On contact interactions realised as Friedrichs systems

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Abstract. We realise the Hamiltonians of contact interactions in quantum mechanics within the framework of abstract Friedrichs systems. In particular, we show that the construction of the self-adjoint (or even only closed) operators of contact interaction supported at a fixed point can be associated with the construction of the bijective realisations of a suitable pair of abstract Friedrich operators. In this respect, the Hamiltonians of contact interaction provide novel examples of abstract Friedrich systems.

Mathematics Subject Classification (2010). 35F45, 35Q40, 81Q10.

Keywords. Friedrichs systems on a Hilbert space, quantum Hamiltonians of point interaction.

1. Introduction

In this work we make a bridge between two seemingly distant mathematical subjects, which are instead much closer than what has appeared so far in the literature: the Hamiltonians of contact (or 'point', or 'zero-range') interactions in quantum mechanics on the one hand, and the Friedrichs systems of partial differential equations on the other.

The former are so-called singular perturbations of elliptic differential operators on $\mathbb{R}^d$, which serve as models for quantum particles subject to an interaction – the actual 'perturbation' of the free Hamiltonian – supported on manifolds of positive co-dimension. The latter are systems of first order coupled partial differential equations that are equivalent to certain boundary value problems on domains with suitable boundary conditions.

It is in fact the understanding of the role of the boundary, as we shall show, that allows one to recognise contact interaction Hamiltonians as special classes, and in this respect *novel examples*, of Friedrichs systems. In turn, we

This work was partially supported by the Croatian Science Foundation project 9780 WeConMApp and by the 2014-2017 MIUR-FIR grant no. RBFR13WAET.

will give evidence of how one can exploit properties of abstract Friedrichs systems, recently discussed in the literature, in order to qualify the self-adjoint realisation of a contact interaction Hamiltonian, thus providing yet another (and new) approach to the subject, in addition to the well-established alternative approaches based on operator theory, quadratic forms, non-standard analysis, renormalisation, etc.

We organise our material starting with a background Section on Friedrichs systems on a Hilbert space (Section 2) followed by a background Section on quantum Hamiltonians of contact interactions (Section 3). In Section 4 we establish the general setting for our correspondence between the two classes of operators, and in Sections 5 and 6 we develop such a correspondence in detail for the contact interaction Hamiltonians of δ-type and of δ'-type in one dimension. In complete analogy, in Section 7 we discuss three-dimensional contact interaction Hamiltonians realised as abstract Friedrichs systems. We conclude our presentation with a brief mention to further examples and with a few final remarks in Section 8.

In order to illustrate the intimate bridge announced at the beginning, we choose a presentation where step by step the application of Friedrichs system methods is discussed, so as to reconstruct the considered family of Hamiltonians of contact interactions. For the benefit of the reader, we then recapitulate all the intermediate results in a couple of review statements, see Summary 5.5 and Summary 6.5 below.

Before proceeding, let us fix a few details on our (otherwise essentially standard) notation. By L we denote a complex Hilbert space with scalar product $\langle \cdot \mid \cdot \rangle_L$, which we take to be linear in the first and anti-linear in the second entry. For a densely defined linear operator $A : L \to L$ we denote by $\operatorname{dom} A$, $\overline{A}$, A^* its *domain*, *closure* (if it exists), and *adjoint*, respectively. For $V \subseteq L$, the *restriction* of A to V is denoted by $A|_V$. If $A = A^*$, then A is said to be *self-adjoint*, while the infimum of its spectrum is called the *bottom*. The *identity* operator is denoted by $\mathbb{1}$. $\mathbf{1}_\Lambda$ denotes instead the characteristic function of the set Λ. For a *direct* sum between two vector spaces we use the symbol $\dot{+}$. We write $\ominus$ for the *orthogonal difference* in order to express in which Hilbert space the orthogonal complement is taken. The total derivative of function u, $\frac{\mathrm{d}}{\mathrm{d}x}u$, in short we denote by $\dot{u}$.

2. Background: Friedrichs systems on a Hilbert space

In this Section we focus on the notion of the Friedrichs system on a Hilbert space.

What is today customarily referred to as *Friedrichs systems* is a wide variety of differential equations of mathematical physics, including classical elliptic, parabolic, and hyperbolic equations, which can be re-written in a suitable form, originally identified by Friedrichs in his research on symmetric positive systems [21]. More precisely, for a given open and bounded set $\Omega \subseteq \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$, let the matrix functions $\mathbf{A}_k \in W^{1,\infty}(\Omega)^{r\times r}$

and $\mathbf{C} \in L^\infty(\Omega)^{r\times r}$ satisfy $\mathbf{A}_k = \mathbf{A}_k^*$ and

$$(\exists\, \mu_0 > 0) \quad \mathbf{C} + \mathbf{C}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k \geqslant 2\mu_0 \mathbf{I} \qquad \text{a.e. on } \Omega\,.$$

Then the first-order differential operator $T : L^2(\Omega)^r \longrightarrow \mathcal{D}'(\Omega)^r$ defined by

$$T\mathsf{u} \;:=\; \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u} \tag{2.1}$$

is called *the (classical) Friedrichs operator*, while (for given $\mathsf{f} \in L^2(\Omega)^r$) the first-order system of partial differential equations $T\mathsf{u} = \mathsf{f}$ is called *the (classical) Friedrichs system.*

The general problem for such systems is the well-posedness in a suitable regularity class and for suitable boundary conditions, thus the existence and uniqueness of the solution, as well as its continuous dependence on given data.

Recently, this has become of particular relevance in numerical analysis [24, 25, 17], as Friedrichs systems turned out to provide a conveniently unified framework for numerical solutions to partial differential equations of different types. This aim of ample versatility has also naturally led to formulate the differential problem relative to classical Friedrichs systems in an abstract form on a Hilbert space [20, 4] (see Definition 2.1 below), in order to exploit powerful and general operator-theoretic methods, applicable to each concrete version. Important recent results concern well-posedness results [20, 2, 4], the representations of boundary conditions [2], the connection with the classical theory [3, 4, 5, 6], applications to various initial or boundary value problems of elliptic, hyperbolic, and parabolic type [7, 14, 16, 17, 26], and the development of different numerical schemes [12, 13, 15, 17, 18, 19].

Let us revisit the main features of such an abstract formulation of Friedrichs systems on a Hilbert space.

Definition 2.1 ([20, 4])**.** A densely defined linear operator T on a complex Hilbert space L is called an *abstract Friedrichs operator* if it admits another (densely defined) linear operator $\widetilde{T}$ on L with the following properties:

(T1) T and $\widetilde{T}$ have a common domain $\mathcal{D}$, which is dense in L, satisfying

$$\langle T\phi \mid \psi \rangle_L \;=\; \langle \phi \mid \widetilde{T}\psi \rangle_L\,, \qquad \phi, \psi \in \mathcal{D}\,;$$

(T2) there is a constant $c > 0$ for which

$$\|(T + \widetilde{T})\phi\|_L \;\leqslant\; c\|\phi\|_L\,, \qquad \phi \in \mathcal{D}\,;$$

(T3) there exists a constant $\mu_0 > 0$ such that

$$\langle (T + \widetilde{T})\phi \mid \phi \rangle_L \;\geqslant\; 2\mu_0 \|\phi\|_L^2\,, \qquad \phi \in \mathcal{D}\,.$$

The pair $(T, \widetilde{T})$ is called a *joint pair of abstract Friedrichs operators.* (The definition is indeed symmetric in T and $\widetilde{T}$.)

In fact (see, e.g., [8, Theorem 7]), T and $\widetilde{T}$ in the definition above are closable operators with

$$\operatorname{dom}\overline{T} \;=\; \operatorname{dom}\overline{\widetilde{T}} \qquad \text{and} \qquad \operatorname{dom}\widetilde{T}^* \;=\; \operatorname{dom}T^*\,. \tag{2.2}$$

Theorem 2.2. (see, e.g., [8, Theorem 8]**).** *Let T and $\widetilde{T}$ be two densely defined linear operators on a complex Hilbert space L. Then $(T,\widetilde{T})$ is a joint pair of abstract Friedrichs operators on L if and only if $T\subseteq\widetilde{T}^*$, $\widetilde{T}\subseteq T^*$, and $\overline{T+\widetilde{T}}$ is an everywhere defined, bounded, self-adjoint operator in L with strictly positive bottom. The sole conditions $T\subseteq\widetilde{T}^*$ and $\widetilde{T}\subseteq T^*$ are actually equivalent to condition (T1).*

Thus, for $A_0:=\overline{T}$ and $A_0':=\overline{\widetilde{T}}$, where $(T,\widetilde{T})$ is a joint pair of abstract Friedrichs operators, one has $A_1 := (A_0')^* = \widetilde{T}^*$ and $A_1' := (A_0)^* = T^*$. Based on this and on (2.2), one naturally defines the following.

