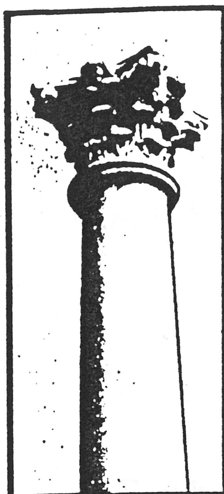


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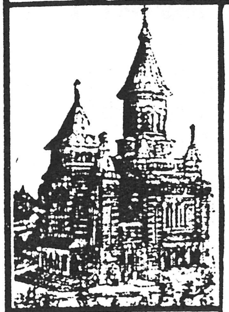
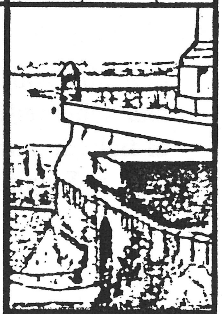

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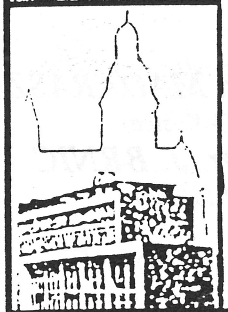
NUMERICAL MODELLING OF FREE-SURFACE FLOWS with variable bottom topography

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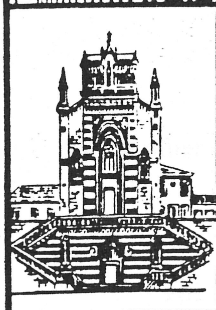
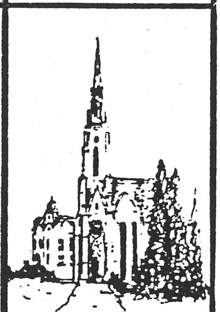

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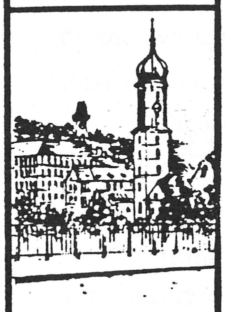


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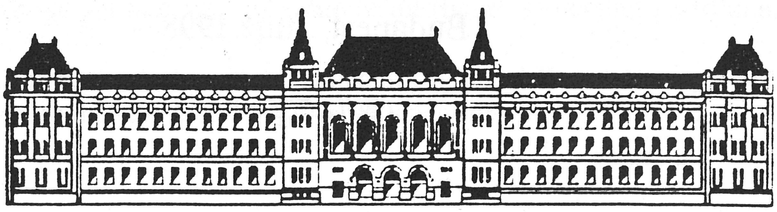
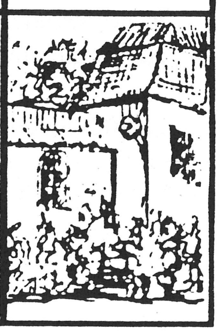

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BULLETINS for APPLIED & COMPUTER MATHEMATICS
(B A M)

B A M 1446 - 1474/'98 - LXXXV-A
Lectured at the PC - 121 / '98 - B.almádi

ISSN 0133 - 3526

- Manuscript -

Prepared for publication
at the
P A M M - C e n t r e

Budapest, June 1998

(BULLFEDH)

NUMERICAL MODELLING OF FREE-SURFACE FLOWS WITH VARIABLE BOTTOM TOPOGRAPHY

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Abstract

This paper presents possible ways of dealing with the problem of correct handling the source term in the case when finite difference upwind method based on approximate Roe's Riemann solver has been applied for the numerical solution of the one-dimensional free-surface flow with variable bottom topography. The possibility of writing the continuity equation in two different ways, with water depth or with water surface elevation as dependent variable, has been considered. These two variations were combined with two approaches of discretizing the source term : centred discretization of the source term and upwinding of the source term in the same manner as the physical flux. Numerical results on the test case that simulates the propagation of the tidal wave in the channel with variable bottom topography shows that the upwinding of the source term can be avoided if the water surface elevation is used as dependent variable in the continuity equation.

