect to its standard Fréchet locally convex topology. Hence, a subset of $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ is bounded if and only if it is bounded in the sense of seminorms which generate the corresponding locally convex topology (which is a stronger notion then metric boundedness).

An oscillating sequence of functions $u_n(\mathbf{x}) := e^{2\pi i\frac{\mathsf{k}}{\varepsilon_n}\cdot\mathbf{x}}$ is (ω_n)-pure for any $\omega_n\to0^+$ such that $\lim_n\frac{\varepsilon_n}{\omega_n}$ exists in $[0,\infty]$, but the associated semiclassical measure depends on the choice of a scale:

$$\mu_{sc}^{(\omega_n)} = \lambda\boxtimes\begin{cases}0 & , \quad \lim_n\frac{\varepsilon_n}{\omega_n}=0\\ \delta_{\frac{\mathsf{k}}{c}} & , \quad \lim_n\frac{\varepsilon_n}{\omega_n}=c\in\langle0,\infty\rangle\\ \delta_{\mathsf{0}} & , \quad \lim_n\frac{\varepsilon_n}{\omega_n}=\infty\end{cases}\,.$$

Here we can see that in some situations by semiclassical measures one can really recover both the direction (i.e. k) and the frequency (i.e. $\frac{1}{\varepsilon_n}$) of oscillations, what is not the case with H-measures. However, in the case where (ω_n) is not of the same order of convergence as (ε_n) the loss of information takes place. Indeed, in the case $\lim_n\frac{\varepsilon_n}{\omega_n}=0$ the loss of energy at infinity occurs, and that phenomenon is described by the (ω_n)-*oscillatory* property which we present in the next section. On the other hand, the mixture of various pieces of information at the origin takes place in the case $\lim_n\frac{\varepsilon_n}{\omega_n}=\infty$.

From this scholarly example we notice that, unlike H-measures, strong convergence of a (sub)sequence under consideration does not necessarily follow from a trivial semiclassical measure, and in general one requires additional information in order to obtain compactness results.

A problem of the recovery and loss of information by microlocal defect tools has been addressed in [2], while here we want to provide more detailed insight into the problem. To this effect in the next section we introduce the new notion denominated as (ω_n)-*concentratory* property, being in some sense a counterpart to the already existing (ω_n)-oscillatory property and preventing the loss of information at the origin of the frequency domain. In the third section we demonstrate that semiclassical measures attain the best performance level if both the properties are satisfied simultaneously for some scale, which we define as a *characteristic scale* of a sequence, and we address the existence problem of such a scale for an arbitrary bounded sequence. Section 4 contains some (strong) compactness results based on the above properties, followed by concluding remarks closing the paper.

2. (ω_n)-concentratory sequences

We recall (cf. [4, Def. 3.3] and [7, Def. 1.6]) the definition of the (ω_n)-oscillatory property.

Definition 1. Let (u_n) be a sequence in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ for an open $\Omega\subseteq\mathbf{R}^d$, and (ω_n) be a sequence of positive numbers converging to zero. We say that (u_n) is (ω_n)-*oscillatory* if

$$(2)\qquad (\forall\,\varphi\in\mathrm{C}^\infty_c(\Omega))\qquad \lim_{R\to\infty}\limsup_n\int\limits_{|\boldsymbol{\xi}|\geqslant\frac{R}{\omega_n}}|\widehat{\varphi\mathsf{u}_n}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}=0\,.$$

∎

An interpretation of this condition is that the frequencies of the observed sequence do not converge to infinity *faster* then $\frac{1}{\omega_n}$.

Note that the last property resembles the one assumed by the Kolmogorov-Riesz compactness theorem [8, Cor. 7], which can be obtained from (2) by inserting $\omega_n=1$. Of course, the latter assumption is a stronger one, providing strong convergence of the sequence (u_n), while here in general we consider weakly convergent sequences.

The notion of the (ω_n)-oscillatory property is tightly related to semiclassical measures, so it has appeared already in the very first articles on semiclassical measures. At the beginning it was stated without a localisation by test functions (cf. [6, Section 3] and [10, Theorem III.1(3)]), while in more recent papers one can find the same definition as given here (cf. [4, Def. 3.3] and [7, Def. 1.6]). It has an important role in most of successful applications of semiclassical measures, since it is necessary in order to obtain a relation between defect and semiclassical measures. More precisely the following result holds (cf. [4, Lemma 3.4(i)] and [7, Prop. 1.7(i)]).

Lemma 1. *Let (u_n) be bounded in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ and (ω_n)-pure, and let $\mathsf{u}_n\otimes\mathsf{u}_n$ converge weakly$\ast$ to a measure $\boldsymbol{\nu}$ in $\mathcal{M}(\Omega;\mathrm{M_r}(\mathbf{C}))$. The sequence (u_n) is (ω_n)-oscillatory if and only if for any $\varphi\in\mathrm{C}^\infty_c(\Omega)$ it holds:*

$$\langle\boldsymbol{\nu},\varphi\rangle=\left\langle\boldsymbol{\mu}_{sc}^{(\omega_n)},\varphi\boxtimes1\right\rangle\,.$$

∎

The last relation ensures that no portion of macroscopic energy of the original sequence is lost by its semiclassical measure. Specially, assuming the (ω_n)-oscillatory property, a trivial semiclassical measure implies strong convergence $\mathsf{u}_n\longrightarrow0$.

Since test functions in the definition of semiclassical measures are taken from $\mathcal{S}(\mathbf{R}^d_{\boldsymbol{\xi}})$, the only place where some energy can be lost is at infinity in the dual space. Thus the (ω_n)-oscillatory property ensures that the semiclassical measure with scale (ω_n) captures all energy of the original sequence. In particular, it implies the sequence does not exhibit oscillations on a frequency faster than $\frac{1}{\omega_n}$.

At this level we find suitable to describe different relations between sequences of positive numbers converging to zero by using standard asymptotic behaviour notions. For zero sequences (ω_n) and $(\tilde\omega_n)$ we say that:

- (ω_n) is *faster* then $(\tilde\omega_n)$ if $\omega_n=o(\tilde\omega_n)$, i.e. $\lim_n\frac{\omega_n}{\tilde\omega_n}=0$;
- (ω_n) is *not slower* then $(\tilde\omega_n)$ if $\omega_n=O(\tilde\omega_n)$, i.e. $\limsup_n\frac{\omega_n}{\tilde\omega_n}<\infty$;

- (ω_n) is *of the same order* as $(\tilde\omega_n)$ if $\omega_n = \Theta(\tilde\omega_n)$, i.e. $\omega_n = O(\tilde\omega_n)$ and $\tilde\omega_n = O(\omega_n)$;
- (ω_n) is *slower* then $(\tilde\omega_n)$ if $(\tilde\omega_n)$ is faster then (ω_n);
- (ω_n) is *not faster* then $(\tilde\omega_n)$ if $(\tilde\omega_n)$ is not slower then (ω_n).

The same terminology we also apply for sequences converging to infinity, just commuting positions of ω_n and $\tilde\omega_n$ in the properties defining each notion. Moreover, for simplicity, the sequences of positive numbers converging to zero we refer to as *(semiclassical) scales*.

As demonstrated in the example of an oscillating sequence in the introduction, the other place where the information on the original sequence is partially lost is the origin of the dual space, where slow oscillations are mixed together, and one cannot recover neither their direction nor exact frequencies. In order to prevent such a scenario, one needs to apply a scale that will prevent concentration effects at the origin of the dual space. Intuitively, this would mean that the information of the sequence propagates on the scale not slower then $\frac{1}{\omega_n}$, which leads us to the following definition.

Definition 2. We say that sequence (u_n) in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ is (ω_n)-*concentratory* if

$$(\forall\,\varphi\in \mathrm{C}^\infty_c(\Omega))\qquad \lim_{R\to\infty}\limsup_n \int\limits_{|\boldsymbol\xi|\leqslant\frac{1}{R\omega_n}} |\widehat{\varphi\mathsf{u}_n}(\boldsymbol\xi)|^2\,d\boldsymbol\xi = 0\,. \tag{3}$$

∎

This notion, which has not yet been studied according to our best knowledge, can be seen as a counterpart of the (ω_n)-oscillatory property, both of them prevailing extreme propagations, which are either too slow or too fast.

When being applied to semiclassical measures, the introduced notion indeed prevents concentration effects at the origin of the dual space, as demonstrated by the next result.

Theorem 3. *If (u_n) is bounded in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ and (ω_n)-pure, then $\mathsf{tr}\boldsymbol\mu_{sc}^{(\omega_n)}(\Omega\times\{0\})=0$ if and only if (u_n) is (ω_n)-concentratory.*

Dem. Let us first notice that the statement is meaningful since $\mathsf{tr}\boldsymbol\mu_{sc}$ is a non-negative functional, hence by the Riesz representation theorem we can treat it as a (classical) measure on the Borel σ-algebra. Therefore, for any $\phi\in\mathrm{C}_c(\Omega\times\mathbf{R}^d)$ we have

$$\langle \mathsf{tr}\boldsymbol\mu_{sc},\phi\rangle = \int\limits_{\mathbf{R}^d}\phi(\mathbf{x},\boldsymbol\xi)\,d\mathsf{tr}\boldsymbol\mu_{sc}\,,$$

implying that $\mathsf{tr}\boldsymbol\mu_{sc}(\Omega\times\{0\})=0$ is equivalent to

$$(\forall\,\varphi\in\mathrm{C}_c(\Omega))\qquad \int\limits_{\Omega\times\mathbf{R}^d}|\varphi(\mathbf{x})|^2\chi_{\{0\}}(\boldsymbol\xi)\,d\mathsf{tr}\boldsymbol\mu_{sc}(\mathbf{x},\boldsymbol\xi)=0\,,$$

where $\chi_{\{0\}}$ is equal to 1 at the origin and 0 otherwise.

By $\zeta\in\mathrm{C}^\infty_c(\mathbf{R}^d)$ we denote a smooth cutoff function identically equal to 1 on $\mathrm{K}[0,1]$ such that $0\leqslant\zeta\leqslant 1$, while $\operatorname{supp}\zeta\subseteq\mathrm{K}(0,2)$. Further, we define $\zeta_m := \zeta(m\,\cdot)$.