Definition 2.3 ([8])**.** A *joint pair of closed abstract Friedrichs operators* on a Hilbert space L is a pair (A_0,A_0') of *closed* operators on L satisfying

$$A_0 \;\subseteq\; (A_0')^* \;=:\; A_1 \qquad \text{and} \qquad A_0' \;\subseteq\; (A_0)^* \;=:\; A_1'\,, \tag{2.3}$$

and such that A_0+A_0' is bounded on L and extends to an everywhere defined, bounded, self-adjoint operator in L with strictly positive bottom. The corresponding domains are denoted by

$$W_0 \;:=\; \operatorname{dom}A_0 \;=\; \operatorname{dom}A_0' \qquad \text{and} \qquad W \;:=\; \operatorname{dom}A_1 \;=\; \operatorname{dom}A_1'\,. \tag{2.4}$$

The *abstract boundary value problem* for a Friedrichs operator is determined by a subspace $V\subseteq L$, with $\mathcal{D}\subseteq W_0\subseteq V\subseteq W$, on which the problem $(A_1|_V)u=\widetilde{T}^*u=f$ has a unique solution $u\in V$, for each given $f\in L$.

The choice of V is the direct analogue, for a concrete Friedrichs system (2.1), to the choice of a boundary condition on $\partial\Omega$. More precisely, in [20, 4] it was recognised that conditions on V that are sufficient to make $A_1|_V : V\to L$ an isomorphism (with V equipped with the graph-norm topology of A_1), and thus, equivalently, to make $A_1|_V : V\to L$ a bijection with bounded and everywhere defined inverse on L, correspond to selecting boundary conditions for a concrete Friedrichs system (2.1) that give rise to maximally chosen boundary maps with a definite sign. This is precisely the type of boundary condition occurring in a vast class of classical (concrete) Friedrichs systems of interest, thus making this type of choice particularly relevant.

Elaborating further such abstract boundary conditions in the framework of Definitions 2.1 and 2.3 above one sets the following Definitions 2.4 and 2.6.

Definition 2.4 ([8])**.** Let (A_0,A_0') be a joint pair of closed abstract Friedrichs operators on a Hilbert space L. The *boundary form* associated with (A_0,A_0') is the map $D:W\times W\to\mathbb{C}$ defined by

$$(\forall\, w,w'\in W) \qquad D[w,w'] \;:=\; \langle\, A_1 w \mid w'\,\rangle_L - \langle\, w \mid A_1' w'\,\rangle_L\,.$$

Lemma 2.5. **(**[20, Lemma 2.3 and 2.4] **and** [4, Lemma 1]**).** *The boundary form is symmetric, i.e.,*

$$(\forall\, w, w' \in W) \qquad D[w, w'] \;=\; \overline{D[w', w]}$$

(whence, in particular, $D[w,w] \in \mathbb{R}$*), and*

$$(\forall\, w_0 \in W_0)(\forall\, w' \in W) \qquad D[w_0, w'] \;=\; D[w', w_0] \;=\; 0\,.$$

Definition 2.6 **(**[8]**).** Let (A_0, A_0') be a joint pair of closed abstract Friedrichs operators on a Hilbert space L. Let V and $\widetilde{V}$ be subspaces of L such that

$$W_0 \;\subseteq\; V \;\subseteq\; W \qquad \text{and} \qquad W_0 \;\subseteq\; \widetilde{V} \;\subseteq\; W\,,$$

and with the additional properties that $A_1|_V$ and $A_1'|_{\widetilde{V}}$ are mutually adjoint (thus, in particular, $A_1|_V$ and $A_1'|_{\widetilde{V}}$ are closed operators on L) and that the boundary form satisfies

$$\begin{aligned} (\forall\, u \in V) \qquad D[u,u] \;&=\; \langle\, A_1 u \mid u\,\rangle_L - \langle\, u \mid A_1' u\,\rangle_L \;\geqslant\; 0\,,\\ (\forall\, v \in \widetilde{V}) \qquad D[v,v] \;&=\; \langle\, A_1 v \mid v\,\rangle_L - \langle\, v \mid A_1' v\,\rangle_L \;\leqslant\; 0\,. \end{aligned} \tag{2.5}$$

Then the pair $(A_1|_V, A_1'|_{\widetilde{V}})$ is called an *adjoint pair of bijective realisations with signed boundary map* relative to the given (A_0, A_0').

One may observe that if $V = \widetilde{V}$, i.e., if the domains of two operators of the considered adjoint pair of bijective realisations with signed boundary map are equal, then in (2.5) we have equalities.

In [8], in collaboration with N. Antonić, we solved the until then unanswered problem, given a joint pair of closed abstract Friedrichs operators (A_0, A_0'), of whether bijective realisations of A_0 with signed boundary map *do* exist, with which multiplicity, and possibly according to which general classification. The essentially complete answer is the following.

Theorem 2.7. **(**[8, Theorem 13]**).** *Let* (A_0, A_0') *be a joint pair of closed abstract Friedrichs operators on the Hilbert space L and denote by* (A_1, A_1') *the corresponding pair of adjoint operators* (2.3)*.*

- (i) *There exists an adjoint pair of bijective realisations with signed boundary map relative to* (A_0, A_0')*. Moreover, there is an adjoint pair* $(A_{\mathrm{r}}, A_{\mathrm{r}}^*)$ *of bijective realisations with signed boundary map relative to* (A_0, A_0') *such that*

$$W_0 + \ker A_1' \;\subseteq\; \operatorname{dom} A_{\mathrm{r}} \qquad \textit{and} \qquad W_0 + \ker A_1 \;\subseteq\; \operatorname{dom} A_{\mathrm{r}}^*\,.$$

- (ii) *If both* $\ker A_1 \neq \{0\}$ *and* $\ker A_1' \neq \{0\}$*, then the pair* (A_0, A_0') *admits uncountably many adjoint pairs of bijective realisations with signed boundary map. On the other hand, if either* $\ker A_1 = \{0\}$ *or* $\ker A_1' = \{0\}$*, then there is exactly one adjoint pair of bijective realisations with signed boundary map relative to* (A_0, A_0')*. Such a pair is precisely* (A_1, A_0') *when* $\ker A_1 = \{0\}$*, and* (A_0, A_1') *when* $\ker A_1' = \{0\}$*.*
- (iii) *An explicit (constructive) classification of all adjoint pairs of bijective realisations with signed boundary map relative to* (A_0, A_0') *is given by Theorem* 2.9 *below.*

Our actual formulation of the classification mentioned in Theorem 2.7(iii) requires the language of Grubb's extension theory of densely defined and closed operators on Hilbert space [22]. For convenience, we cast the main features of this theory in the following Theorem.

Theorem 2.8. ([22, Chapter II] **and** [23, Chapter 13])**.** *Let (A_0, A_0') and (A_1, A_1') be two pairs of mutually adjoint, closed and densely defined operators in L satisfying*

$$A_0 \subseteq (A_0')^* = A_1 \qquad \textit{and} \qquad A_0' \subseteq (A_0)^* = A_1' ,$$

which admit a further pair $(A_{\mathrm{r}}, A_{\mathrm{r}}^*)$ of reference operators that are closed, satisfy $A_0 \subseteq A_{\mathrm{r}} \subseteq A_1$, equivalently $A_0' \subseteq A_{\mathrm{r}}^* \subseteq A_1'$, and are invertible with everywhere defined bounded inverses A_{r}^{-1} and $(A_{\mathrm{r}}^*)^{-1}$.