1. Introduction

In recent years many researches have been made in application of hyperbolic partial differential equations describing free-surface flows. For practical applications a great importance have problems with variable bottom topography that results in the presence of the bottom slope term as a part of the source term in the mathematical model. Variable bottom topography and consequent presence of the source term can, if not handled correctly, cause a lot of numerical difficulties.

In the chapter 2 the mathematical model of free-surface flow with variable bottom topography is described. The reason of eliminating excessive numerical diffusion leads to the rewriting of the continuity equation with water elevation as dependent variable instead of water depth because on the variable topography the water elevation is normally much smoother than the water depth.

In the chapter 3 is described the finite difference upwind numerical scheme based on flux-difference splitting technique which uses Roe's approximate Riemann solver as a building block for spatial discretizing of the flux term. Because it is well known that centred discretization of the source term combined with flux-difference or flux-vector splitting techniques for the flux term gives bad results, the way of upwinding the source term in the same manner as the flux term is described.

Chapter 4 describes the test case that simulates the propagation of the tidal wave in the channel with variable bottom topography and gives numerical results for the two approaches of writing the continuity equation given in chapter 2, combined with the two ways of discretizing the source term described in the chapter 3.

2. Mathematical model

Free-surface flows of an incompressible fluid when the depth of the fluid is small when compared to the characteristic dimension of the problem are governed by the Saint-Venant shallow-water equations. These equations are obtained from the incompressible Euler equations with assumption of the hydrostatic pressure distribution and with neglecting the dissipative effects. They have the form of non-linear hyperbolic system of partial differential equations so the conservation form is preferred to the non-conservation one because non-linearity produces discontinuities in the solution. For the one-dimensional case we have system of two equations, which consists of the continuity equation (1) and the momentum equation (2) :

$$\frac{\partial h}{\partial t}+\frac{\partial (uh)}{\partial x}=0 \tag{1}$$

$$\frac{\partial (uh)}{\partial t}+\frac{\partial}{\partial x}\left(u^2h+0.5gh^2\right)=gh(S_b-S_f) \tag{2}$$

Here $h = h(x,t)$ represents water depth, $u = u(x,t)$ depth averaged velocity and g is a constant gravity acceleration. On the left hand of equation (2) we have the source term which consists of the bottom slope term $S_b=-\frac{dZ}{dx}$, where $Z = Z(x)$ is the bottom elevation measured from the fixed reference level (Figure 1), and the friction slope term S_f which is usually calculated by the use of Manning's formula :

$$S_f=\frac{n^2u|u|}{h^{4/3}} \tag{3}$$

where n is Manning's roughness coefficient. In our numerical calculation we will neglect the friction term according to the test case described in chapter 4 and will consider only the variable bottom topography.

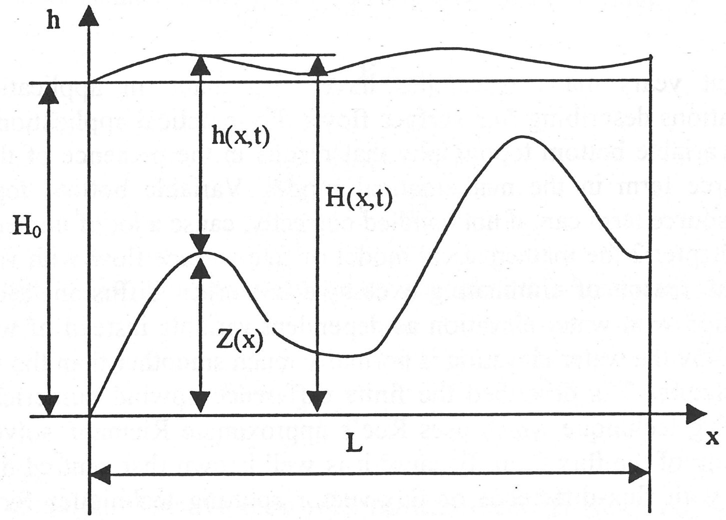

Figure 1.