By the Lebesgue dominated convergence theorem, non-negativity of diagonal elements of matrix $\boldsymbol\mu_{sc}$, and the definition of semiclassical measures, for any $i\in 1..d$ we have

$$\begin{aligned}
0 = \int\limits_{\Omega\times\mathbf{R}^d}|\varphi(\mathbf{x})|^2\chi_{\{0\}}(\boldsymbol\xi)\,d\mu^{ii}_{sc}(\mathbf{x},\boldsymbol\xi) &= \lim_m \int\limits_{\Omega\times\mathbf{R}^d}|\varphi(\mathbf{x})|^2\zeta_m(\boldsymbol\xi)\,d\mu^{ii}_{sc}(\mathbf{x},\boldsymbol\xi)\\
&= \lim_m\lim_n\int\limits_{\mathbf{R}^d}|\widehat{\varphi u^i_n}(\boldsymbol\xi)|^2\zeta_m(\omega_n\boldsymbol\xi)\,d\boldsymbol\xi\\
&\geqslant \limsup_m\limsup_n \int\limits_{|\omega_n\boldsymbol\xi|\leqslant\frac{1}{m}}|\widehat{\varphi u^i_n}(\boldsymbol\xi)|^2\,d\boldsymbol\xi\geqslant 0\,,
\end{aligned}$$

where in the last step we have used the fact that ζ_m is equal to 1 on $\mathrm{K}[0, \frac{1}{m}]$. Therefore,

$$(\forall\, i \in 1..d) \qquad \lim_m \limsup_n \int\limits_{|\boldsymbol{\xi}|\leqslant \frac{1}{m\omega_n}} |\widehat{\varphi u^i_n}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} = 0\,,$$

and hence (u^i_n) is (ω_n)-concentratory, implying that the whole sequence is (ω_n)-concentratory.

The opposite implication follows by the estimate

$$(\forall\, i \in 1..d) \qquad \int\limits_{\mathbf{R}^d} |\widehat{\varphi u^i_n}(\boldsymbol{\xi})|^2 \zeta_m(\omega_n \boldsymbol{\xi}) \, d\boldsymbol{\xi} \leqslant \int\limits_{|\boldsymbol{\xi}|\leqslant \frac{2}{m\omega_n}} |\widehat{\varphi u^i_n}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi}\,,$$

and the equivalence established at the beginning of the proof.

Q.E.D.

In [10, Remark III.9] there is given a characterisation of the (ω_n)-oscillatory property stating that a sequence (u_n) possesses this property if for any test function $\varphi \in \mathrm{C}^\infty_c(\Omega)$ there exist $\boldsymbol{\alpha} \in \mathbf{N}^d_0 \setminus \{0\}$ and $C > 0$ such that $\omega_n^{|\boldsymbol{\alpha}|} \|\partial_{\boldsymbol{\alpha}}(\varphi \mathsf{u}_n)\|_{\mathrm{L}^2(\Omega;\mathbf{C}^r)} \leqslant C$. Here we present an extension of this sufficient condition which will also cover the just introduced (ω_n)-concentratory property.

Lemma 2. *A sequence (u_n) in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ is (ω_n)-oscillatory (concentratory) if for any test function $\varphi \in \mathrm{C}^\infty_c(\Omega)$ there exist $s > 0$ ($s < 0$) and $C > 0$ such that $\omega_n^s \|\varphi \mathsf{u}_n\|_{\mathrm{H}^s(\mathbf{R}^d;\mathbf{C}^r)} \leqslant C$.*

Dem. Let $\varphi \in \mathrm{C}^\infty_c(\Omega)$ be an arbitrary test function. Moreover, let us first study the case $s > 0$ and the (ω_n)-oscillatory property. By the assumption we have

$$\begin{aligned}
\int\limits_{|\boldsymbol{\xi}|\geqslant \frac{R}{\omega_n}} |\widehat{\varphi \mathsf{u}_n}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} &= R^{-2s} \int\limits_{|\boldsymbol{\xi}|\geqslant \frac{R}{\omega_n}} R^{2s} |\widehat{\varphi \mathsf{u}_n}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \leqslant R^{-2s} \int\limits_{|\boldsymbol{\xi}|\geqslant \frac{R}{\omega_n}} |\omega_n \boldsymbol{\xi}|^{2s} |\widehat{\varphi \mathsf{u}_n}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \\
&\leqslant R^{-2s} \omega_n^{2s} \int\limits_{\mathbf{R}^d} (1 + |\boldsymbol{\xi}|^2)^s |\widehat{\varphi \mathsf{u}_n}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \leqslant \frac{C^2}{R^{2s}}\,.
\end{aligned}$$

Since the estimate above is independent of n, and tends to zero as $R \to \infty$, we have that (u_n) is (ω_n)-oscillatory.

Furthermore, for $s < 0$ similarly as above we have the following estimate

$$\begin{aligned}
\int\limits_{|\boldsymbol{\xi}|\leqslant \frac{1}{R\omega_n}} |\widehat{\varphi \mathsf{u}_n}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} &= (\omega_n^2 + R^{-2})^{-s} \int\limits_{|\boldsymbol{\xi}|\leqslant \frac{1}{R\omega_n}} (\omega_n^2 + R^{-2})^s |\widehat{\varphi \mathsf{u}_n}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \\
&\leqslant (\omega_n^2 + R^{-2})^{-s} \int\limits_{|\boldsymbol{\xi}|\leqslant \frac{1}{R\omega_n}} (\omega_n^2 + |\omega_n \boldsymbol{\xi}|^2)^s |\widehat{\varphi \mathsf{u}_n}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \\
&\leqslant C^2 (\omega_n^2 + R^{-2})^{-s}\,,
\end{aligned} \tag{4}$$

which implies that (u_n) is (ω_n)-concentratory.

Q.E.D.

Now let us go back to the introductory example of an oscillating sequence and check whether it possesses (ω_n)-oscillatory and/or concentratory property.

Example 1. (oscillation) For $\varepsilon_n \to 0^+$ and $\mathsf{k} \in \mathbf{Z}^d \setminus \{0\}$ let us define $u_n(\mathbf{x}) := e^{2\pi i \frac{\mathsf{k}}{\varepsilon_n} \cdot \mathbf{x}}$. It is well known that this sequence converges weakly to zero in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$, but not strongly.

For an arbitrary $\varphi \in \mathrm{C}^\infty_c(\Omega)$, using $\widehat{\varphi u_n}(\boldsymbol{\xi}) = \hat{\varphi}\Big(\boldsymbol{\xi} - \frac{\mathsf{k}}{\varepsilon_n}\Big)$ and the transformation of variable $\boldsymbol{\eta} = \boldsymbol{\xi} - \frac{\mathsf{k}}{\varepsilon_n}$, for $s \in \mathbf{R}$ we have

$$\omega_n^{2s} \|\varphi u_n\|^2_{\mathrm{H}^s(\mathbf{R}^d)} = \int_{\mathbf{R}^d} |\hat{\varphi}(\boldsymbol{\eta})|^2 \left(\omega_n^2 + \left|\omega_n \boldsymbol{\eta} + \frac{\omega_n}{\varepsilon_n}\mathsf{k}\right|^2\right)^s d\boldsymbol{\eta}\,. \tag{5}$$

Therefore, for $s=1$ and (ω_n) not slower then (ε_n) we obtain (for n large enough)

$$\omega_n\|\varphi u_n\|_{\mathrm{H}^1(\mathbf{R}^d)} \leqslant C\|\varphi\|_{\mathrm{H}^1(\mathbf{R}^d)} < \infty\,,$$

thus by the previous lemma we have that (u_n) is (ω_n)-oscillatory for (ω_n) not slower then (ε_n).

In the next step we take $s=-d$ and assume that (ω_n) is not faster then (ε_n). Moreover, we separate the integral in (5) into two parts, taken over the closed ball $\mathrm{K}[0,\frac{|\mathsf{k}|}{2\varepsilon_n}]$ and its complement. For the first part we have

$$\int_{\mathrm{K}[0,\frac{|\mathsf{k}|}{2\varepsilon_n}]} \frac{|\hat\varphi(\boldsymbol\eta)|^2}{\Bigl(\omega_n^2+|\omega_n\boldsymbol\eta+\frac{\omega_n}{\varepsilon_n}\mathsf{k}|^2\Bigr)^d}\,d\boldsymbol\eta \leqslant \Bigl(\frac{1}{C|\mathsf{k}|}\Bigr)^{2d}\|\varphi\|^2_{\mathrm{L}^2(\mathbf{R}^d)} < \infty\,,$$

where we have used that $|\omega_n\boldsymbol\eta+\frac{\omega_n}{\varepsilon_n}\mathsf{k}| \geqslant C|\mathsf{k}|$ on $\mathrm{K}[0,\frac{|\mathsf{k}|}{2\varepsilon_n}]$ for some constant $C>0$.

For the complement we shall use that $\hat\varphi\in\mathcal{S}(\mathbf{R}^d)$, implying that $|\boldsymbol\eta|^d|\hat\varphi(\boldsymbol\eta)|$ is a bounded function. Therefore, we have

$$\begin{aligned}
\int_{c\mathrm{K}[0,\frac{|\mathsf{k}|}{2\varepsilon_n}]} \frac{|\hat\varphi(\boldsymbol\eta)|^2}{\Bigl(\omega_n^2+|\omega_n\boldsymbol\eta+\frac{\omega_n}{\varepsilon_n}\mathsf{k}|^2\Bigr)^d}\,d\boldsymbol\eta &= \int_{c\mathrm{K}[0,\frac{|\mathsf{k}|}{2\varepsilon_n}]} \frac{|\boldsymbol\eta|^{2d}|\hat\varphi(\boldsymbol\eta)|^2}{|\boldsymbol\eta|^{2d}\Bigl(\omega_n^2+|\omega_n\boldsymbol\eta+\frac{\omega_n}{\varepsilon_n}\mathsf{k}|^2\Bigr)^d}\,d\boldsymbol\eta\\
&\leqslant C_\varphi\int_{c\mathrm{K}[0,\frac{|\mathsf{k}|}{2\varepsilon_n}]} \frac{d\boldsymbol\eta}{(\frac{|\mathsf{k}|}{2})^{2d}(\frac{\omega_n}{\varepsilon_n})^{2d}\Bigl(1+|\boldsymbol\eta+\frac{\mathsf{k}}{\varepsilon_n}|^2\Bigr)^d}\\
&\leqslant \tilde C_\varphi\int_{\mathbf{R}^d}\frac{d\boldsymbol\xi}{(1+|\boldsymbol\xi|^2)^d} < \infty\,,
\end{aligned}$$

resulting that $\omega_n^{-d}\|\varphi u_n\|_{\mathrm{H}^{-d}(\mathbf{R}^d)}$ is bounded. Hence, by the previous lemma we have that (u_n) is (ω_n)-concentratory for (ω_n) not faster then (ε_n).