(i) *There are decompositions*

$$\mathrm{dom}\, A_1 = \mathrm{dom}\, A_{\mathrm{r}} \dot{+} \ker A_1 \qquad \textit{and} \qquad \mathrm{dom}\, A_1' = \mathrm{dom}\, A_{\mathrm{r}}^* \dot{+} \ker A_1' , \tag{2.6}$$

the corresponding projections

$$\begin{aligned} p_{\mathrm{r}} &: \mathrm{dom}\, A_1 \to \mathrm{dom}\, A_{\mathrm{r}} , & p_{\mathrm{k}} &: \mathrm{dom}\, A_1 \to \ker A_1 , \\ p_{\mathrm{r}'} &: \mathrm{dom}\, A_1' \to \mathrm{dom}\, A_{\mathrm{r}}^* , & p_{\mathrm{k}'} &: \mathrm{dom}\, A_1' \to \ker A_1' , \end{aligned} \tag{2.7}$$

satisfying

$$\begin{aligned} p_{\mathrm{r}} &= A_{\mathrm{r}}^{-1} A_1 , & p_{\mathrm{r}'} &= (A_{\mathrm{r}}^*)^{-1} A_1' , \\ p_{\mathrm{k}} &= \mathbb{1} - p_{\mathrm{r}} , & p_{\mathrm{k}'} &= \mathbb{1} - p_{\mathrm{r}'} , \end{aligned} \tag{2.8}$$

and being continuous with respect to the graph norms.

(ii) *There is a one-to-one correspondence between all pairs of mutually adjoint operators (A, A^*) with $A_0 \subseteq A \subseteq A_1$, equivalently $A_0' \subseteq A^* \subseteq A_1'$, and all pairs of densely defined mutually adjoint operators $B : \mathcal{V} \to \mathcal{W}$ and $B^* : \mathcal{W} \to \mathcal{V}$, with domains* $\mathrm{dom}\, B \subseteq \mathcal{V}$ *and* $\mathrm{dom}\, B^* \subseteq \mathcal{W}$, *where $\mathcal{V}$ and $\mathcal{W}$ run through all closed subspaces of* $\ker A_1$ *and* $\ker A_1'$. *The correspondence is given by*

$$\begin{aligned} \mathrm{dom}\, A &= \Big\{ u \in \mathrm{dom}\, A_1 \,:\, p_{\mathrm{k}} u \in \mathrm{dom}\, B\,,\ P_{\mathcal{W}}(A_1 u) = B(p_{\mathrm{k}} u) \Big\} , \\ \mathrm{dom}\, A^* &= \Big\{ v \in \mathrm{dom}\, A_1' \,:\, p_{\mathrm{k}'} v \in \mathrm{dom}\, B^*\,,\ P_{\mathcal{V}}(A_1' v) = B^*(p_{\mathrm{k}'} v) \Big\} , \end{aligned} \tag{2.9}$$

and conversely, by

$$\begin{aligned} \mathrm{dom}\, B &= p_{\mathrm{k}}\, \mathrm{dom}\, A , & \mathcal{V} &= \overline{\mathrm{dom}\, B} , & B(p_{\mathrm{k}} u) &= P_{\mathcal{W}}(A_1 u) , \\ \mathrm{dom}\, B^* &= p_{\mathrm{k}'}\, \mathrm{dom}\, A^* , & \mathcal{W} &= \overline{\mathrm{dom}\, B^*} , & B^*(p_{\mathrm{k}'} v) &= P_{\mathcal{V}}(A_1' v) , \end{aligned} \tag{2.10}$$

where $P_{\mathcal{V}}$ and $P_{\mathcal{W}}$ are the orthogonal *projections from L onto $\mathcal{V}$ and $\mathcal{W}$.*

(iii) *In the correspondence above, A is injective, resp. surjective, resp. bijective, if and only if so is B.*

(iv) *When* A_B *corresponds to* B *as above, then*

$$\operatorname{dom} A_B \;=\; \left\{ w_0 + (A_{\mathrm{r}})^{-1}(B\nu + \nu') + \nu \;\middle|\; \begin{array}{c} w_0 \in \operatorname{dom} A_0 \\ \nu \in \operatorname{dom} B \\ \nu' \in \ker A_1' \ominus \mathcal{W} \end{array} \right\}, \tag{2.11}$$

$$A_B(w_0 + (A_{\mathrm{r}})^{-1}(B\nu + \nu') + \nu) \;=\; A_0 w_0 + B\nu + \nu'$$

and

$$\operatorname{dom}(A_B)^* \;=\; \left\{ w_0' + (A_{\mathrm{r}}^*)^{-1}(B^*\mu' + \mu) + \mu' \;\middle|\; \begin{array}{c} w_0' \in \operatorname{dom} A_0' \\ \mu' \in \operatorname{dom} B^* \\ \mu \in \ker A_1 \ominus \mathcal{V} \end{array} \right\}, \tag{2.12}$$

$$(A_B)^*(w_0' + (A_{\mathrm{r}}^*)^{-1}(B^*\mu' + \mu) + \mu') \;=\; A_0' w_0' + B^*\mu' + \mu\,,$$

and, moreover,

$$(A_B)^* \;=\; A_{B^*}\,.$$

Observe that our notation is chosen in such a way that p denotes the projection induced by a direct sum decomposition, whereas P denotes the *orthogonal* projection onto a *closed* subspace. Furthermore, the non-primed ν's or μ's all belong to $\ker A_1$, whereas their primed counterparts belong to $\ker A_1'$. Let us also remark that $\ker A_1' \ominus \mathcal{W}$ denotes the orthogonal complement of $\mathcal{W}$ in $\ker A_1'$, and respectively for $\ker A_1 \ominus \mathcal{V}$. For the trivial choice $\mathcal{V} = \mathcal{W} = \{0\}$ one has $A_B = A_{\mathrm{r}}$.

We can now provide the complete formulation of Theorem 2.7(iii).

Theorem 2.9. **(**[8, Theorem 18]**).** *Let* (A_0, A_0') *be a joint pair of closed abstract Friedrichs operators, and let* $(A_{\mathrm{r}}, A_{\mathrm{r}}^*)$ *be an adjoint pair of bijective realisations of* (A_0, A_0') *with signed boundary map. Let* (A_B, A_B^*) *be a generic pair of closed extensions* $A_0 \subseteq A_B \subseteq A_1$, $A_0' \subseteq A_B^* \subseteq A_1'$, *according to the notation of the parametrisation given in Theorem* 2.8. *Let* p_{k} *and* $\mathrm{p}_{\mathrm{k}'}$ *be the projections* (2.7) *identified by direct decompositions* (2.6). *With reference to the following two sets of* 'mirror' *conditions, namely*

$$\begin{array}{ll} (\forall\, \nu \in \operatorname{dom} B) & \\ (\forall\, \nu' \in \ker A_1' \ominus \mathcal{W}) & \end{array} \left\{ \begin{array}{r} \langle\, \nu \mid A_1'\nu \,\rangle_L - 2\,\mathfrak{Re}\langle\, \mathrm{p}_{\mathrm{k}'}\nu \mid B\nu \,\rangle_L \;\leqslant\; 0 \\ \langle\, \mathrm{p}_{\mathrm{k}'}\nu \mid \nu' \,\rangle_L \;=\; 0 \end{array} \right. \tag{2.13}$$

and

$$\begin{array}{ll} (\forall\, \mu' \in \operatorname{dom} B^*) & \\ (\forall\, \mu \in \ker A_1 \ominus \mathcal{V}) & \end{array} \left\{ \begin{array}{r} \langle\, A_1\mu' \mid \mu' \,\rangle_L - 2\,\mathfrak{Re}\langle\, B^*\mu' \mid \mathrm{p}_{\mathrm{k}}\mu' \,\rangle_L \;\leqslant\; 0 \\ \langle\, \mu \mid \mathrm{p}_{\mathrm{k}}\mu' \,\rangle_L \;=\; 0\,, \end{array} \right. \tag{2.14}$$

one has these conclusions.

(i) *Any of the following three facts,*
  (a) *conditions* (2.13) *and* (2.14) *hold true, or*
  (b) *condition* (2.13) *holds true and* $B : \operatorname{dom} B \to \mathcal{W}$ *is a bijection, or*
  (c) *condition* (2.14) *holds true and* $B^* : \operatorname{dom} B^* \to \mathcal{V}$ *is a bijection,*

is sufficient *for* (A_B, A_B^*) *to be another adjoint pair of bijective realisations with signed boundary map relative to* (A_0, A_0').