The set of PDEs (1),(2) is usually written in vector form and in case of $S_f = 0$ we have :

$$\boldsymbol{U}_t+\boldsymbol{F}_x+\boldsymbol{S}=0 \tag{4}$$

where $\boldsymbol{U} = \boldsymbol{U}(x,t)$ denotes the vector of unknowns, $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{U})$ is the flux vector, and $\boldsymbol{S} = \boldsymbol{S}(x,t)$ is the source term :

$$U = \begin{bmatrix} u \\ uh \end{bmatrix} \qquad F = \begin{bmatrix} uh \\ u^2 h + 0.5gh^2 \end{bmatrix} \qquad S = \begin{bmatrix} 0 \\ gh\dfrac{dZ}{dx} \end{bmatrix}$$

The conservative hyperbolic system (4) is characterised by the Jacobian matrix A of the flux F:

$$A = \frac{\partial F}{\partial U} = \begin{bmatrix} 0 & 1 \\ -u^2 + c^2 & 2u \end{bmatrix} \tag{5}$$

where $c = \sqrt{gh}$ represents the celerity i.e. the velocity of propagation of gravitational waves of small amplitude in the fluid continuum. Jacobian matrix A has two real and distinct eigenvalues

$$\begin{aligned} \lambda_1 &= u + \sqrt{gh} = u + c \\ \lambda_2 &= u - \sqrt{gh} = u - c \end{aligned} \tag{6}$$

which are, in fact, absolute velocities of propagation of small disturbances in the fluid continuum. Many numerical methods for solving the hyperbolic system (4) are based on tracking the propagation of discontinuities in right direction, so the diagonal form of Jacobian A is usually used :

$$A = X \Lambda X^{-1} \tag{7}$$

where Λ represents the diagonal matrix of eigenvalues of A , and X is the matrix of corresponding eigenvectors having the form

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad X = \begin{bmatrix} 1 & 1 \\ u + c & u - c \end{bmatrix}.$$

For the shallow water flows with variable bottom topography it is much more convenient to take the water elevation

$$H(x,t) = h(x,t) + Z(x) \tag{8}$$

as dependent variable in the continuity equation (1), so after putting (5) in (1) we get

$$\frac{\partial H}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{9}$$

without changing the momentum equation (2). For the following numerical calculation we will designate the system of equations (1) and (2) with the corresponding vector of unknowns $U = [\ h \ \ uh\ ]^T$ as a mathematical model I , and the system of equations (9) and (2) with the $U = [\ H \ \ uh\ ]^T$ as a mathematical model II.

3. Numerical model

For the numerical integration of the system (4) with appropriate boundary and initial conditions, the finite difference method will be used, so the domain will be discretized with the uniform mesh characterised by points $x_i = i\Delta x$. For the time discretization will be used explicit two-step predictor-corrector scheme that belongs to the family of Runge-Kutta methods with second order accuracy. If we denote the vector of unknowns in time step t^n and at particular computational point i as U^n_i then the following scheme gives after the predictor step U^*_i and after the corrector step we get the vector of unknowns in time step t^{n+1} :

$$U_i^* = U_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right) - \Delta t \cdot S_i^n$$
$$U_i^{n+1} = \frac{1}{2}\left(U_i^n + U_i^* - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^* - F_{i-1/2}^*\right) - \Delta t \cdot S_i^*\right). \tag{10}$$

