Periodic approximation of exceptional Lyapunov exponents for semi-invertible operator cocycles

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Preliminaries

Let (M, d) be a compact metric space and let $f \colon M \to M$ be a homeomorphism. We say that f satisfies the **Anosov closing property** if there exist $C_1, \varepsilon_0, \theta > 0$ such that if $z \in M$ satisfies $d(f^n(z), z) < \varepsilon_0$ then there exists a periodic point $p \in M$ such that $f^n(p) = p$ and

$$d(f^j(z), f^j(p)) \leq C_1 e^{-\theta \min\{j, n-j\}} d(f^n(z), z),$$

for every $j = 0, 1, \ldots, n$.

We note that subshifts of finite type and Anosov diffeomorphisms satisfy this property.

Preliminaries

Let $X = (X, \|\cdot\|)$ be an arbitrary Banach space and let $B(X)$ denote the space of all bounded linear operators on X. Finally, let $A : M \to B(X)$ be an α-**Hölder continuous map**. We recall that this means that there exists a constant $C_2 > 0$ such that

$$\|A(q_1) - A(q_2)\| \leq C_2 d(q_1, q_2)^\alpha,$$

for all $q_1, q_2 \in M$. We consider a **linear cocycle** $\mathcal{A}$ over (M, f) with generator A. Recall that for $q \in M$ and $n \in \mathbb{N}_0$,

$$\mathcal{A}(q, n) = \begin{cases} A(f^{n-1}(q)) \cdots A(q) & \text{for } n > 0; \\ \mathrm{Id} & \text{for } n = 0. \end{cases}$$

Preliminaries

Let μ be an **ergodic f-invariant** Borel probability measure. We have the existence of the **largest Lyapunov exponent** $\lambda(\mu) \in [-\infty, \infty)$ such that

$$\lambda(\mu) = \lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(q,n)\| \quad \text{for } \mu\text{-a.e. } q \in M,$$

and of the **index of the compactness** $\kappa(\mu) \in [-\infty, \infty)$ such that

$$\kappa(\mu) = \lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(q,n)\|_{ic} \quad \text{for } \mu\text{-a.e. } q \in M,$$

where for $T \in B(X)$, $\|T\|_{ic}$ is the infimum over all $r > 0$ such that $T(B_X(0,1))$ can be covered by finitely many open balls of radius r.

Lyapunov exponents

We assume that the cocycle $\mathcal{A}$ is **quasicompact** which means that $\kappa(\mu) < \lambda(\mu)$. Then, we have the following:

- there exists $l \in \mathbb{N} \cup \{\infty\}$ and a sequence of numbers $(\lambda_i(\mu))_{i=1}^{l}$ such that
$$\lambda(\mu) = \lambda_1(\mu) > \lambda_2(\mu) > \dots \lambda_i(\mu) > \kappa(\mu).$$
  If $l = \infty$, then $\lambda_i(\mu) \to \kappa(\mu)$;
- there exists a Borel measurable $\mathcal{R}^{\mu} \subset M$ such that $\mu(\mathcal{R}^{\mu}) = 1$ and for each $q \in \mathcal{R}^{\mu}$ and $i \in \mathbb{N} \cap [1, l]$ a measurable decomposition

Lyapunov exponents

$$X = \bigoplus_{j=1}^{i} E_j(q) \oplus V_{i+1}(q),$$

where $E_j(q)$ are finite-dimensional subspaces of X,

$$A(q)E_j(q) = E_j(f(q)) \quad \text{and} \quad A(q)V_{i+1}(q) \subset V_{i+1}(f(q)).$$

- for $q \in \mathcal{R}^\mu$ and $v \in E_j(q) \setminus \{0\}$,

$$\lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(q,n)v\| = \lambda_j(\mu).$$

Furthermore, for $v \in V_{i+1}(q)$ we have that

$$\limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(q,n)v\| \le \lambda_{i+1}(\mu).$$

Lyapunov exponents

The numbers $\lambda_i(\mu)$ are called **Lyapunov exponents** of the cocycle $\mathcal{A}$ with respect to μ. Furthermore, subspaces $E_i(q)$ are called **Oseledets subspaces** of $\mathcal{A}$ w.r.t. μ. The number $d_i(\mu) := \dim E_i(q)$ is the multiplicity of the exponent $\lambda_i(q)$.