(ii) *Assume further that* $\operatorname{dom} A_{\mathrm{r}} = \operatorname{dom} A_{\mathrm{r}}^*$. *Then the following properties are* equivalent*:*

(a) (A_B, A_B^*) *is another adjoint pair of bijective realisations with signed boundary map relative to* (A_0, A_0')*;*
(b) *the mirror conditions* (2.13) *and* (2.14) *are satisfied.*

Summarising, the whole class of adjoint pairs of bijective realisations of a given joint pair (A_0, A_0') of abstract Friedrichs operators is fully characterised by Theorem 2.8 with respect to a reference pair $(A_\mathrm{r}, A_\mathrm{r}^*)$ (which may be chosen conveniently, and in any case it exists, as guaranteed by Theorem 2.7(i)). In addition, the qualification of the special (and relevant) sub-class of bijective realisations of (A_0, A_0') *with signed boundary map* is provided by Theorem 2.9, if $(A_\mathrm{r}, A_\mathrm{r}^*)$ is taken to be a pair of bijective realisations of (A_0, A_0') with signed boundary map and with the property $\mathrm{dom}\, A_\mathrm{r} = \mathrm{dom}\, A_\mathrm{r}^*$.

3. Quantum particle subject to a one centre contact interaction

Next to the previous background on abstract Friedrichs systems, let us concisely revisit in this Section the mathematical model for a quantum particle in $\mathbb{R}^d$, $d \in \{1, 2, 3\}$, subject to an interaction of zero range supported in a point $x_0 \in \mathbb{R}^d$. For convenience, we take $x_0 = 0$. Our discussion extends with natural changes to the case of finitely or infinitely many centres $x_0, x_1, x_2, ... \in \mathbb{R}^d$ and to keep it as clean as possible we only consider the one-centre case.

As long as the particle is a point particle without internal degrees of freedom, a quantum Hamiltonan of zero range interaction localised at the point $x_0 = 0$ is a *self-adjoint* extension, with respect to the Hilbert space $L^2(\mathbb{R}^d)$, of the restriction of negative Laplacian on smooth functions compactly supported away from the origin $-\Delta|_{C^\infty_c(\mathbb{R}^d \setminus \{0\})}$, so as to model a particle that moves freely as long as its wave function is supported away from the interaction centre. In fact, if $d \geqslant 4$, then $-\Delta|_{C^\infty_c(\mathbb{R}^d \setminus \{0\})}$ is already essentially self-adjoint and its unique self-adjoint realisation, its operator closure, is the self-adjoint negative Laplacian with domain $H^2(\mathbb{R}^d)$, whereas when $d \in \{1, 2, 3\}$ such an operator admits non-trivial self-adjoint extensions, each of which describes a physically inequivalent model.

The construction, the classification, and the study of such self-adjoint realisations is completely understood in the literature, by a number of alternative means, ranging from operator and extension theory to quadratic form theory, limiting procedures of Schrödinger operators with finite range potentials ‘shrinking’ to a delta-like profile, renormalisation procedures of the coupling constant of the delta-like interaction, as well as methods of non-standard analysis. A comprehensive overview may be found in [1] and in the end-of-chapter notes and references therein.

We refer to [1] for the formulation of the Theorems below. Also, we discuss explicitly dimensions $d = 1$ and $d = 3$, which correspond to deficiency indices $(2, 2)$ and $(1, 1)$ respectively, while the case $d = 2$ resembles very much the case $d = 3$.

Theorem 3.1 (One-dimensional case).

(i) *With respect to the Hilbert space* $L^2(\mathbb{R})$*,*

$$\mathring{H} \;:=\; -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,, \qquad \mathrm{dom}\,\mathring{H} \;:=\; C_c^\infty(\mathbb{R}\backslash\{0\}) \tag{3.1}$$

defines a symmetric, positive, and densely defined operator with the adjoint given by

$$\mathring{H}^* \;:=\; -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,, \qquad \mathrm{dom}\,\mathring{H}^* \;:=\; H^2(\mathbb{R}\backslash\{0\})$$

and with deficiency indices $(2,2)$*, i.e.,*

$$\dim\ker(\mathring{H}^* \pm \mathrm{i}\mathbb{1}) \;=\; \dim\ker(\mathring{H}^* + \mathbb{1}) \;=\; 2\,.$$

In fact,

$$\ker(\mathring{H}^* + \mathbb{1}) \;=\; \mathrm{span}\{\psi_1,\psi_2\}\,, \qquad \begin{array}{rcl} \psi_1(x) & := & \mathbf{1}_{[0,+\infty)}(x)\,e^{-x} \\ \psi_2(x) & := & \mathbf{1}_{(-\infty,0]}(x)\,e^{x}\,. \end{array}$$

Thus, $\mathring{H}$ *admits a four (real) parameter family of self-adjoint extensions.*

(ii) *The one-parameter family* $\{-\Delta_\alpha\,|\,\alpha\in(-\infty,+\infty]\}$ *of operators defined by*

$$\begin{aligned} -\Delta_\alpha \;&:=\; -\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,, \\ \mathrm{dom}(-\Delta_\alpha) \;&:=\; \left\{ \begin{array}{c} g\in H^2(\mathbb{R}\backslash\{0\})\cap H^1(\mathbb{R}) \textit{ such that} \\ \dot g(0^+)-\dot g(0^-) \;=\; \alpha g(0) \end{array} \right\} \end{aligned} \tag{3.2}$$

is a sub-family of self-adjoint extensions of $\mathring{H}$ *(the so-called '*δ*-type extensions'). For each* $-\Delta_\alpha$ *one has*

$$\sigma_{\mathrm{ess}}(-\Delta_\alpha) \;=\; \sigma_{\mathrm{ac}}(-\Delta_\alpha) \;=\; [0,+\infty)\,, \qquad \sigma_{\mathrm{sc}}(-\Delta_\alpha) \;=\; \emptyset\,,$$

$$\sigma_{\mathrm{p}}(-\Delta_\alpha) \;=\; \begin{cases} \{-\alpha^2/4\} & \textit{if } \alpha\in(-\infty,0) \\ \emptyset & \textit{if } \alpha\notin(-\infty,0)\,, \end{cases}$$

the corresponding eigenvalue when $\alpha\in(-\infty,0)$ *being non-degenerate with normalised eigenfunction* $\sqrt{-\frac{\alpha}{2}}\,e^{\alpha|x|/2}$*. The special case* $\alpha=0$ *corresponds to the self-adjoint negative Laplacian* $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ *with domain* $H^2(\mathbb{R})$*, whereas the case* $\alpha=+\infty$ *yields the homogeneous Dirichlet boundary condition at the origin and corresponds to the 'decoupled' Hamiltonian*

$$\begin{aligned} \mathrm{dom}(-\Delta_\infty) \;&=\; \{g\in H^2(\mathbb{R}\backslash\{0\})\cap H^1(\mathbb{R}))\;:\;g(0)=0\} \\ &=\; H^2_0(\mathbb{R}^-)\oplus H^2_0(\mathbb{R}^+)\,, \\ -\Delta_\infty \;&=\; -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\oplus-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,, \end{aligned}$$

i.e., $-\Delta_\infty$ *is the direct sum of the negative Dirichlet Laplacians on each half line.*

(iii) *The one-parameter family* $\{\Xi_\beta \,|\, \beta \in (-\infty,+\infty]\}$ *of operators defined by*

$$\Xi_\beta \;:=\; -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,, \qquad \mathrm{dom}(\Xi_\beta) \;:=\; \left\{ \begin{array}{c} g \in H^2(\mathbb{R}\backslash\{0\}) \textit{ such that} \\ \dot g(0^+) = \dot g(0^-) =: \dot g(0) \\ g(0^+) - g(0^-) \;=\; \beta \dot g(0) \end{array} \right\} \tag{3.3}$$

is a sub-family of self-adjoint extensions of $\mathring{H}$ *(the so-called '*δ'*-type extensions'). For each* Ξ_β *one has*

$$\begin{aligned} \sigma_{\mathrm{ess}}(\Xi_\beta) \;&=\; \sigma_{\mathrm{ac}}(\Xi_\beta) \;=\; [0,+\infty)\,, \qquad \sigma_{\mathrm{sc}}(\Xi_\beta) \;=\; \emptyset\,, \\ \sigma_{\mathrm{p}}(\Xi_\beta) \;&=\; \begin{cases} \{-4/\beta^2\} & \textit{if } \beta \in (-\infty,0) \\ \quad \emptyset & \textit{if } \beta \notin (-\infty,0)\,, \end{cases} \end{aligned}$$

the corresponding eigenvalue when $\beta \in (-\infty,0)$ *being non-degenerate with normalised eigenfunction* $\sqrt{-\frac{\beta}{8}}\,\mathrm{sgn}(x)\,e^{2|x|/\beta}$*. The special case* $\beta = 0$ *corresponds to the self-adjoint negative Laplacian* $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ *with domain* $H^2(\mathbb{R})$*, whereas the case* $\beta = +\infty$ *yields the homogeneous Neumann boundary condition at the origin and corresponds to the 'decoupled' Hamiltonian*

$$\begin{aligned} \mathrm{dom}(\Xi_\infty) \;&=\; \{g \in H^2(\mathbb{R}\backslash\{0\}) \,:\, \dot g(0^+) = \dot g(0^-) = 0\} \\ &=\; \{g \in H^2(\mathbb{R}^-) \,:\, \dot g(0^-) = 0\} \oplus \{g \in H^2(\mathbb{R}^+) \,:\, \dot g(0^+) = 0\}\,, \\ \Xi_\infty \;&=\; -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \oplus -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\,, \end{aligned}$$

i.e., Ξ_∞ *is the direct sum of the negative Neumann Laplacians on each half line.*