Here $F_{i+1/2} = F(U_i, U_{i+1}) = F(U_L, U_R)$ represents numerical flux through the cell face between cells i and $i+1$, and the indexes L and R denote the values at the cells left and right of the corresponding cell face. For the Roe's approximate Riemann solver that belongs to a family of upwind schemes based on the flux-difference splitting, numerical flux has the form :

$$F_{i+1/2}^* = \frac{1}{2}\left[F_R + F_L - \left|\widetilde{A}_{i+1/2}\right|(U_R - U_L)\right] \tag{11}$$

where is $F_R = F(U_R) = F(U_{i+1})$ and $F_L = F(U_L) = F(U_i)$. $\widetilde{A}_{i+1/2}$ is the Jacobian matrix of the flux evaluated at some average state $\widetilde{U}$ between U_L and U_R :

$$U_L = \begin{bmatrix} h_L \\ u_L h_L \end{bmatrix} \qquad U_R = \begin{bmatrix} h_R \\ u_R h_R \end{bmatrix} \qquad \widetilde{U} = \begin{bmatrix} \widetilde{h} \\ \widetilde{u}\widetilde{h} \end{bmatrix}$$

$$\widetilde{A}_{i+1/2} = A(\widetilde{U}) = \begin{bmatrix} 0 & 1 \\ -\widetilde{u}^2 + \widetilde{c}^2 & 2\widetilde{u} \end{bmatrix} \tag{12}$$

where is for Roe's scheme

$$\widetilde{c} = \sqrt{g\left(\frac{h_L + h_R}{2}\right)} \qquad \widetilde{h} = \sqrt{h_L h_R} \qquad \widetilde{u} = \frac{\sqrt{h_L}u_L + \sqrt{h_R}u_R}{\sqrt{h_L} + \sqrt{h_R}} \tag{13}$$

After that Jacobian matrix is taken in diagonalized form with absolute values of its eigenvalues :

$$|A| = X|\Lambda|X^{-1}$$

$$|\Lambda| = \begin{bmatrix} |\lambda_1| & 0 \\ 0 & |\lambda_2| \end{bmatrix}.$$

The source term in (10) can be discretized by central differences in the following way :

$$S_i^n = \begin{bmatrix} 0 \\ g\sqrt{h_{i+1}h_{i-1}}\dfrac{Z(x_{i+1}) - Z(x_{i-1})}{2\Delta x} \end{bmatrix} \tag{14}$$

But use of this discretization of source term with upwind methods based on flux-difference or flux-vector splitting to the case of free-surface flow with variable bottom topography gives poor results if the classical mathematical model I with continuity equation (1) is used. In that case the source term should be upwinded in the same way as the flux. Bermudez and Vazquez [1] gave a general methodology of upwinding the source term for class of flux-difference or flux-vector splitting schemes. Numerical source term is divided in the left and right numerical source term :

$$S_i = S_{i-1/2} + S_{i+1/2} \tag{15}$$

with

$$S_{i-1/2} = \frac{1}{2}\left(I + \left|\widetilde{A}_{i-1/2}\right|\widetilde{A}_{i-1/2}^{-1}\right)\cdot \widetilde{S}_{i-1/2}$$
$$S_{i+1/2} = \frac{1}{2}\left(I - \left|\widetilde{A}_{i+1/2}\right|\widetilde{A}_{i+1/2}^{-1}\right)\cdot \widetilde{S}_{i+1/2} \qquad (16)$$

where is $\widetilde{S} = S(x_L, x_R, U_L, U_R)$ an approximation of source term on corresponding cell face :

$$\widetilde{S}_{i+1/2} = \begin{bmatrix} 0 \\ g\sqrt{h_L h_R}\,\dfrac{Z(x_R) - Z(x_L)}{\Delta x} \end{bmatrix} \qquad (17)$$

4. Test case and numerical results

Two mathematical models described in chapter 2 will be applicated with centred discretization of the source term or with upwinding of the source term to the test case described by Bermudez and Vazquez [1]. Test case simulates propagation of a tidal wave of a 43200 second period from the left to the right of the channel of length L = 648000 m (Figure 1) with a variable bottom elevation Z = Z(x) defined by :

$$Z(x) = 10.5 + \frac{40x}{L} - 10\sin\left[\pi\left(\frac{4x}{L} + \frac{1}{2}\right)\right] \qquad (18)$$