Those conclusions are ensured by the version of MET established by Froyland, Lloyd and Quas building on the earlier work of Ruelle, Mañé, Thieullen, Lian and Lu. Further extensions/alternative proofs: Blumenthal, González-Tokman-Quas.

Main result

We will denote by

$$\gamma_1(\mu) \geq \gamma_2(\mu) \geq \dots$$

Lyapunov exponents of $\mathcal{A}$ with respect to μ **counted with their multiplicities**. For example, $\gamma_i(\mu) = \lambda_1(\mu) \quad \text{for } i = 1, \dots, d_1(\mu)$.

Theorem

For $s \in \mathbb{N} \cap [1, l]$ there exist a sequence of periodic points $(p_k)_k$ such that

$$\lim_{k\to\infty} \gamma_i(p_k) = \gamma_i(\mu),$$

for $i \in \{1, \dots, d_1(\mu) + \dots + d_s(\mu)\}$.

Previous work

- X finite-dimensional and $A(q)$ invertible: Kalinin;
- X finite-dimensional and $A(q)$ not necessarily invertible: Backes;
- X Banach space and $A(q)$ invertible: Kalinin and Sadovskaya;
- Basis (M, f) nonuniformly hyperbolic system: Kalinin and Sadovskaya.

Also, Kalinin and Sadovskaya showed that **without quasicompactness we don't have desired approximation** of the largest Lyapunov exponent.

Idea of the proof

Step 1: Lyapunov norms

For each $q \in \mathcal{R}^\mu$, we have the decomposition

$$X = E_1(q) \oplus \dots E_s(q) \oplus V_{s+1}(q).$$

For $q \in \mathcal{R}^\mu$ and $\delta > 0$ sufficiently small we can construct a norm $\|\cdot\|_q$ on X such that:

- $e^{(\lambda_i-\delta)n}\|u\|_q \le \|\mathcal{A}(q,n)u\|_{f^n(q)} \le e^{(\lambda_i+\delta)n}\|u\|_q$ for $n \in \mathbb{N}$ and $u \in E_i(q)$;
- $\|\mathcal{A}(q,n)u\|_{f^n(q)} \le e^{(\lambda_s-\delta)n}\|u\|_q$ for $n \in \mathbb{N}$ and $u \in V_{s+1}(q)$;
- $\|\mathcal{A}(q,n)u\|_{f^n(q)} \le e^{(\lambda_1+\delta)n}\|u\|_q$ for $n \in \mathbb{N}$ and $u \in X$;

Idea of the proof

- there exists a Borel-measurable function $K_\delta \colon \mathcal{R}^\mu \to (0, \infty)$ such that
$$\|u\| \le \|u\|_q \le K_\delta(q)\|u\|,$$
for $q \in \mathcal{R}^\mu$ and $u \in X$;
- for $q \in \mathcal{R}^\mu$ and $n \in \mathbb{N}$,
$$K_\delta(q)e^{-\delta n} \le K_\delta(f^n(q)) \le e^{\delta n} K_\delta(q).$$
- $\|\cdot\|_q$ depends measurably on q;

For $N \in \mathbb{N}$, let $\mathcal{R}^\mu_{\delta,N}$ be the set of all $q \in \mathcal{R}^\mu$ such that $K_\delta(q) \le N$. WLOG: this is a compact set and Lyapunov norm and Osel. splitting depend continuously on it.