It is worth remarking that the operator closure of $\mathring{H}$ has domain

$$\mathrm{dom}(\overline{\mathring{H}}) = H^2_0(\mathbb{R}\backslash\{0\}) = \{g \in H^2(\mathbb{R}) \,:\, g(0) = \dot g(0) = 0\}\,.$$

The 'δ-type extensions' $-\Delta_\alpha$ extend also the closed symmetric operator acting as $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on the domain $\{g \in H^2(\mathbb{R}) \,:\, g(0) = 0\}$, which is larger than $\mathrm{dom}(\overline{\mathring{H}})$, and whose adjoint is the operator acting as $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on the domain $H^2(\mathbb{R}\backslash\{0\}) \cap H^1(\mathbb{R})$, which is smaller than $\mathrm{dom}(\mathring{H}^*)$.

Theorem 3.2 (Three-dimensional case).

(i) *With respect to the Hilbert space* $L^2(\mathbb{R}^3)$*,*

$$\mathring{H} \;:=\; -\Delta\,, \qquad \mathrm{dom}\,\mathring{H} \;:=\; C^\infty_c(\mathbb{R}^3\backslash\{0\}) \tag{3.4}$$

defines a symmetric, positive, and densely defined operator with the adjoint given by

$$\mathring{H}^* \;:=\; -\Delta\,, \qquad \mathrm{dom}\,\mathring{H}^* \;:=\; \{g \in H^2_{\mathrm{loc}}(\mathbb{R}^3\backslash\{0\}) \cap L^2(\mathbb{R}^3) \,:\, \Delta g \in L^2(\mathbb{R}^3)\}$$

and with deficiency indices $(1,1)$*, i.e.,*

$$\dim\ker(\mathring{H}^* \pm \mathrm{i}\mathbb{1}) \;=\; \dim\ker(\mathring{H}^* + \mathbb{1}) \;=\; 1\,.$$

In fact,

$$\ker(\mathring{H}^* + \mathbb{1}) \;=\; \mathrm{span}\{\psi_0\}\,, \qquad \psi_0(x) \;:=\; \frac{e^{-|x|}}{|x|}\,.$$

Thus, $\mathring{H}$ admits a one (real) parameter family of self-adjoint extensions.

(ii) *With respect to canonical isomorphism (partial wave decomposition)*

$$L^2(\mathbb{R}^3) \;\cong\; \bigoplus_{\ell=0}^{\infty} U^{-1}L^2(\mathbb{R}^+,\mathrm{d}r)\otimes \mathrm{span}\{Y_{\ell,-\ell},\dots,Y_{\ell,\ell}\}\,,$$

where $x \equiv r\omega$, $r := |x|$, $\omega \in \mathbb{S}^2$ give the polar coordinates for $x \in \mathbb{R}^3$, the $Y_{\ell,m}$'s form the orthonormal basis of spherical harmonics for $L^2(\mathbb{S}^2)$, and $U : L^2(\mathbb{R}^+, r^2\mathrm{d}r) \to L^2(\mathbb{R}^+,\mathrm{d}r)$ is the unitary operator defined by $(Uf)(r) := rf(r)$, the operator $\mathring{H}$ reduces as

$$\mathring{H} \;=\; \bigoplus_{\ell=0}^{\infty} U^{-1}h_\ell\, U \otimes \mathbb{1}\,,$$

where each h_ℓ acts on $L^2(\mathbb{R}^+,\mathrm{d}r)$ as

$$h_\ell \;=\; -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\ell(\ell+1)}{r^2}$$

with domain

$$\mathrm{dom}\, h_0 \;=\; H^2_0(\mathbb{R}^+) \;=\; \left\{\phi \in L^2(\mathbb{R}^+) \,\middle|\, \begin{array}{c} \phi,\dot\phi \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}^+),\ \ddot\phi \in L^2(\mathbb{R}^+), \\ \phi(0^+) = \dot\phi(0^+) = 0 \end{array}\right\},$$

$$\mathrm{dom}\, h_\ell \;=\; \left\{\phi \in L^2(\mathbb{R}^+) \,\middle|\, \begin{array}{c} \phi,\dot\phi \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}^+), \\ -\ddot\phi + \ell(\ell+1)r^{-2}\phi \in L^2(\mathbb{R}^+) \end{array}\right\}, \qquad \ell \geqslant 1\,.$$

For $\ell \geqslant 1$ each h_ℓ is self-adjoint and in fact $\oplus_{\ell=1}^{\infty} h_\ell$ is precisely the component of the self-adjoint negative Laplacian on the sector of angular momentum $\ell \geqslant 1$. h_0 is symmetric, positive, and densely defined in $L^2(\mathbb{R}^+)$ with deficiency indices $(1,1)$, hence admits a one-parameter family of self-adjoint extensions.

(iii) *The self-adjoint extensions of h_0 form the family $\{h_{0,\alpha}\,|\,\alpha \in (-\infty,+\infty]\}$, where*

$$\begin{aligned} h_{0,\alpha} \;&=\; -\frac{\mathrm{d}^2}{\mathrm{d}r^2}\,, \\ \mathrm{dom}\, h_{0,\alpha} \;&=\; \left\{\phi \in L^2(\mathbb{R}^+) \,\middle|\, \begin{array}{c} \phi,\dot\phi \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}^+),\ \ddot\phi \in L^2(\mathbb{R}^+), \\ \dot\phi(0^+) = 4\pi\alpha\,\phi(0^+) \end{array}\right\}. \end{aligned} \tag{3.5}$$

In turn, $\{-\Delta_\alpha\,|\,\alpha \in (-\infty,+\infty]\}$, with $-\Delta_\alpha := h_{0,\alpha} \oplus (\oplus_{\ell=1}^{\infty} h_\ell)$, is the family of all self-adjoint extensions of $\mathring{H}$ on $L^2(\mathbb{R}^3)$. For each $-\Delta_\alpha$ one has

$$\sigma_{\mathrm{ess}}(-\Delta_\alpha) \;=\; \sigma_{\mathrm{ac}}(-\Delta_\alpha) \;=\; [0,+\infty)\,, \qquad \sigma_{\mathrm{sc}}(-\Delta_\alpha) \;=\; \emptyset\,,$$

$$\sigma_{\mathrm{p}}(-\Delta_\alpha) \;=\; \begin{cases} \{-(4\pi\alpha)^2\} & \textit{if } \alpha \in (-\infty,0) \\ \quad\emptyset & \textit{if } \alpha \notin (-\infty,0)\,, \end{cases}$$

the corresponding eigenvalue when $\alpha \in (-\infty, 0)$ *being non-degenerate with normalised eigenfunction* $\sqrt{-\alpha}\, e^{4\pi\alpha|x|}/|x|$. *The special case* $\alpha = +\infty$ *corresponds to the self-adjoint negative Laplacian on* $L^2(\mathbb{R}^3)$ *with domain* $H^2(\mathbb{R}^3)$.

4. Contact interactions realised as Friedrichs systems: set-up

We are going to establish the following connection between the subjects of Sections 2 and 3, which we find new and particularly informative. Let us first formulate it in an informal way.

Outline:

(i) The operator $\mathring{H}$ considered in Theorem 3.1 or 3.2 is in one-to-one correspondence to a joint pair of abstract Friedrichs operators;

(ii) the contact interaction Hamiltonians, namely the self-adjoint extensions of $\mathring{H}$, are in one-to-one correspondence to an adjoint pair of abstract *closed* Friedrichs operators;

(iii) the quest, by means of Friedrichs systems methods, of the bijective realisations of the joint pair of abstract Friedrichs operators corresponding to $\mathring{H}$ produces, in a novel and independent way, the family of invertible self-adjoint extensions of $\mathring{H}$;

(iv) moreover, the bijective realisations *with signed boundary map* of the joint pair of abstract Friedrichs operators corresponding to $\mathring{H}$ yield the subclass of self-adjoint extensions of $\mathring{H}$ with empty point spectrum, that is, the *repulsive* (*non-confining*) Hamiltonians of contact interactions.