Water in the channel is initially at rest, so the initial conditions are :

$$H(x,0) = H_0(x) = H_0 = 60.5\,\text{m} \qquad \text{or} \qquad h(x,0) = h_0(x) = H_0 - Z(x)$$
$$u(x,0) = u_0(x) = 0$$

The tidal wave is defined as boundary condition at the left end of the channel (x = 0) by the sinusoidal function $\varphi(t)$:

$$h(0,t) = h_0(0) + \varphi(t) = h_0(0) + 4 + 4\sin\left[\pi\left(\frac{2t}{43200} - \frac{1}{2}\right)\right].$$

At the other end of the channel (x = L) is applied boundary condition

$$u(L,t) = 0\,.$$

The simulation is performed for the time interval T = 10800 s. If we assume that wave travels by the average constant speed $\bar{c} = \sqrt{g\bar{h}}$ with $\bar{h} \approx 40\,\text{m}$ then it can propagate into the channel no more than $c_{\max} \cdot T \approx 214000$ m .

Figure 2 shows surface elevation H, and velocity u, after the time T=10800 s computed with the mathematical model I, that is with the continuity equation (1), with centred discretization of the source term according to (14). It is clearly seen that the wave has perturbed the whole domain and that this combination of mathematical and numerical model gives a poor result.

If the upwinding of the source term is performed on the same mathematical model then we get very accurate result (figure 3) .

If we apply mathematical model II with continuity equation (9) with centred discretization of the source term then we get also very accurate result (figure 4) without performing upwinding of the source term.

Results on the figure 5 show that mathematical model II is not compatible with upwinding of the source term because that combination gives very poor result.

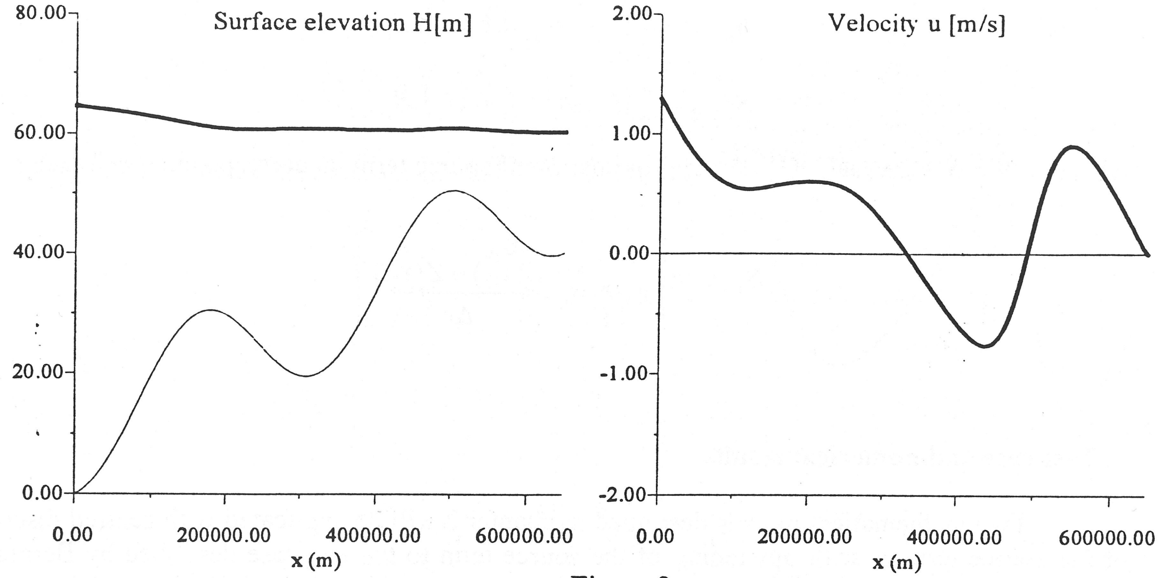

Figure 2.