The above analysis implies that the oscillating sequence (u_n) possesses both the (ω_n)-oscillatory and the (ω_n)-concentratory property if (ω_n) is of the same order as (ε_n). Since the previous lemma provides only sufficient conditions, in order to conclude that there is no other scale implying both the properties simultaneously, we still need to use Theorem 4 below under the fact that (u_n) is not strongly convergent. ∎

As concentration provides the second standard prototype of a sequences converging weakly but not strongly, we consider concentrating sequences as the next example. Like in the previous one, we study for which scale (ω_n) such a sequence is (ω_n)-oscillatory and/or (ω_n)-concentratory, but unlike the approach based on Lemma 2 here we shall use a direct one, based on the very definitions.

Example 2. (concentration) For given $v\in\mathrm{L}^2(\mathbf{R}^d)$, $\varepsilon_n\to0^+$, and $\mathbf{x}_0\in\mathbf{R}^d$ we define

$$u_n(x) := \varepsilon_n^{-d/2}v\Bigl(\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon_n}\Bigr)\,.$$

It is easy to see that (u_n) is bounded in $\mathrm{L}^2(\mathbf{R}^d)$ and that converges weakly to zero. Hence, let us examine for which scale (ω_n) this sequence is (ω_n)-oscillatory and/or (ω_n)-concentratory.

For $\varphi\in\mathrm{C}_c^\infty(\mathbf{R}^d)$ by the Lebesgue dominated convergence theorem we have $\varphi u_n-\varphi(\mathbf{x}_0)u_n \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^d)$. Indeed, under the change of variables given by $\mathbf{y}=\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon_n}$ we have

$$\int_{\mathbf{R}^d}|\varphi(\mathbf{x})-\varphi(\mathbf{x}_0)|^2|u_n(\mathbf{x})|^2\,d\mathbf{x} = \int_{\mathbf{R}^d}|\varphi(\varepsilon_n\mathbf{y}+\mathbf{x}_0)-\varphi(\mathbf{x}_0)|^2|v(\mathbf{y})|^2\,d\mathbf{y}\,,$$

and the application of the Lebesgue dominated convergence theorem is justified as $\varphi(\varepsilon_n\mathbf{y}+\mathbf{x}_0)-\varphi(\mathbf{x}_0)\longrightarrow0$ pointwise, and $\|\varphi\|^2_{\mathrm{L}^\infty(\mathbf{R}^d)}|v(\cdot)|^2$ is integrable.

This implies that for the (ω_n)-oscillatory property it is sufficient to study

$$\int\limits_{|\boldsymbol{\xi}|\geqslant\frac{R}{\omega_n}} |\hat{u}_n(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} = \varepsilon_n^d \int\limits_{|\boldsymbol{\xi}|\geqslant\frac{R}{\omega_n}} |\hat{v}(\varepsilon_n\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} = \int\limits_{|\boldsymbol{\eta}|\geqslant R\frac{\varepsilon_n}{\omega_n}} |\hat{v}(\boldsymbol{\eta})|^2\,d\boldsymbol{\eta}\,,$$

where in the first equality we have used $\hat{u}_n(\boldsymbol{\xi}) = \varepsilon_n^{d/2} e^{-2\pi i\boldsymbol{\xi}\cdot\mathbf{x}_0}\hat{v}(\varepsilon_n\boldsymbol{\xi})$. Therefore, by the last integral it is clear that (u_n) is (ω_n)-oscillatory for and only for scales (ω_n) which are not slower then (ε_n). Analogously, (u_n) is (ω_n)-concentratory if and only if (ω_n) is not faster then (ε_n). ∎

From the last two examples it is obvious that both notions, (ω_n)-oscillatory and (ω_n)-concentratory, turn out to be a bit confusing, as an oscillating sequence may posses both properties, same as a concentrating one. However, as the notion of (ω_n)-oscillatory sequences is already well established, we adjust to the existing framework, and denominate the new notion as (ω_n)-concentratory.

We finish this section by two lemmas providing some basic properties of the considered notions. The first result follows directly from the very Definitions 1 and 2.

Lemma 3. *If $\mathsf{u}_n \in \mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ is (ω_n)-oscillatory (concentratory) then it is also $(\tilde{\omega}_n)$-oscillatory (concentratory) for any scale $(\tilde{\omega}_n)$ which is not slower (not faster) then (ω_n).* ∎

Next result implies linearity of (ω_n)-oscillatory and concentratory properties.

Lemma 4. *Let (u_n) and (v_n) be both (ω_n)-oscillatory (concentratory). Then the sum $(\mathsf{u}_n + \mathsf{v}_n)$ is also (ω_n)-oscillatory (concentratory).*

Dem. The claim trivially follows by the triangular inequality. Indeed, as both (u_n) and (v_n) are (ω_n)-oscillatory, by the estimate

$$\int_{|\boldsymbol{\xi}|\leqslant\frac{R}{\omega_n}} |\widehat{\varphi\mathsf{u}_n} + \widehat{\varphi\mathsf{v}_n}|^2\,d\boldsymbol{\xi} \leqslant \left(\sqrt{\int_{|\boldsymbol{\xi}|\leqslant\frac{R}{\omega_n}} |\widehat{\varphi\mathsf{u}_n}|^2\,d\boldsymbol{\xi}} + \sqrt{\int_{|\boldsymbol{\xi}|\leqslant\frac{R}{\omega_n}} |\widehat{\varphi\mathsf{v}_n}|^2\,d\boldsymbol{\xi}}\right)^2\,,$$

we have that $(\mathsf{u}_n + \mathsf{v}_n)$ is (ω_n)-oscillatory.

Analogously for the (ω_n)-concentratory property.

Q.E.D.

3. Characteristic scale of a sequence

Since the introduction of semiclassical measures a natural question arose on their relation to H-measures — whether one object can be reconstructed from the other one, an issue which launched interesting academic discussions (cf. [13, Chapter 32], [10, Remark III.11]). Here we provide a well known result on the relation between the above objects (cf. [4, Lemma 3.4(ii)]), restated in terms of an introduced notion by means of Theorem 3.

Corollary 1. *Let (u_n) be a bounded sequence in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ such that it is (ω_n)-pure, (ω_n)-oscillatory and (ω_n)-concentratory for some $\omega_n \to 0^+$, and let $\boldsymbol{\mu}_{sc} = \boldsymbol{\mu}_{sc}^{(\omega_n)}$ be the corresponding semiclassical measure with the scale (ω_n).*

Then $\mathsf{u}_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$, it is pure, and for any choice of test functions $\varphi \in \mathrm{C}^\infty_c(\Omega)$ and $\psi \in \mathrm{C}^\infty(\mathrm{S}^{d-1})$ we have

$$\langle \boldsymbol{\mu}_H, \varphi \boxtimes \psi\rangle = \langle \boldsymbol{\mu}_{sc}, \varphi \boxtimes (\psi\circ\boldsymbol{\pi})\rangle\,,$$

where $\boldsymbol{\mu}_H$ is the H-measure associated to (u_n). ∎

In previous papers this result required semiclassical measure not to be supported in the origin of the dual space, i.e. that $\mathsf{tr}\boldsymbol{\mu}_{sc}^{(\omega_n)}(\Omega\times\{0\}) = 0$ — an assumption, as shown by Theorem 3, which is equivalent to the introduced (ω_n)-concentratory property. Unlike the first one, the latter property is stated in terms of a sequence only, and does not require knowledge of a measure, just like the (ω_n)-oscillatory property.

The last corollary, as well as Lemma 1 and Theorem 3, demonstrates that the best choice of a semiclassical measure is obtained by taking its scale (ω_n) such that the sequence under consideration is both (ω_n)-oscillatory and (ω_n)-concentratory, as such selection will prevent leaking of energy at infinity, as well as mixing of information at the origin of the dual space.

For this reason we define such a scale as a *characteristic scale of a sequence* (u_n). Next theorem shows that the notion is well defined, and that a sequence cannot have two characteristic scales of different orders, unless it converges strongly to zero.

Theorem 4. *Let $\mathsf{u}_n \in \mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ be (ω_n)-oscillatory and (ω_n)-concentratory for $\omega_n \to 0^+$. The following is equivalent:*

a) There exists $\tilde\omega_n \to 0^+$ slower then (ω_n) for which (u_n) is $(\tilde\omega_n)$-oscillatory.

b) There exists $\tilde\omega_n \to 0^+$ faster then (ω_n) for which (u_n) is $(\tilde\omega_n)$-concentratory.

c) $\mathsf{u}_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$.

Dem. The condition (c) trivially implies (a) and (b) by the following inequalities:

$$\int_{|\boldsymbol{\xi}|\geqslant\frac{R}{\omega_n}} |\widehat{\varphi\mathsf{u}_n}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} \leqslant \|\varphi\mathsf{u}_n\|^2_{\mathrm{L}^2(\Omega;\mathbf{C}^r)} \qquad\text{and}\qquad \int_{|\boldsymbol{\xi}|\leqslant\frac{1}{R\omega_n}} |\widehat{\varphi\mathsf{u}_n}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} \leqslant \|\varphi\mathsf{u}_n\|^2_{\mathrm{L}^2(\Omega;\mathbf{C}^r)}\,,$$

where $\varphi\in\mathrm{C}^\infty_c(\Omega)$ is arbitrary.