Let us develop the above programme in the one-dimensional case first, and set up the needed preliminaries.

4.1. Order reduction

Step (i) requires an obvious but crucial *reduction of the order of the differential operator*, in order to match the conditions (T1) and (T2) of Definition 2.1. Indeed, (3.1) defines a *second order*, *symmetric*, and *unbounded* differential operator $\mathring{H}$ on $L^2(\mathbb{R}^3)$: as such, the pair $(T, \widetilde{T}) = (\mathring{H}, \mathring{H})$ clearly satisfies (T1), for (T1) is tantamount as $T \subseteq \widetilde{T}^*$ and $\widetilde{T} \subseteq T^*$ (Theorem 2.2), but fails to satisfy (T2), which is a boundedness requirement on $T + \widetilde{T}$ and is only possible if in that sum an exact cancellation occurs between the unbounded parts of T and $\widetilde{T}$. This latter phenomenon is rather typical of suitable odd-order differential operators T for which (T1) reads $\widetilde{T} = -T + B$ for some bounded operator B – which is precisely what happens in the prototypical case of formula (2.1).

This suggests to rather think of the *second order*, *scalar* differential problem $f = \mathring{H}u$, where $u \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$, as the *first order* and *matrix-valued* differential problem

$$\begin{pmatrix} \dot{u} \\ f \end{pmatrix} = S \begin{pmatrix} -\dot{u} \\ u \end{pmatrix} = \begin{pmatrix} 0 & \frac{\mathrm{d}}{\mathrm{d}x} \\ \frac{\mathrm{d}}{\mathrm{d}x} & 0 \end{pmatrix} \begin{pmatrix} -\dot{u} \\ u \end{pmatrix}, \qquad S := \sigma \frac{\mathrm{d}}{\mathrm{d}x},$$

where $\sigma = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the first 2×2 Pauli matrix. This way, the pair $(T, \widetilde{T}) = (S, -S)$ obviously satisfies conditions (T1) and (T2). Condition (T3) of Definition 2.1 only fails to hold because $T + \widetilde{T} = \mathbb{O}$, which would be immediately cured by a non-restrictive shift of S, e.g., considering the pair $(T, \widetilde{T}) = (S + \mathbb{1}, -S + \mathbb{1})$, which can be obtained by shifting $\mathring{H}$ to $\mathring{H} + \mathbb{1}$ on the first place. In fact, it will be convenient in our discussion to exploit the freedom of a generic shift of S.

Let us therefore introduce on the Hilbert space

$$L := L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \tag{4.1}$$

the densely defined operators $T_\lambda : L \to L$ and $\widetilde{T}_\lambda : L \to L$, for arbitrary $\lambda > 0$, defined by

$$\begin{aligned} T_\lambda &:= \sigma \frac{\mathrm{d}}{\mathrm{d}x} + \lambda \mathbb{1} \\ \widetilde{T}_\lambda &:= -\sigma \frac{\mathrm{d}}{\mathrm{d}x} + \lambda \mathbb{1} \\ \operatorname{dom} T_\lambda &:= \operatorname{dom} \widetilde{T}_\lambda := C_c^\infty(\mathbb{R} \setminus \{0\}) \oplus C_c^\infty(\mathbb{R} \setminus \{0\}) . \end{aligned} \tag{4.2}$$

Thus, for a generic $\mathsf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in L$,

$$T_\lambda \mathsf{u} = \begin{pmatrix} \dot{u}_2 + \lambda u_1 \\ \dot{u}_1 + \lambda u_2 \end{pmatrix} .$$

Next, we introduce the linear map

$$\begin{aligned} \Phi &: \mathfrak{L}(L^2(\mathbb{R}) \oplus L^2(\mathbb{R})) \longrightarrow \mathfrak{L}(L^2(\mathbb{R})) , \\ \operatorname{dom} \Phi(A) &:= \left\{ u \in L^2(\mathbb{R}) : (\exists!\, v_u \in L^2(\mathbb{R})) \quad \begin{pmatrix} v_u \\ u \end{pmatrix} \in \operatorname{dom} A \cap \ker P_1 A \right\} , \\ \Phi(A)\, u &:= P_2 A \begin{pmatrix} v_u \\ u \end{pmatrix} , \end{aligned} \tag{4.3}$$

where $\mathfrak{L}(X)$ is the space of *linear* (not necessarily bounded) maps on the vector space X and $P_j : L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $j \in \{1, 2\}$, is the orthogonal projection onto the j-th component of L.

Then we have the following Lemma, whose proof is straightforward and thus omitted.

Lemma 4.1. *Let* $\lambda > 0$.

(i) *The pair* $(T_\lambda, \widetilde{T}_\lambda)$ *is a joint pair of abstract Friedrichs operators on the Hilbert space* L.

(ii) *Given* $u \in L^2(\mathbb{R})$, $u \in \operatorname{dom}\Phi(T_\lambda)$ *if and only if* $u \in C_c^\infty(\mathbb{R}\backslash\{0\})$, *and*

$$\Phi(T_\lambda)u \;=\; \lambda^{-1}(\mathring{H} + \lambda^2 \mathbb{1})u\,.$$

In other words, the equality

$$\Phi(T_\lambda) \;=\; \lambda^{-1}(\mathring{H} + \lambda^2 \mathbb{1})$$

is valid as an identity of operators on L. Therefore:

- the correspondence $\frac{1}{\lambda}(\mathring{H} + \lambda^2\mathbb{1}) \mapsto T_\lambda$ amounts to pass from the second order differential operator of interest, to a first order differential operator that is part of a joint pair of abstract Friedrichs operators on L;
- the converse procedure of this order reduction is provided by the map $T_\lambda \mapsto \Phi(T_\lambda) = \frac{1}{\lambda}(\mathring{H} + \lambda^2\mathbb{1})$.

Next, let us search for self-adjoint extensions of $\mathring{H}$ on $L^2(\mathbb{R})$ (and in fact, more generally, for closed extensions of $\mathring{H}$) by reducing $\mathring{H}$ to the pair $(T_\lambda, \widetilde{T}_\lambda)$, determining the relevant adjoint realisations of such a pair on L, and then lifting, via the map Φ, each such realisation to an extension of $\mathring{H}$.

To this aim, let us set up in the following Subsection the analysis of (the operator closure of) the pair $(T_\lambda, \widetilde{T}_\lambda)$.

4.2. Analysis of the auxiliary joint pair of abstract Friedrichs operators

Let us define

$$\begin{aligned} A_{\lambda,0} &:= \overline{T_\lambda}\,, & A_{\lambda,1} &:= \widetilde{T}_\lambda^*\,, \\ A'_{\lambda,0} &:= \overline{\widetilde{T}_\lambda}\,, & A'_{\lambda,1} &:= T_\lambda^*\,. \end{aligned} \tag{4.4}$$

Then

$$\begin{aligned} \operatorname{dom} A_{\lambda,0} &= \operatorname{dom} A'_{\lambda,0} = H_0^1(\mathbb{R}\backslash\{0\}) \oplus H_0^1(\mathbb{R}\backslash\{0\}) =: W_0 \\ \operatorname{dom} A_{\lambda,1} &= \operatorname{dom} A'_{\lambda,1} = H^1(\mathbb{R}\backslash\{0\}) \oplus H^1(\mathbb{R}\backslash\{0\}) =: W\,, \end{aligned} \tag{4.5}$$

and $(A_{\lambda,0}, A'_{\lambda,0})$ is a joint pair of closed abstract Friedrichs operators according to Definition 2.3.