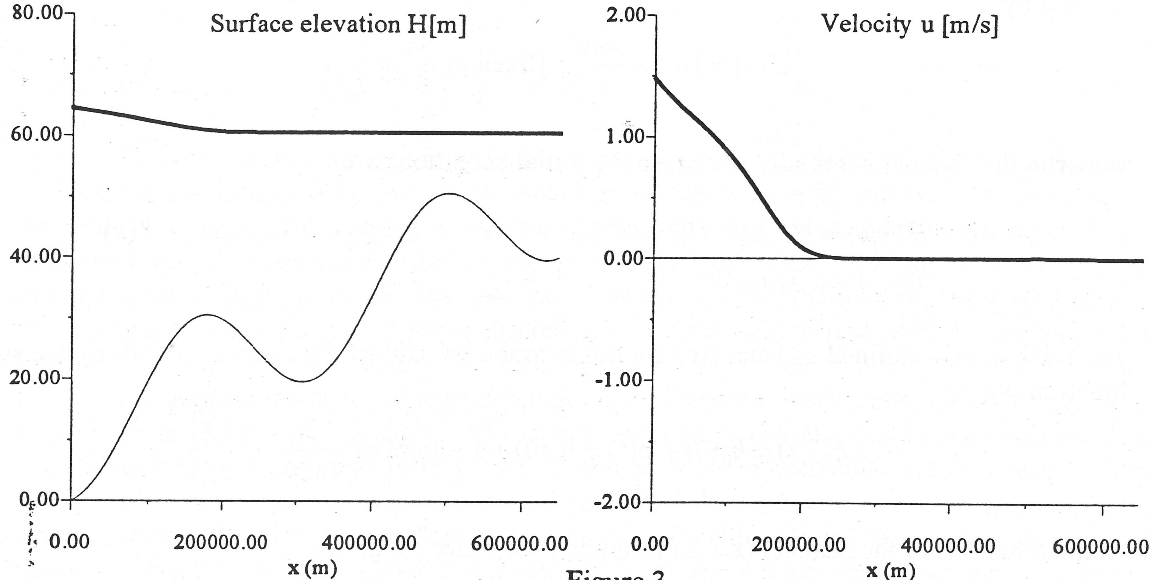

Figure 3.