Let us prove that (a) implies (c). For an arbitrary $\varepsilon>0$ and $\varphi\in\mathrm{C}^\infty_c(\Omega)$ there exist $R>0$ such that

$$\limsup_n \int\limits_{|\boldsymbol{\xi}|\geqslant\frac{R}{\tilde\omega_n}} |\widehat{\varphi\mathsf{u}_n}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} < \frac{\varepsilon}{2} \qquad\text{and}\qquad \limsup_n \int\limits_{|\boldsymbol{\xi}|\leqslant\frac{1}{R\omega_n}} |\widehat{\varphi\mathsf{u}_n}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} < \frac{\varepsilon}{2}\,.$$

Further on, let $n_0\in\mathbf{N}$ be such that for any $n\geqslant n_0$ we have $\frac{\omega_n}{\tilde\omega_n} < \frac{1}{R^2}$. Hence, for any $n\geqslant n_0$ we get

$$\int\limits_{|\boldsymbol{\xi}|\geqslant\frac{R}{\tilde\omega_n}} |\widehat{\varphi\mathsf{u}_n}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} = \int\limits_{|\boldsymbol{\xi}|\geqslant\frac{R}{\omega_n}\frac{\omega_n}{\tilde\omega_n}} |\widehat{\varphi\mathsf{u}_n}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} \geqslant \int\limits_{|\boldsymbol{\xi}|\geqslant\frac{1}{R\omega_n}} |\widehat{\varphi\mathsf{u}_n}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}\,,$$

implying

$$\limsup_n \int\limits_{|\boldsymbol{\xi}|\geqslant\frac{1}{R\omega_n}} |\widehat{\varphi\mathsf{u}_n}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} \leqslant \limsup_n \int\limits_{|\boldsymbol{\xi}|\geqslant\frac{R}{\tilde\omega_n}} |\widehat{\varphi\mathsf{u}_n}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} < \frac{\varepsilon}{2}\,.$$

Finally, we have

$$\limsup_n \|\varphi\mathsf{u}_n\|^2_{\mathrm{L}^2(\Omega;\mathbf{C}^r)} \leqslant \limsup_n \int\limits_{|\boldsymbol{\xi}|\leqslant\frac{1}{R\omega_n}} |\widehat{\varphi\mathsf{u}_n}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} + \limsup_n \int\limits_{|\boldsymbol{\xi}|\geqslant\frac{1}{R\omega_n}} |\widehat{\varphi\mathsf{u}_n}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} < \varepsilon\,,$$

so the claim follows by the arbitrariness of ε and φ.

The proof that (b) implies (c) goes in the same manner.

Q.E.D.

A natural question arising at this point is whether an arbitrary bounded L^2 sequence possess its characteristic scale, whose positive answer would provide the existence of a corresponding semiclassical measure capable to capture all the microlocal information. However, the next example demonstrates this is not the case.

Example 3. For $\varepsilon_n \to 0^+$ and $\mathsf{k}, \mathsf{s} \in \mathbf{Z}^d \setminus \{0\}$ let us define sequences $u_n(\mathbf{x}) := e^{2\pi i \frac{\mathsf{k}}{\varepsilon_n}\cdot \mathbf{x}}$ and $v_n(\mathbf{x}) := e^{2\pi i \frac{\mathsf{s}}{\varepsilon_n^2}\cdot \mathbf{x}}$. Let us show that the sum (u_n+v_n) does not have a characteristic scale, i.e. that we cannot find $\omega_n \to 0^+$ such that $(u_n + v_n)$ is both (ω_n)-oscillatory and (ω_n)-concentratory.

According to Example 1 we have that a characteristic scale of (u_n) is (ε_n), while for (v_n) we can take (ε_n^2). Moreover, by Lemma 3 (u_n) is also (ε_n^2)-oscillatory, while (v_n) is (ε_n)-concentratory, implying that, by Lemma 4, $(u_n + v_n)$ is (ε_n^2)-oscillatory and (ε_n)-concentratory.

Let us assume that $(u_n + v_n)$ has a characteristic scale (ω_n). As $(u_n + v_n)$ does not converge strongly, by Theorem 4 there should exist constants $m, M > 0$ such that for n large enough we have $m\varepsilon_n^2 \leqslant \omega_n \leqslant M\varepsilon_n$.

For such (ω_n), again by Lemma 3, (u_n) is (ω_n)-oscillatory and (v_n) is (ω_n)-concentratory. Applying Lemma 4 to simple identities $u_n = (u_n + v_n) + (-v_n)$ and $v_n = (u_n + v_n) + (-u_n)$, we obtain that (u_n) is (ω_n)-concentratory and (v_n) is (ω_n)-oscillatory. However, by Example 1 we have that it is necessary that (ω_n) is not faster then (ε_n) and that it is not slower then (ε_n^2), which leads to an obvious contradiction.

∎

For semiclassical measures associated to the sequence (u_n+v_n) from the above example it can be shown that by choosing different characteristic scales, they can capture at most one frequency scale, ε_n or ε_n^2, but not both of them simultaneously (cf. [2, Example 2]).

Even a more interesting example provides the next sequence that incorporates an infinite number of frequency scales, for which associated semiclassical measures turn out as incapable to capture even a single one, regardless of a chosen scale.

Example 4. Let a sequence (u_n) be defined by the relation

$$u_n(\mathbf{x}) := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} e^{2\pi i n^j \mathsf{k}\cdot\mathbf{x}}\,, \tag{6}$$

for $\mathsf{k} \in \mathbf{Z}^d \setminus \{0\}$.

We split analysis of this sequence into several parts.

I. The sequence converges weakly to zero in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$, but not strongly.

As u_n are periodic functions, the boundedness follows easily by the computation:

$$\int_{[0,1]^d} |u_n(\boldsymbol{\xi})|^2\, d\mathbf{x} = \frac{1}{n}\sum_{j,l=1}^{n} \int_{[0,1]^d} e^{2\pi i (n^j - n^l)\mathsf{k}\cdot\mathbf{x}}\, d\mathbf{x} = \frac{1}{n}\sum_{j=1}^{n} 1 = 1\,.$$

Moreover, from the above it is clear that (u_n) does not converge strongly to zero.

By integration by parts for $\varphi \in \mathrm{C}^\infty_c(\mathbf{R}^d)$ and $\mathsf{s} \in \mathbf{Z} \setminus \{0\}$ we have

$$|\langle e^{2\pi i \mathsf{s}\cdot\mathbf{x}}, \varphi\rangle| \leqslant \frac{C_\varphi}{|\mathsf{s}|}\,,$$

where $C_\varphi = \frac{\sqrt{d}\|\nabla\varphi\|_{\mathrm{L}^1(\mathbf{R}^d)}}{2\pi}$. Therefore,

$$|\langle u_n, \varphi\rangle| \leqslant \frac{1}{\sqrt{n}}\sum_{j=1}^{n} |\langle e^{2\pi i n^j \mathsf{k}\cdot\mathbf{x}}, \varphi\rangle| \leqslant \frac{C_\varphi}{\sqrt{n}}\sum_{j=1}^{n}\frac{1}{|n^j\mathsf{k}|} \leqslant \frac{C_\varphi}{|\mathsf{k}|\sqrt{n}}\sum_{j=1}^{n}\frac{1}{n} = \frac{C_\varphi}{|\mathsf{k}|\sqrt{n}}$$

tends to zero as $n \to \infty$. Thus, by the density of $\mathrm{C}^\infty_c(\mathbf{R}^d)$ in $\mathrm{L}^2_c(\mathbf{R}^d)$ and the boundedness of (u_n), we have $u_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$.

Therefore, this sequence is suitable for applications of both semiclassical and H-measures, and below we present a derivation of these objects.

II. Approximation of the terms $\int_{\mathbf{R}^d}|\widehat{\varphi u_n}(\boldsymbol{\xi})|^2\psi(\omega_n\boldsymbol{\xi})\,d\boldsymbol{\xi}$ and $\int_{\mathbf{R}^d}|\widehat{\varphi u_n}(\boldsymbol{\xi})|^2\psi\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big)\,d\boldsymbol{\xi}$.

Let $\varphi\in\mathrm{C}_c^\infty(\mathbf{R}^d)$, $\varphi\neq 0$, and $\psi\in\mathcal{S}(\mathbf{R}^d)$. For $\omega_n\to 0^+$ we have

$$\begin{aligned}\int_{\mathbf{R}^d}|\widehat{\varphi u_n}(\boldsymbol{\xi})|^2\psi(\omega_n\boldsymbol{\xi})\,d\boldsymbol{\xi}=&\frac{1}{n}\sum_{j,l=1}^n\int_{\mathbf{R}^d}\hat\varphi(\boldsymbol{\xi}-n^j\mathsf{k})\bar{\hat\varphi}(\boldsymbol{\xi}-n^l\mathsf{k})\psi(\omega_n\boldsymbol{\xi})\,d\boldsymbol{\xi}\\ =&\frac{1}{n}\sum_{j,l=1}^n\int_{\mathbf{R}^d}\hat\varphi(\boldsymbol{\eta})\bar{\hat\varphi}\Big(\boldsymbol{\eta}+(n^j-n^l)\mathsf{k}\Big)\psi(\omega_n\boldsymbol{\eta}+\omega_n n^j\mathsf{k})\,d\boldsymbol{\eta}\,,\end{aligned}$$

where in the second equality we have used the change of variables $\boldsymbol{\eta}=\boldsymbol{\xi}-n^j\mathsf{k}$.