As a weak differential operator, $A_{\lambda,1}$, respectively $A'_{\lambda,1}$, acts formally as T_λ, respectively $\widetilde{T}_\lambda$:

$$A_{\lambda,1}\mathsf{u} \;=\; \sigma\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{u} + \lambda\mathsf{u}\,, \qquad A'_{\lambda,1}\mathsf{u} \;=\; -\sigma\frac{\mathrm{d}}{\mathrm{d}x}\mathsf{u} + \lambda\mathsf{u}\,.$$

The boundary form associated with the pair $(A_{\lambda,0}, A'_{\lambda,0})$ (Definition 2.4) is obtained by integration by parts: for any $\mathsf{u} \equiv \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\mathsf{v} \equiv \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in W,

$$\begin{aligned} D[\mathsf{u},\mathsf{v}] &= \langle\, A_{\lambda,1}\mathsf{u} \mid \mathsf{v}\,\rangle_L - \langle\, \mathsf{u} \mid A'_{\lambda,1}\mathsf{v}\,\rangle_L \\ &= -\big(u_2(0^+)\bar{v}_1(0^+) - u_2(0^-)\bar{v}_1(0^-)\big) \\ &\qquad - \big(u_1(0^+)\bar{v}_2(0^+) - u_1(0^-)\bar{v}_2(0^-)\big)\,, \end{aligned}$$

where $u(0^\pm) := \lim_{x\to 0^\pm} u(x)$. Clearly, D does not depend on λ.

There is a natural choice for an adjoint pair $(A_{\lambda,\mathrm{r}}, A^*_{\lambda,\mathrm{r}})$ of *bijective realisations with signed boundary map* relative to $(A_{\lambda,0}, A'_{\lambda,0})$, in the sense of Definition 2.6: indeed, choosing the subspaces

$$V \;:=\; \widetilde{V} \;:=\; H^1(\mathbb{R}) \oplus H^1(\mathbb{R})\,, \tag{4.6}$$

it is immediately seen that $W_0 \subseteq V \subseteq W$, $W_0 \subseteq \widetilde{V} \subseteq W$, that correspondingly $A_{\lambda,1}|_V$ and $A'_{\lambda,1}|_V$ are mutually adjoint operators, and that the boundary form D vanishes on V, this latter fact following from the continuity of any $H^1(\mathbb{R})$-function. Therefore,

$$A_{\lambda,\mathrm{r}} \;:=\; A_{\lambda,1}|_V \qquad \text{and} \qquad A^*_{\lambda,\mathrm{r}} \;=\; A'_{\lambda,1}|_V \tag{4.7}$$

form an adjoint pair of bijective realisations with signed boundary map relative to $(A_{\lambda,0}, A'_{\lambda,0})$, with $\operatorname{dom} A_{\lambda,\mathrm{r}} = \operatorname{dom} A^*_{\lambda,\mathrm{r}} = V$.

As an advantage of the abstract theory reviewed in Section 2, given the data $(A_{\lambda,0}, A'_{\lambda,0})$ and $(A_{\lambda,\mathrm{r}}, A^*_{\lambda,\mathrm{r}})$,

- the whole class of adjoint pairs of (bijective) realisations of $(A_{\lambda,0}, A'_{\lambda,0})$ is constructively given by the classification of Theorem 2.8(ii);
- the whole sub-class of adjoint pairs of bijective realisations *with signed boundary map* relative to $(A_{\lambda,0}, A'_{\lambda,0})$ is constructively given by the classification of Theorem 2.9(ii), i.e., by the fulfilment of the mirror conditions (2.13)–(2.14).

In order to apply Theorems 2.8 and 2.9 to the present case, let us determine the relevant kernels and projections. It is straightforward to see that

$$\begin{aligned} \ker A_{\lambda,1} \;&=\; \operatorname{span}\{\boldsymbol{\nu}_{\lambda,1}, \boldsymbol{\nu}_{\lambda,2}\} \\ \ker A'_{\lambda,1} \;&=\; \operatorname{span}\{\boldsymbol{\nu}'_{\lambda,1}, \boldsymbol{\nu}'_{\lambda,2}\}\,, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} \boldsymbol{\nu}_{\lambda,1}(x) \;&:=\; \mathbf{1}_{(-\infty,0)} \begin{pmatrix} -e^{\lambda x} \\ e^{\lambda x} \end{pmatrix} \qquad & \boldsymbol{\nu}'_{\lambda,1}(x) \;&:=\; \mathbf{1}_{(-\infty,0)} \begin{pmatrix} e^{\lambda x} \\ e^{\lambda x} \end{pmatrix} \\ \boldsymbol{\nu}_{\lambda,2}(x) \;&:=\; \mathbf{1}_{(0,+\infty)} \begin{pmatrix} e^{-\lambda x} \\ e^{-\lambda x} \end{pmatrix} \qquad & \boldsymbol{\nu}'_{\lambda,2}(x) \;&:=\; \mathbf{1}_{(0,+\infty)} \begin{pmatrix} -e^{-\lambda x} \\ e^{-\lambda x} \end{pmatrix}. \end{aligned} \tag{4.9}$$

The vectors $\boldsymbol{\nu}_{\lambda,1}, \boldsymbol{\nu}_{\lambda,2}, \boldsymbol{\nu}'_{\lambda,1}, \boldsymbol{\nu}'_{\lambda,2}$ are pairwise orthogonal in L and

$$\|\boldsymbol{\nu}_{\lambda,1}\|_L \;=\; \|\boldsymbol{\nu}_{\lambda,2}\|_L \;=\; \|\boldsymbol{\nu}'_{\lambda,1}\|_L \;=\; \|\boldsymbol{\nu}'_{\lambda,2}\|_L \;=\; \frac{1}{\sqrt{\lambda}}\,.$$

Furthermore, with respect to the choice (4.7) for $A_{\lambda,\mathrm{r}}$, the (non-orthogonal) projections $p_{\lambda,\mathrm{k}} : \operatorname{dom} A_{\lambda,1} \to \ker A_{\lambda,1}$ and $p_{\lambda,\mathrm{k}'} : \operatorname{dom} A'_{\lambda,1} \to \ker A'_{\lambda,1}$ defined in (2.6)–(2.7) act in the present case as

$$\begin{aligned} p_{\lambda,\mathrm{k}}\mathsf{u} \;&=\; C_1(\mathsf{u})\boldsymbol{\nu}_{\lambda,1} + C_2(\mathsf{u})\boldsymbol{\nu}_{\lambda,2} \\ p_{\lambda,\mathrm{k}'}\mathsf{u} \;&=\; -C_2(\mathsf{u})\boldsymbol{\nu}'_{\lambda,1} - C_1(\mathsf{u})\boldsymbol{\nu}'_{\lambda,2}\,, \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} C_1(\mathsf{u}) \;&:=\; \tfrac{1}{2}(u_1(0^+) - u_1(0^-)) - \tfrac{1}{2}(u_2(0^+) - u_2(0^-)) \\ C_2(\mathsf{u}) \;&:=\; \tfrac{1}{2}(u_1(0^+) - u_1(0^-)) + \tfrac{1}{2}(u_2(0^+) - u_2(0^-))\,. \end{aligned} \tag{4.11}$$

In particular, (4.9) and (4.10) imply

$$\begin{aligned} p_{\lambda,\mathrm{k}}(\boldsymbol{\nu}'_{\lambda,1}) \;&=\; -\boldsymbol{\nu}_{\lambda,2}\,, \qquad & p_{\lambda,\mathrm{k}}(\boldsymbol{\nu}'_{\lambda,2}) \;&=\; -\boldsymbol{\nu}_{\lambda,1}\,, \\ p_{\lambda,\mathrm{k}'}(\boldsymbol{\nu}_{\lambda,1}) \;&=\; -\boldsymbol{\nu}'_{\lambda,2}\,, \qquad & p_{\lambda,\mathrm{k}'}(\boldsymbol{\nu}_{\lambda,2}) \;&=\; -\boldsymbol{\nu}'_{\lambda,1}\,. \end{aligned} \tag{4.12}$$

Following the extension scheme of Theorems 2.8 and 2.9, the next step is the qualification of the pairs (B_λ, B_λ^*) of densely defined and mutually adjoint operators $B_\lambda : \mathcal{V}_\lambda \to \mathcal{W}_\lambda$ and $B_\lambda^* : \mathcal{W}_\lambda \to \mathcal{V}_\lambda$, with domains $\operatorname{dom} B_\lambda \subseteq \mathcal{V}_\lambda$ and $\operatorname{dom} B_\lambda^* \subseteq \mathcal{W}_\lambda$, where $\mathcal{V}_\lambda$ and $\mathcal{W}_\lambda$ are closed subspaces of $\ker A_{\lambda,1}$ and $\ker A'_{\lambda,1}$.