Step 2: There exist $q \in \mathcal{R}^{\mu}_{\delta,N}$ and a sequence $(p_k)_k$ of periodic points (n_k=period of p_k)

$$d(f^j(q), f^j(p_k)) \text{ small for } 1 \le j \le n_k \text{ and } k \in \mathbb{N},$$

$$f^{n_k}(q) \in B(q, \frac{1}{k}) \cap \mathcal{R}^{\mu}_{\delta,N},$$

and $\mu_{p_k} \to \mu$, where

$$\mu_{p_k} = \frac{1}{n_k} \sum_{j=0}^{n_k - 1} \delta_{f^j(p_k)}.$$

Step 3: We have

$$\limsup_{k \to \infty} (\gamma_1(p_k) + \ldots \gamma_i(p_k)) \le \gamma_1(\mu) + \ldots + \gamma_i(\mu),$$

for $i \in \{1, \ldots, d_1(\mu) + \ldots d_s(\mu)\}$.

Idea of the proof

For $k \in \mathbb{N}$, $1 \leq j \leq n_k$ and $\gamma \in [0,1)$, set

$$C_\gamma^{j,1} = \left\{ u+v \in E_1(f^j(q)) \oplus V_2(f^j(q)) : \|v\|_{f^j(q)} \leq (1-\gamma)\|u\|_{f^j(q)} \right\}.$$

Step 4: For $1 \leq j \leq n_k$ and $u \in C_0^{j,1}$,

$$\|((A(f^j(p_k))u))_E^{j+1}\|_{f^{j+1}(q)} \geq e^{\lambda_1 - 2\delta}\|u_E^j\|_{f^j(q)}.$$

Moreover, for k large, there exist $\gamma \in (0,1)$ such that

$$A(f^j(p_k)(C_0^{j,1}) \subset C_\gamma^{j+1,1}.$$

For k large, $C_\gamma^{n_k,1} \subset C_0^{0,1}$ and thus $\mathcal{A}(p_k, n_k)C_0^{0,1} \subset C_0^{0,1}$. Then,

$$\|\mathcal{A}(p_k, n_k)u\|_{f^{n_k}(q)} \geq \frac{1}{2}e^{n_k(\lambda_1 - 2\delta)}\|u\|_q \geq \frac{1}{4}e^{n_k(\lambda_1 - 2\delta)}\|u\|_{f^{n_k}(q)},$$

for $u \in C_0^{0,1}$.

Step 5:

$$\lambda(p_k, u) \geq \lambda_1 - 3\delta \quad \text{for } u \in C_0^{0,1}.$$

Let

$$i^k := \max\{i : V_i(p_k) \cap C_0^{0,1} \neq \{0\}\}.$$

Then,

$$\lambda_{i^k}(p_k) \geq \lambda_1 - 3\delta.$$

Step 6: For large k, we have that

$$\dim(E_1(p_k) \oplus \ldots \oplus E_{i^k}(p_k)) = \dim E_1(q).$$

Then,

$$\gamma_i(p_k) \geq \gamma_i(\mu) - 3\delta \quad \text{for } i = 1, \ldots, d_1(\mu).$$

and thus

$$\gamma_i(p_k) \to \gamma_i(\mu) \quad \text{for } i = 1, \ldots, d_1(\mu).$$

Next, using $X = E_1(f^j(q)) \oplus E_2(f^j(q)) \oplus V_3(f^j(q))$,

$$C_\gamma^{j,2} := \left\{ u \in X : \|u_V\|_{f^j(q)} \leq (1-\gamma)\|u_{E_2}\|_{f^j(q))} \right\}.$$

Application

Theorem

Assume that $\mathcal{A}$ is quasicompact w.r.t. every ergodic f-invariant Borel probability measure μ and that there exists $c > 0$ such that

$$\lambda(p) \le -c \quad \text{for each periodic } p \in M.$$

Then, there exists $D, \lambda > 0$ such that

$$\|\mathcal{A}(q,n)\| \le De^{-\lambda n} \quad \text{for } q \in M \text{ and } n \in \mathbb{N}.$$