Let us first prove that terms for $j\neq l$ tends to zero as n tends to infinity. For $j\neq l$ and $|\boldsymbol{\eta}|\leqslant\frac{n|\mathsf{k}|}{2}$ we have $|\boldsymbol{\eta}+(n^j-n^l)\mathsf{k}|\geqslant n|\mathsf{k}|-|\boldsymbol{\eta}|\geqslant\frac{n|\mathsf{k}|}{2}$, which together with the integrability of $|\boldsymbol{\eta}|^4|\hat\varphi(\boldsymbol{\eta})|^2$ (as $\hat\varphi\in\mathcal{S}(\mathbf{R}^d)$), by the Hölder inequality implies

$$\begin{aligned}\frac{1}{n}\sum_{j\neq l}\int_{\mathrm{K}[0,\frac{n|\mathsf{k}|}{2}]}\frac{|\hat\varphi(\boldsymbol{\eta})|}{|\boldsymbol{\eta}+(n^j-n^l)\mathsf{k}|^2}\Big||\boldsymbol{\eta}+(n^j-n^l)\mathsf{k}|^2\hat\varphi\Big(\boldsymbol{\eta}+(n^j-n^l)\mathsf{k}\Big)\Big||\psi(\omega_n\boldsymbol{\eta}+\omega_n n^j\mathsf{k})|\,d\boldsymbol{\eta}&\\ \leqslant\frac{C}{n^3}\sum_{j\neq l}1\leqslant\frac{C}{n}\,,&\end{aligned}$$

where C depends only on φ,ψ and k. It is left to estimate the above terms when the integration is over the complement of $\mathrm{K}[0,\frac{n|\mathsf{k}|}{2}]$, for which we analogously get

$$\begin{aligned}\frac{1}{n}\sum_{j\neq l}\int_{c\mathrm{K}[0,\frac{n|\mathsf{k}|}{2}]}\frac{1}{|\boldsymbol{\eta}|^2}\Big||\boldsymbol{\eta}|^2\hat\varphi(\boldsymbol{\eta})\Big|\Big|\hat\varphi\Big(\boldsymbol{\eta}+(n^j-n^l)\mathsf{k}\Big)\Big||\psi(\omega_n\boldsymbol{\eta}+\omega_n n^j\mathsf{k})|\,d\boldsymbol{\eta}&\\ \leqslant\frac{C}{n^3}\sum_{j\neq l}1\leqslant\frac{C}{n}\,.&\end{aligned}$$

Therefore, it is sufficient to study the limit of

$$\frac{1}{n}\sum_{j=1}^n\int_{\mathbf{R}^d}|\hat\varphi(\boldsymbol{\eta})|^2\psi(\omega_n\boldsymbol{\eta}+\omega_n n^j\mathsf{k})\,d\boldsymbol{\eta}\,.$$

Moreover, we can further simplify the above expression by considering only the integration over a compact set. Indeed, for $M>0$ we have

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^n\int_{c\mathrm{K}[0,M]}|\hat\varphi(\boldsymbol{\eta})|^2|\psi(\omega_n\boldsymbol{\eta}+\omega_n n^j\mathsf{k})|\,d\boldsymbol{\eta}=\frac{1}{n}\sum_{j=1}^n\int_{c\mathrm{K}[0,M]}\frac{1}{|\boldsymbol{\eta}|^2}\Big|\boldsymbol{\eta}|\hat\varphi(\boldsymbol{\eta})\Big|^2|\psi(\omega_n\boldsymbol{\eta}+\omega_n n^j\mathsf{k})|\,d\boldsymbol{\eta}&\\ \leqslant\frac{C}{M^2}\,,&\end{aligned}$$

where we have again used that $\hat\varphi\in\mathcal{S}(\mathbf{R}^d)$, which implies that $|\cdot|\hat\varphi(\cdot)$ is square integrable.

Finally, in order to compute associated semiclassical measures it is left to study

$$\frac{1}{n}\sum_{j=1}^n\int_{\mathrm{K}[0,M]}|\hat\varphi(\boldsymbol{\eta})|^2\psi(\omega_n\boldsymbol{\eta}+\omega_n n^j\mathsf{k})\,d\boldsymbol{\eta}\,,\tag{7}$$

for some $M>0$.

Similarly, we can show that for $\varphi\in\mathrm{C}_c^\infty(\Omega)$ and $\psi\in\mathrm{C}(\mathrm{S}^{d-1})$ the limit of

$$\int_{\mathbf{R}^d}|\widehat{\varphi u_n}(\boldsymbol{\xi})|^2\psi\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big)\,d\boldsymbol{\xi}$$

is arbitrarily close (by choosing $M > 0$ large enough) to the limit of

$$\frac{1}{n}\sum_{j=1}^{n}\int_{\mathrm{K}[0,M]}|\hat{\varphi}(\boldsymbol{\eta})|^2\psi\Big(\frac{\boldsymbol{\eta}+n^j\mathsf{k}}{|\boldsymbol{\eta}+n^j\mathsf{k}|}\Big)\,d\boldsymbol{\eta}\,. \tag{8}$$

III. Derivation of semiclassical measures.

Let us first assume the case where there exists $j_0 \in \mathbf{N}$ such that (ω_n) is slower than $(\frac{1}{n^{j_0}})$, i.e. that $\lim_n n^{j_0}\omega_n = \infty$. Fix $\varepsilon > 0$ and take $R > 0$ such that for $|\boldsymbol{\xi}| > R$ we have $|\psi(\boldsymbol{\xi})| \leqslant \frac{\varepsilon}{\|\varphi\|^2_{\mathrm{L}^2(\mathbf{R}^d)}}$. Moreover, for n large enough we have $n^{j_0}\omega_n > \frac{R+M}{|\mathsf{k}|}$ and $\omega_n < 1$. Thus for $j \geqslant j_0$ and $|\boldsymbol{\eta}| \leqslant M$ it holds

$$|\omega_n\boldsymbol{\eta}+\omega_n n^j\mathsf{k}| \geqslant \omega_n n^j|\mathsf{k}| - \omega_n|\boldsymbol{\eta}| > R+M-M = R\,,$$

implying $|\psi(\omega_n\boldsymbol{\eta}+\omega_n n^j\mathsf{k})| \leqslant \frac{\varepsilon}{\|\varphi\|^2_{\mathrm{L}^2(\mathbf{R}^d)}}$. Therefore,

$$\begin{aligned}
\lim_n\Big|\frac{1}{n}\sum_{j=1}^{n}\int_{\mathrm{K}[0,M]}|\hat{\varphi}(\boldsymbol{\eta})|^2\psi(\omega_n\boldsymbol{\eta}+\omega_n n^j\mathsf{k})\,d\boldsymbol{\eta}\Big| &\leqslant \limsup_n\frac{1}{n}\sum_{j=1}^{j_0-1}\int_{\mathrm{K}[0,M]}|\hat{\varphi}(\boldsymbol{\eta})|^2|\psi(\omega_n\boldsymbol{\eta}+\omega n^j\mathsf{k})|\,d\boldsymbol{\eta}\\
&\quad+\limsup_n\frac{1}{n}\sum_{j=j_0}^{n}\int_{\mathrm{K}[0,M]}|\hat{\varphi}(\boldsymbol{\eta})|^2|\psi(\omega_n\boldsymbol{\eta}+\omega n^j\mathsf{k})|\,d\boldsymbol{\eta}\\
&\leqslant \lim_n\frac{j_0-1}{n}\|\psi\|_{\mathrm{L}^\infty(\mathbf{R}^d)}\|\varphi\|^2_{\mathrm{L}^2(\mathbf{R}^d)}+\lim_n\frac{n-j_0+1}{n}\varepsilon = \varepsilon\,,
\end{aligned}$$

resulting in the trivial semiclassical measure, $\mu_{sc} = 0$.

It is left to examine the case in which (ω_n) is faster then any (n^{-j}) scale, i.e. for any $j \in \mathbf{N}$ we have $\lim_n n^j\omega_n = 0$. For $\varepsilon > 0$ we choose $R, r > 0$ such that for $|\boldsymbol{\xi}| > R$ we have $|\psi(\boldsymbol{\xi})| \leqslant \frac{\varepsilon}{2\|\varphi\|^2_{\mathrm{L}^2(\mathbf{R}^d)}}$, while for $|\boldsymbol{\xi}| < r$ we have $|\psi(\boldsymbol{\xi})-\psi(\mathsf{0})| \leqslant \frac{\varepsilon}{2\|\varphi\|^2_{\mathrm{L}^2(\mathbf{R}^d)}}$.

Take n large enough such that $n\omega_n < \frac{r}{2|k|}$, and define $j_n := \max\{j \in 1..n : n^j\omega_n < \frac{r}{2|\mathsf{k}|}\}$. As for every $j \in \mathbf{N}$ we have $n^j\omega_n \longrightarrow 0$, it follows that $j_n \to \infty$. Since for n large enough we have $\omega_n < \frac{r}{2M}$, for $j \leqslant j_n$ it follows $|\omega_n\boldsymbol{\eta}+\omega_n n^j\mathsf{k}| < r$, implying $|\psi(\omega_n\boldsymbol{\eta}+\omega_n n^j\mathsf{k})-\psi(\mathsf{0})| \leqslant \frac{\varepsilon}{2\|\varphi\|^2_{\mathrm{L}^2(\mathbf{R}^d)}}$. On the other hand, if $j_n < n$, then $n^{j_n+2}\omega_n \geqslant \frac{r}{2|\mathsf{k}|}n$, and for $j \geqslant j_n+2$ we have $|\omega_n\boldsymbol{\eta}+\omega_n n^j\mathsf{k}| > R$, hence $|\psi(\omega_n\boldsymbol{\eta}+\omega_n n^j\mathsf{k})| \leqslant \frac{\varepsilon}{2\|\varphi\|^2_{\mathrm{L}^2(\mathbf{R}^d)}}$.