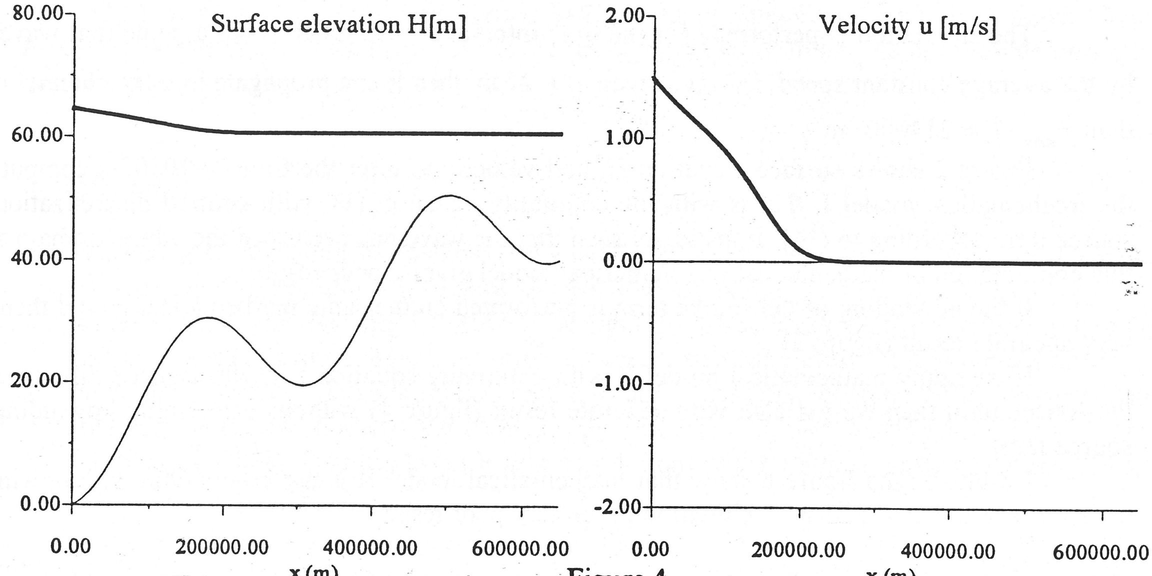

Figure 4.

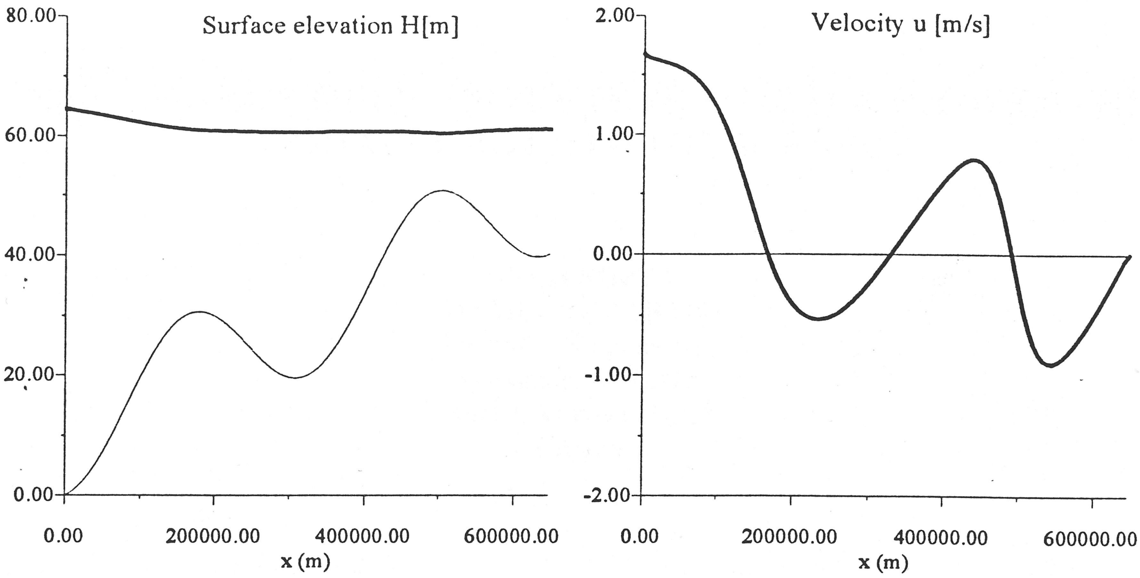

Figure 5.

5. Conclusion

Finite difference upwind method based on approximate Riemann solver has been applied for the numerical solution of the one-dimensional free-surface flow with variable bottom topography. The possibility of writing the continuity equation in two different ways, with water depth or with water surface elevation as dependent variable, has been considered. These two variations were designated as mathematical models I and II and they were tested in the combination with two approaches of discretizing the source term : centred discretization of the source term and upwinding of the source term in the same manner as the physical flux.

It is well known that centred discretization of the source term combined with flux-difference or flux-vector splitting techniques for the flux term give bad results and that the upwinding of the source term is necessary. That is not valid when the second form of continuity equation is used. Numerical results on the test case that simulates the propagation of the tidal wave in the channel with variable bottom topography show that the application of second mathematical model gives the opposite result when combined with two different discretizations of the source term. We can conclude that the upwinding of the source term can be avoided if the water surface elevation is used as dependent variable in the continuity equation. Other alternative for avoiding very complicate application of upwinding of the source term and approximate Riemann solvers is the use of much simpler Lax-Friedrichs solver that allows extracting of the part of the flux term and its appropriate discretization as described by Nujic [3].

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