Therefore,

$$\begin{aligned}
&\lim_n\Big|\frac{1}{n}\sum_{j=1}^{n}\int_{\mathrm{K}[0,M]}|\hat{\varphi}(\boldsymbol{\eta})|^2\psi(\omega_n\boldsymbol{\eta}+\omega_n n^j\mathsf{k})\,d\boldsymbol{\eta}-\frac{j_n}{n}\psi(\mathsf{0})\|\varphi\|^2_{\mathrm{L}^2(\mathbf{R}^d)}\Big|\\
&\qquad\leqslant \limsup_n\frac{1}{n}\sum_{j=1}^{j_n}\int_{\mathrm{K}[0,M]}|\hat{\varphi}(\boldsymbol{\eta})|^2|\psi(\omega_n\boldsymbol{\eta}+\omega_n n^j\mathsf{k})-\psi(\mathsf{0})|\,d\boldsymbol{\eta}\\
&\qquad\quad+\limsup_n\frac{1}{n}\int_{\mathrm{K}[0,M]}|\hat{\varphi}(\boldsymbol{\eta})|^2|\psi(\omega_n\boldsymbol{\eta}+\omega_n n^{j_n+1}\mathsf{k})|\,d\boldsymbol{\eta}\\
&\qquad\quad+\limsup_n\frac{1}{n}\sum_{j=j_n+2}^{n}\int_{\mathrm{K}[0,M]}|\hat{\varphi}(\boldsymbol{\eta})|^2|\psi(\omega_n\boldsymbol{\eta}+\omega_n n^j\mathsf{k})|\,d\boldsymbol{\eta}\\
&\qquad\leqslant \limsup_n\frac{j_n}{n}\frac{\varepsilon}{2}+\lim_n\frac{1}{n}\|\psi\|_{\mathrm{L}^\infty(\mathbf{R}^d)}\|\varphi\|^2_{\mathrm{L}^2(\mathbf{R}^d)}+\limsup_n\frac{n-j_n-1}{n}\frac{\varepsilon}{2}\leqslant\varepsilon\,,
\end{aligned}$$

where the third term after the first inequality is equal to zero if $j_n \geqslant n-1$. Hence, it is sufficient to study $\frac{j_n}{n}\psi(\mathsf{0})\|\varphi\|^2_{\mathrm{L}^2(\mathbf{R}^d)}$. As $(\frac{j_n}{n})$ is bounded (by 1), we can pass to a subsequence (not relabeled) converging to $C\psi(\mathsf{0})\|\varphi\|^2_{\mathrm{L}^2(\mathbf{R}^d)}$, where $C := \lim_n\frac{j_n}{n} \in [0,1]$, obtaining semiclassical measure $\mu_{sc} = C\lambda \boxtimes \delta_0$. Moreover, if $\lim_n \frac{j_n}{n} > 0$ we can tell that (ω_n) is not slower than $(n^{-\tilde{C}n})$, for any $0 < \tilde{C} < C$, while it is slower of any $(n^{-\tilde{C}n})$, $\tilde{C} > 0$, if $\lim_n \frac{j_n}{n} = 0$.

At the end, we can summarise that all semiclassical measures associated to the sequence (6) are of the form $C\psi(0)\|\varphi\|^2_{\mathrm{L}^2(\mathbf{R}^d)}$, where $C \in [0,1]$. Thus, in any case we cannot recover the direction of oscillations from semiclassical measures.

IV. Derivation of the H-measure.

Let us take φ as above and $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$ arbitrarily. As for $|\boldsymbol{\eta}| \leqslant M$ and $n > \frac{2M}{|\mathsf{k}|}$ we have

$$
\begin{aligned}
\Big|\frac{\boldsymbol{\eta}+n^j\mathsf{k}}{|\boldsymbol{\eta}+n^j\mathsf{k}|} - \frac{\mathsf{k}}{|\mathsf{k}|}\Big| &\leqslant \Big|\frac{\boldsymbol{\eta}+n^j\mathsf{k}}{|\boldsymbol{\eta}+n^j\mathsf{k}|} - \frac{n^j\mathsf{k}}{|\boldsymbol{\eta}+n^j\mathsf{k}|}\Big| + \Big|\frac{n^j\mathsf{k}}{|\boldsymbol{\eta}+n^j\mathsf{k}|} - \frac{n^j\mathsf{k}}{|n^j\mathsf{k}|}\Big| \\
&\leqslant 2\frac{|\boldsymbol{\eta}|}{n^j|\mathsf{k}| - |\boldsymbol{\eta}|} \\
&\leqslant 2\frac{M}{n|\mathsf{k}| - M} \leqslant \frac{4M}{n|\mathsf{k}|}\,,
\end{aligned}
$$

by the uniform continuity of ψ it follows

$$
\begin{aligned}
\lim_n \Big|\frac{1}{n}\sum_{j=1}^n \int_{\mathrm{K}[0,M]} |\hat{\varphi}(\boldsymbol{\eta})|^2 \psi\Big(\frac{\boldsymbol{\eta}+n^j\mathsf{k}}{|\boldsymbol{\eta}+n^j\mathsf{k}|}\Big)\, d\boldsymbol{\eta} - \psi\Big(\frac{\mathsf{k}}{|\mathsf{k}|}\Big)\|\varphi\|^2_{\mathrm{L}^2(\mathbf{R}^d)}\Big| \\
\leqslant \|\varphi\|^2_{\mathrm{L}^2(\mathbf{R}^d)} \lim_n \max_{\boldsymbol{\eta}\in\mathrm{K}[0,M]} \max_{j\in\mathbf{N}} \Big|\psi\Big(\frac{\boldsymbol{\eta}+n^j\mathsf{k}}{|\boldsymbol{\eta}+n^j\mathsf{k}|}\Big) - \psi\Big(\frac{\mathsf{k}}{|\mathsf{k}|}\Big)\Big| = 0\,,
\end{aligned}
$$

implying that (u_n) is pure and the associated H-measure is given by $\lambda \boxtimes \delta_{\frac{\mathsf{k}}{|\mathsf{k}|}}$. Since the H-measure contains information about the direction of oscillations k, which was not the case with semiclassical measures, this is an example in which we cannot, by any means, recover the H-measure starting from corresponding semiclassical measures. ∎

Remark. The same conclusions derived above for semiclassical measures remain valid if directions of oscillations in (6) do not coincide, while an associated H-measure in that case has its support within $\mathbf{R}^d \times K$, where K is a closure in the unit sphere of the set $\{\frac{\mathsf{k}_j}{|\mathsf{k}_j|}, j \in \mathbf{N}\}$, comprising all propagation directions. ∎

Corollary 1 and the above examples demonstrate that semiclassical measures are a preferable tool for study of a sequences which possess characteristic scale. In that case they capture essential microlocal properties (amplitude, frequency and direction of propagations), and H-measures can be reconstructed from them by averaging information along the rays in the frequency domain.

However, if a sequence under consideration does not allow for a characteristic scale, or if it is unknown, H-measures turn out as a better tool capturing all the information but frequencies.

Having demonstrated that characteristic scale providing simultaneously (ω_n)-oscillatory and concentratory property in general does not exist for an arbitrary bounded sequence, one may wonder whether it is possible to achieve any of the mentioned properties by a suitable choice of a semiclassical scale. The positive result is obtained by Theorem 5 below, for which we need the following two lemmas.

Lemma 5. *For any countable family of scales there exists a scale faster (slower) then all scales in the corresponding family.*

Dem. Let us denote by $\Big\{\Big(\omega_n^{(1)}\Big), \Big(\omega_n^{(2)}\Big), \dots\Big\}$ corresponding countable family of scales.

Let us first show that there exists scale $(\tilde{\omega}_n)$ which is not slower then any scale in the family. One such sequence we can get by the diagonal argument, i.e. by defining $\tilde{\omega}_n := \min_{k\leqslant n} \omega_n^{(k)}$. Indeed, for any $k \in \mathbf{N}$ and all $n \geqslant k$ we have $\tilde{\omega}_n \leqslant \omega_n^{(k)}$, implying $\tilde{\omega}_n \to 0^+$ and $\limsup_n \frac{\tilde{\omega}_n}{\omega_n^{(k)}} < \infty$. Finally, by $\omega_n := \tilde{\omega}_n^2$ we get a scale faster then any in the family.

To construct a slower scale we need to be slightly more careful in the diagonal argument in order not to slow a constructed sequence too much, losing the convergence to zero. Indeed, an analogue approach would be by taking $\omega_n := \max_{k\leqslant n}\omega_n^{(k)}$, but it is not difficult to find a family for which we would not have $\omega_n \to 0^+$.

Therefore, we continue in the different manner by constructing in an inductive manner an auxiliary sequence of positive integers (b_m) for which we define $\omega_n := \frac{1}{m+1}$, for $b_m \leqslant n < b_{m+1}$. Let us define $b_0 := 1$. For any $m \in \mathbf{N}$, by taking into account the zero convergence of the scales, there exists $j_m \in \mathbf{N}$ such that for all $k \leqslant m$ and all $n \geqslant j_m$ we have $\omega_n^{(k)} \leqslant \frac{1}{(m+1)^2}$. Then we define $b_m := \max\{j_m, b_{m-1}+1\}$. It is obvious that (b_m) is strictly increasing sequence, and $b_m \geqslant m+1 \to \infty$. As we have announced before, we define $\omega_n := \frac{1}{m+1}$ for $b_m \leqslant n < b_{m+1}$, which is well defined since (b_m) is strictly increasing and unbounded. It is left to show that (ω_n) converges to zero and that it is slower then any scale in the family.

As for any $n \geqslant b_m$ we have $\omega_n \leqslant \frac{1}{m+1}$, it is immediate that $\omega_n \to 0^+$. On the other hand, since for m large enough $b_m \leqslant n < b_{m+1}$ implies

$$\frac{\omega_n}{\omega_n^{(k)}} \geqslant \frac{(m+1)^2}{m+1} = m+1\,,$$

it is easy to see that (ω_n) is slower then any $(\omega_n^{(k)})$.

Q.E.D.

The existence of a slower scale is a consequence of a more general result from the set theory and the Hausdorff gaps, while in the previous lemma we presented only one possible construction.

The last result paves the path to the following lemma which provides the (ω_n)-oscillatory (or concentratory) property by assuming relation (2) (or (3)) holds just for a countable number of test functions, each of them associated with a different scale.

Lemma 6. *Let (u_n) be a bounded sequence in $\mathrm{L}^2_{\rm loc}(\Omega;\mathbf{C}^r)$. If for any test function φ_k from a countable dense subset $\mathcal{G} \subseteq \mathrm{C}^\infty_c(\Omega)$ (in the corresponding topology of strict inductive limit) there exists scale $(\omega_n^{(k)})$ such that*

$$\lim_{R\to\infty}\limsup_n \int_{A_{n,R}} |\widehat{\varphi_k\mathsf{u}_n}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi} = 0\,, \tag{9}$$

where $A_{n,R} = \Big\{|\boldsymbol{\xi}| \geqslant R/\omega_n^{(k)}\Big\}$ $\Big(A_{n,R} = \Big\{|\boldsymbol{\xi}| \leqslant 1/R\omega_n^{(k)}\Big\}\Big)$, then there exists scale (ω_n) such that (u_n) is (ω_n)-oscillatory (concentratory).

Dem. By the previous lemma there exists a scale (ω_n) such that all functions in $\mathcal{G}$ satisfy the statement condition (9) with $\omega_n^{(k)}$ replaced by ω_n.

Therefore, it is left to prove that (9) is also valid (with $\omega_n^{(k)}$ replaced by ω_n) for an arbitrary test function from $\mathrm{C}^\infty_c(\Omega)$. Let $\varphi \in \mathrm{C}^\infty_c(\Omega)$. Then for any $\varepsilon > 0$ there exists an integer $k \in \mathbf{N}$ and a compact $K \subseteq \Omega$ such that $\operatorname{supp}\varphi, \operatorname{supp}\varphi_k \subseteq K$, $\|\varphi - \varphi_k\|_{\mathrm{L}^\infty(K)} < \varepsilon$. As in the estimate

$$\limsup_n \int\limits_{A_{n,R}} |\widehat{\varphi\mathsf{u}_n}|^2\,d\boldsymbol{\xi} \leqslant \limsup_n \int\limits_{A_{n,R}} \Big(|\widehat{\varphi\mathsf{u}_n}|^2 - |\widehat{\varphi_k\mathsf{u}_n}|^2\Big)\,d\boldsymbol{\xi} + \limsup_n \int\limits_{A_{n,R}} |\widehat{\varphi_k\mathsf{u}_n}|^2\,d\boldsymbol{\xi}\,,$$

the second term in the right hand side is arbitrarily small for $R > 0$ large enough, it is sufficient to estimate the first term, for which we have

$$\begin{aligned}
\Big|\|\widehat{\varphi\mathsf{u}_n}\chi_{A_{n,R}}\|^2_{\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)} &- \|\widehat{\varphi_k\mathsf{u}_n}\chi_{A_{n,R}}\|^2_{\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)}\Big| \\
=&\Big(\|\widehat{\varphi\mathsf{u}_n}\chi_{A_{n,R}}\|_{\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)} + \|\widehat{\varphi_k\mathsf{u}_n}\chi_{A_{n,R}}\|_{\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)}\Big) \\
&\qquad\Big|\|\widehat{\varphi\mathsf{u}_n}\chi_{A_{n,R}}\|_{\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)} - \|\widehat{\varphi_k\mathsf{u}_n}\chi_{A_{n,R}}\|_{\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)}\Big| \\
\leqslant&(2\|\varphi\|_{\mathrm{L}^\infty(K)} + \varepsilon)\sup_n \|\mathsf{u}_n\|^2_{\mathrm{L}^2(K;\mathbf{C}^r)}\varepsilon\,.
\end{aligned}$$

By the arbitrariness of $\varepsilon > 0$ we conclude that (u_n) is indeed (ω_n)-oscillatory (concentratory).

Q.E.D.

Remark. In the previous lemma we could take $\mathcal{G}$ to be dense in the weaker topology of the space $\mathrm{C}_c(\Omega)$ since in the proof it is sufficient to approximate test functions only in the L^∞ norm, while derivatives play no role here. ∎

Finally, for an arbitrary bounded sequence we are ready to prove that there exists a scale with respect to which the sequence is oscillatory and also, under one more condition, a scale with respect to which it is concentratory.

Theorem 5. *Let (u_n) be an arbitrary bounded sequence in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$.*
a) Then there exists scale (ω_n) for which the sequence is (ω_n)-oscillatory.
b) There exists scale (ω_n) for which the sequence is (ω_n)-concentratory if and only if the sequence converges weakly to zero in the same space.

Dem. By the previous lemma it is sufficient to take an arbitrary $\varphi\in\mathrm{C}^\infty_c(\Omega)$ and to prove the existence of a scale (ω_n) such that

$$\lim_{R\to\infty}\limsup_n\int_{A_{n,R}}|\widehat{\varphi\mathsf{u}_n}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}=0\,, \tag{10}$$

where $A_{n,R}=\Big\{|\boldsymbol{\xi}|\geqslant R/\omega_n\Big\}$ for the oscillatory, while $A_{n,R}=\Big\{|\boldsymbol{\xi}|\leqslant 1/R\omega_n\Big\}$ for the concentratory property.

a) We shall deal first with the (ω_n)-oscillatory property. As $\widehat{\varphi\mathsf{u}_n}\in\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)$, by the continuity from above of measures, for any $k,n\in\mathbf{N}$ there exists $R_{n,k}>1$ such that $\int_{|\boldsymbol{\xi}|\geqslant R_{n,k}}|\widehat{\varphi\mathsf{u}_n}|^2\,d\boldsymbol{\xi}\leqslant\frac{1}{k}$.

Then for any scale satisfying $\omega_n<\frac{1}{R_{n,n}}$ we have

$$\int_{|\boldsymbol{\xi}|\geqslant\frac{1}{\omega_n}}|\widehat{\varphi\mathsf{u}_n}|^2\,d\boldsymbol{\xi}\leqslant\int_{|\boldsymbol{\xi}|\geqslant R_{n,n}}|\widehat{\varphi\mathsf{u}_n}|^2\,d\boldsymbol{\xi}\leqslant\frac{1}{n}\,,$$

implying (10).

b) In the case of the (ω_n)-concentratory property we assume $\mathsf{u}_n\rightharpoonup\mathsf{0}$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$, hence by the Rellich compactness theorem we get that for any $\varphi\in\mathrm{C}^\infty_c(\Omega)$ we have $\varphi\mathsf{u}_n\longrightarrow\mathsf{0}$ in $\mathrm{H}^{-1}(\mathbf{R}^d;\mathbf{C}^r)$. Therefore, by the proof of Lemma 2 (relation (4)) for $\omega_n:=\|\varphi\mathsf{u}_n\|_{\mathrm{H}^{-1}(\mathbf{R}^d;\mathbf{C}^r)}$ we have

$$\lim_{R\to\infty}\limsup_n\int\limits_{|\boldsymbol{\xi}|\leqslant\frac{1}{R\omega_n}}|\widehat{\varphi\mathsf{u}_n}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}=0\,,$$

implying the existence of a scale with respect to which (u_n) is concentratory.

It is left to prove the converse. Let (ω_n) be a scale such that a bounded sequence (u_n) is (ω_n)-concentratory. We need to show that it necessarily implies $\mathsf{u}_n\rightharpoonup\mathsf{0}$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$.

Let $R>0$ and $M>0$ be arbitrary constants. Since $\omega_n\to0^+$, there exists $n_0\in\mathbf{N}$ such that for any $n\geqslant n_0$ we have $\omega_n\leqslant\frac{1}{RM}$. Further on, by the boundedness of (u_n) there exists a weakly converging subsequence $(\mathsf{u}_{n'})$, i.e. $\mathsf{u}_{n'}\rightharpoonup\mathsf{u}$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$. Moreover, using Lebesgue dominated convergence theorem, for any test function $\varphi\in\mathrm{C}^\infty_c(\Omega)$ we have $\widehat{\varphi\mathsf{u}_{n'}}\longrightarrow\widehat{\varphi\mathsf{u}}$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{C}^r)$.

By the estimate

$$\limsup_{n'}\int\limits_{|\boldsymbol{\xi}|\leqslant\frac{1}{R\omega_{n'}}}|\widehat{\varphi\mathsf{u}_{n'}}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}\geqslant\limsup_{n'}\int\limits_{|\boldsymbol{\xi}|\leqslant M}|\widehat{\varphi\mathsf{u}_{n'}}(\boldsymbol{\xi})|^2\,d\boldsymbol{\xi}=\|\widehat{\varphi\mathsf{u}}\|^2_{\mathrm{L}^2(\mathrm{K}(0,M);\mathbf{C}^r)}\,,$$

having in mind that $(\mathsf{u}_{n'})$ is $(\omega_{n'})$-concentratory and that $R>0$ is arbitrary, we end up with $\|\widehat{\varphi\mathsf{u}}\|_{\mathrm{L}^2(\mathrm{K}(0,M);\mathbf{C}^r)}=0$. Now, by the arbitrariness of $M>0$ and φ, using the Plancherel formula, we have $\mathsf{u}\equiv\mathsf{0}$.

Since any weakly convergent subsequence of (u_n) has the same limit, $\mathsf{u}=0$, we have that the whole sequence converges weakly to zero.

Q.E.D.

4. Compactness results

In this section we present some applications of the introduced (ω_n)-concentratory property and of the associated results obtained in preceding two sections.

The next theorem provides a strong convergence result in the spirit of the Kolmogorov-Riesz compactness theorem (cf. [8, Theorem 5]).

Theorem 6. *Let* (u_n) *be a bounded sequence in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$.

a) Assume (u_n) *converges weakly, without passing to a subsequence, to a (maybe unknown) limit* $\mathsf{u}\in \mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$. *Then the sequence converges strongly if and only if it is* (ω_n)*-oscillatory for any* $\omega_n \to 0^+$.

b) The sequence (u_n) *converges strongly to zero if and only if it is* (ω_n)*-concentratory for any* $\omega_n \to 0^+$.

Dem. Let us prove the equivalence for the (ω_n)-oscillatory property, while for the (ω_n)-concentratory property the arguments of the proof follow the same pattern.

If $\mathsf{u}_n \longrightarrow \mathsf{u}$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$, then by writing $\mathsf{u}_n = (\mathsf{u}_n-\mathsf{u})+\mathsf{u}$ and using Lemma 4 we have that (u_n) is (ω_n)-oscillatory for any $\omega_n \to 0^+$. Indeed, by the Lebesgue dominated convergence theorem we have that a constant sequence is (ω_n)-oscillatory for any $\omega_n \to 0^+$, while a sequence converging strongly to zero is trivially (ω_n)-oscillatory for any $\omega_n \to 0^+$.

On the other hand, if (u_n) is (ω_n)-oscillatory for every $\omega_n \to 0^+$, then by the above argument we have that $\mathsf{v}_n := \mathsf{u}_n - \mathsf{u}$ is also (ω_n)-oscillatory for every $\omega_n \to 0^+$. However, by the previous theorem there exists a scale $(\tilde\omega_n)$ such that (v_n) is $(\tilde\omega_n)$-concentratory. Since (v_n) is oscillatory on every scale, and specially on a scale slower than $(\tilde\omega_n)$, the claim follows by Theorem 4.

Q.E.D.

The assumption of the weak convergence in the a) part of the previous theorem is essential to ensure the uniqueness of accumulation points. Indeed, for $\mathsf{u},\mathsf{v}\in \mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ the sequence of functions

$$\mathsf{u}_n := \begin{cases} \mathsf{u} & , \quad 2|n \\ \mathsf{v} & , \quad 2\nmid n \end{cases}$$

is obviously (ω_n)-oscillatory for any $\omega_n \to 0^+$, but it is not strongly (not even weakly) convergent.

It is well known that in order to get a strong precompactness result, apart of a trivial semiclassical measure one needs in addition that the sequence under consideration is (ω_n)-oscillatory at the scale associated to the measure. By means of Theorem 5 we can restate this result in the following form: if a semiclassical measure associated to a bounded sequence in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ is trivial at any scale, then there exist a subsequence converging strongly to zero. Moreover, by the decomposition of semiclassical measures (1) we can extend this statement to sequences with an arbitrary weak limit. Namely, let $\mathsf{u}_n \rightharpoonup \mathsf{u}$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ and let associated semiclassical measures of any scale are equal to $(\mathsf{u}\otimes\mathsf{u})\lambda\boxtimes\delta_0$, then there exists a subsequence $(\mathsf{u}_{n'})$ such that $\mathsf{u}_{n'} \longrightarrow \mathsf{u}$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$.

In most of the previous assertions the weak limit was known or it was assumed to be zero. However, if the corresponding weak limit is not known a priori we can (partially) identify it by means of semiclassical measures. More precisely, as a consequence of Theorem 5 we get the following theorem.

Theorem 7. *Let* (u_n) *be a bounded sequence in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$.

a) If (u_n) *is weakly converging in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$ *to a (maybe unknown) limit* u, *then for any compact* $K\subseteq\Omega$ *we have*

$$\|\mathsf{u}\|^2_{\mathrm{L}^2(K;\mathbf{C}^r)} = \min_{(\omega_n)} \mathsf{tr}\boldsymbol{\mu}_{sc}^{(\omega_n)}(K\times\{0\}),$$

where the minimum is taken over all scales $\omega_n\to 0^+$, *while* $\boldsymbol{\mu}_{sc}^{(\omega_n)}$ *is a semiclassical measure with scale* (ω_n) *associated to* (u_n).

b) If for every $\omega_n \to 0^+$ *the trace of associated semiclassical measures is equal to* $u^2\lambda\boxtimes\delta_0$, *where* u *is a non-negative* $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$ *function, then* $|\mathsf{u}_n| \longrightarrow u$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$.

Dem. a) By the decomposition (1) we have that for any $\omega_n \to 0^+$ the following equality holds:

$$\mathsf{tr}\boldsymbol{\mu}_{sc}^{(\omega_n)} = |\mathsf{u}|^2 \lambda \boxtimes \delta_0 + \mathsf{tr}\boldsymbol{\nu}_{sc}^{(\omega_n)} ,$$

where $\boldsymbol{\mu}_{sc}^{(\omega_n)}$ and $\boldsymbol{\nu}_{sc}^{(\omega_n)}$ are semiclassical measures associated to (u_n) and $(\mathsf{u}_n - \mathsf{u})$. Since $\mathsf{tr}\boldsymbol{\nu}_{sc}^{(\omega_n)}$ is non-negative we have $\mathsf{tr}\boldsymbol{\mu}_{sc}^{(\omega_n)} \geqslant |\mathsf{u}|^2 \lambda \boxtimes \delta_0$, thus for any compact $K \subseteq \Omega$ we get

$$\mathsf{tr}\boldsymbol{\mu}_{sc}^{(\omega_n)}(K \times \{0\}) \geqslant \|\mathsf{u}\|^2_{\mathrm{L}^2(K;\mathbf{C}^r)} .$$

It is left to prove that the inequality above is achieved for some $\omega_n \to 0^+$, which is a simple consequence of Theorem 5. Namely, by Theorem 5 there exists a scale (ω_n) such that sequence $(\mathsf{u}_n - \mathsf{u})$ is concentratory at that scale, hence by Theorem 3 for any compact $K \subseteq \Omega$ we have $\mathsf{tr}\boldsymbol{\nu}_{sc}^{(\omega_n)}(K \times \{0\}) = 0$, thus obtaining an equality in the estimate above.

b) Since (u_n) is bounded we can pass to a weakly converging subsequence $(\mathsf{u}_{n'})$, such that $\mathsf{u}_{n'} \rightharpoonup \mathsf{u}_1$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$, and that subsequence has the same associated semiclassical measures. By a) part of this theorem we have that $|\mathsf{u}_1| = u$, which, by the decomposition (1), implies that for any $\omega_n \to 0^+$ semiclassical measures associated to $(\mathsf{u}_{n'} - \mathsf{u}_1)$ are equal to zero. In particular, by Theorem 5 we can choose (ω_n) such that $(\mathsf{u}_{n'} - \mathsf{u}_1)$ is (ω_n)-oscillatory, thus we get that $\mathsf{u}_{n'} \longrightarrow \mathsf{u}_1$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega;\mathbf{C}^r)$, and then also $|\mathsf{u}_{n'}| \longrightarrow u$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$.

Since the last convergence holds for any weakly converging subsequence of (u_n), we can conclude that the whole sequence $(|\mathsf{u}_n|)$ converges strongly to u.

Q.E.D.

As a special consequence of the a) part of the last theorem it follows that if $\mathsf{tr}\boldsymbol{\mu}_{sc}^{(\omega_n)}(K \times \{0\}) = 0$ for some scale (ω_n), than $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(K;\mathbf{C}^r)$.

5. Conclusion

In this paper we have introduced a notion of the (ω_n)-concentratory property which can be considered as a counterpart to the already existing (ω_n)-oscillatory property. Both notions are inevitably related to semiclassical measures, a microlocal tool depending on an associated semiclassical scale. While the latter prevents semiclassical measures of loosing energy associated to high frequencies, the first property prevail the loss of information related to low frequencies. If a sequence allows both the properties to be satisfied by a same scale, we define it as its characteristic scale.

A semiclassical measure provides the best performance if its scale coincides with the characteristic scale of an associated sequence. In that case it can completely capture important microlocal properties: amplitude, frequency and direction of propagations, while an H-measure associated to the same sequence can be reconstructed from it by averaging information along the rays in the frequency domain. Essentially, this is a well known result from before, but it required assumptions on a semiclassical measure, while here, by means of the introduced (ω_n)-concentratory property, the result is stated solely in terms of a sequence under consideration. In other words, we do not require a semiclassical measure to be constructed first in order to check its performance. Instead, rather by analysing the very sequence, we can deduce a right scale for which the semiclassical measure will attain the best performance level.

However, if a sequence under consideration incorporates two or more frequency scales, and does not allow for a characteristic scale, semiclassical measures fail to recover important part of the information, unlike H-measures which still capture all the information except frequencies. A notable example is given by a sequence (6), incorporating an infinite number of frequency scales, for which all the associated semiclassical measures, regardless of the chosen scale, are either zero or contain their support within the origin of the frequency domain, such loosing information on all frequencies and directions of propagation.

Last section provides compactness results derived by the introduced notion. Theorem 6 provides a result similar to the Kolmogorov-Riesz compactness theorem. The latter one requires relation (2) to hold just for a constant sequence $\omega_n = 1$, while the first one in the a) part considers the same relation with a semiclassical scale, which is a weaker assumption, but it requires (2) to hold for every such scale. However, two assumptions turn out to be equivalent as they are both equivalent to strong precompactness of the considered sequence.

Theorem 6 also resembles to compactness results obtained by H-distributions [1], providing strong convergence if and only if all H-distributions related to a sequence under consideration are equal to zero. However, the compactness result obtained here does not require a microlocal defect object, and is stated in terms of the sequence only.

The b) part of Theorem 6 relies on the (ω_n)-concentratory property that has to be satisfied for every semiclassical scale. The result can be applied for disproving strong convergence of a sequence converging weakly to zero. In that case the strong convergence contradicts the negation of the assumption of the Kolmogorov-Riesz theorem

$$(\exists\, \varphi \in \mathrm{C}_c^\infty(\Omega)) \qquad \limsup_{R\to\infty} \limsup_{n} \int\limits_{|\boldsymbol{\xi}|\geqslant R} |\widehat{\varphi \mathsf{u}_n}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} > 0\,, \tag{11}$$

as well as the negation of the (ω_n)-concentratory property, implying that there exists a semiclassical scale (ω_n) such that

$$(\exists\, \varphi \in \mathrm{C}_c^\infty(\Omega)) \qquad \limsup_{R\to\infty} \limsup_{n} \int\limits_{|\boldsymbol{\xi}|\leqslant \frac{1}{R\omega_n}} |\widehat{\varphi \mathsf{u}_n}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} > 0\,. \tag{12}$$

Relation (11) considers integration over unbounded sets (obtained as complements of enlarging nested family of finite balls), while (12) takes into account integrals over finite sets, whose radius increases boundlessly. Which of the two inequalities is easier to prove depends on a particular sequence and one has both the options at his disposal.

At the end we present how the introduced notion enable us to (partially) recover the unknown limit (weak or strong) of the corresponding sequence via associated semiclassical measures.

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