Since $\dim\ker A_{\lambda,1} = \dim\ker A'_{\lambda,1} = 2$, such $\mathcal{V}_\lambda$ and $\mathcal{W}_\lambda$ can be zero-, one-, or two-dimensional, and B_λ is necessarily bounded. For the present discussion, it is relevant to focus on the case $\dim \mathcal{V}_\lambda = \dim \mathcal{W}_\lambda = 1$. In fact, one has the following.

Lemma 4.2. *Let $\lambda > 0$. For fixed $a_1, a_2 \in \mathbb{C}$ with $|a_1|^2 + |a_2|^2 = 1$, let*

$$\mathcal{V}_\lambda \;:=\; \operatorname{span}\{a_1\boldsymbol{\nu}_{\lambda,1} + a_2\boldsymbol{\nu}_{\lambda,2}\} \tag{4.13}$$

be a generic one-dimensional subspace of $\ker A_{\lambda,1}$*, let $\mathcal{W}_\lambda$ be a one-dimensional subspace of* $\ker A'_{\lambda,1}$*, and let $B_\lambda : \mathcal{V}_\lambda \to \mathcal{W}_\lambda$ be a (bounded) linear operator, with* $\operatorname{dom} B_\lambda := \mathcal{V}_\lambda$*. Then condition* (2.13) *is satisfied if and only if*

$$\mathcal{W}_\lambda \;=\; \operatorname{span}\{a_2\boldsymbol{\nu}'_{\lambda,1} + a_1\boldsymbol{\nu}'_{\lambda,2}\} \tag{4.14}$$

and

$$B_\lambda(a_1\boldsymbol{\nu}_{\lambda,1} + a_2\boldsymbol{\nu}_{\lambda,2}) \;=\; z(a_2\boldsymbol{\nu}'_{\lambda,1} + a_1\boldsymbol{\nu}'_{\lambda,2}) \tag{4.15}$$

with $z \in \mathbb{C}$ such that

$$\mathfrak{Re}\, z \;\leqslant\; -\lambda\,. \tag{4.16}$$

Proof. Let $\boldsymbol{\nu} = a_1\boldsymbol{\nu}_{\lambda,1} + a_2\boldsymbol{\nu}_{\lambda,2} \in \mathcal{V}_\lambda$. Condition (2.13) consists of two requirements: the requirement $\langle\, p_{\lambda,\mathrm{k}'}\boldsymbol{\nu} \mid \boldsymbol{\nu}' \,\rangle_L = 0$, $\boldsymbol{\nu}' \in \ker A'_{\lambda,1} \ominus \mathcal{W}_\lambda$, reads $p_{\lambda,\mathrm{k}'}\boldsymbol{\nu} \perp \ker A'_{\lambda,1} \ominus \mathcal{W}_\lambda$, which is equivalent to $-a_1\boldsymbol{\nu}'_{\lambda,2} - a_2\boldsymbol{\nu}'_{\lambda,1} = p_{\lambda,\mathrm{k}'}\boldsymbol{\nu} \in \mathcal{W}_\lambda$, where we used (4.12). Thus, as $\mathcal{W}_\lambda$ is taken to be one-dimensional, such a requirement is equivalent to $\mathcal{W}_\lambda$ having the form (4.14). As a linear map between one-dimensional spaces, B_λ must then have the form (4.15) for some $z \in \mathbb{C}$. Since

$$\begin{aligned} \langle\, \boldsymbol{\nu} \mid A'_{\lambda,1}\boldsymbol{\nu} \,\rangle_L \;&=\; \langle\, a_1\boldsymbol{\nu}_{\lambda,1} + a_2\boldsymbol{\nu}_{\lambda,2} \mid A'_{\lambda,1}(a_1\boldsymbol{\nu}_{\lambda,1} + a_2\boldsymbol{\nu}_{\lambda,2}) \,\rangle_L \\ &=\; \langle\, a_1\boldsymbol{\nu}_{\lambda,1} + a_2\boldsymbol{\nu}_{\lambda,2} \mid (A_{\lambda,1} + A'_{\lambda,1})(a_1\boldsymbol{\nu}_{\lambda,1} + a_2\boldsymbol{\nu}_{\lambda,2}) \,\rangle_L \\ &=\; 2\lambda\, \|a_1\boldsymbol{\nu}_{\lambda,1} + a_2\boldsymbol{\nu}_{\lambda,2}\|_L^2 \;=\; 2 \end{aligned}$$

(indeed, $A_{\lambda,1} + A'_{\lambda,1} = 2\lambda \cdot \mathbb{1}$) and

$$\begin{aligned} \langle\, p_{\lambda,\mathrm{k}'}\boldsymbol{\nu} \mid B_\lambda\boldsymbol{\nu} \,\rangle_L \;&=\; \langle\, p_{\lambda,\mathrm{k}'}(a_1\boldsymbol{\nu}_{\lambda,1} + a_2\boldsymbol{\nu}_{\lambda,2}) \mid B_\lambda(a_1\boldsymbol{\nu}_{\lambda,1} + a_2\boldsymbol{\nu}_{\lambda,2}) \,\rangle_L \\ &=\; -\overline{z}\, \langle\, a_1\boldsymbol{\nu}'_{\lambda,2} + a_2\boldsymbol{\nu}'_{\lambda,1} \mid a_2\boldsymbol{\nu}'_{\lambda,1} + a_1\boldsymbol{\nu}'_{\lambda,2} \,\rangle_L \;=\; -\frac{\overline{z}}{\lambda} \end{aligned}$$

(having used again (4.12), (4.15), and the orthonormality properties), then the other requirement of condition (2.13), namely $\langle\, \boldsymbol{\nu} \mid A'_{\lambda,1}\boldsymbol{\nu} \,\rangle_L - 2\,\mathfrak{Re}\langle\, p_{\lambda,\mathrm{k}'}\boldsymbol{\nu} \mid B_\lambda\boldsymbol{\nu} \,\rangle_L \leqslant 0$, reads $2(1 + \frac{\mathfrak{Re}\,\overline{z}}{\lambda}) \leqslant 0$, which is equivalent to (4.16). $\square$

Corollary 4.3. *Any triple* $(\mathcal{V}_\lambda, \mathcal{W}_\lambda, B_\lambda)$ *satisfying* (4.13), (4.14), *and* (4.15) *with condition* (4.16) *identifies, through formulas* (2.11)–(2.12), *an adjoint pair* $(A_{B_\lambda}, A^*_{B_\lambda})$ *of bijective realisations with signed boundary map relative to* $(A_{\lambda,0}, A'_{\lambda,0})$.

Proof. It follows from an immediate application of Theorem 2.9, since condition (i)-(b) therein is fulfilled, owing to Lemma 4.2 above. □

In the following, we shall discuss two special classes of operators $B_\lambda : \mathcal{V}_\lambda \to \mathcal{W}_\lambda$: each class gives rise to a distinguished family of bijective realisations $(A_{B_\lambda}, A^*_{B_\lambda})$ of $(A_{\lambda,0}, A'_{\lambda,0})$, and for each such family the map Φ produces a collection of operators $\Phi(A_{B_\lambda})$ that form a distinguished class of closed extensions of $\frac{1}{\lambda}(\mathring{H} + \lambda^2\mathbb{1})$ on $L^2(\mathbb{R})$. This will yield, in particular, the class of self-adjoint 'δ-type' extensions and the class of self-adjoint 'δ'-type' extensions of $\mathring{H}$.

5. Bijective realisations of Friedrichs operators and 1D 'δ-extensions'

Motivated by Corollary 4.3, let us choose $a_1 = a_2$ in Lemma 4.2 and hence let us for fixed $\lambda > 0$ consider the case

$$\begin{aligned} \mathcal{V}_\lambda &= \operatorname{span}\{\boldsymbol{\nu}_{\lambda,1} + \boldsymbol{\nu}_{\lambda,2}\} \\ \mathcal{W}_\lambda &= \operatorname{span}\{\boldsymbol{\nu}'_{\lambda,1} + \boldsymbol{\nu}'_{\lambda,2}\} \\ B_\lambda &: \mathcal{V}_\lambda \to \mathcal{W}_\lambda \\ B_\lambda(\boldsymbol{\nu}_{\lambda,1} + \boldsymbol{\nu}_{\lambda,2}) &= z\,(\boldsymbol{\nu}'_{\lambda,1} + \boldsymbol{\nu}'_{\lambda,2}) \qquad \text{for a fixed } z \in \mathbb{C}\,. \end{aligned} \tag{5.1}$$

Proposition 5.1.

(i) *Associated with the operator* $B_\lambda : \mathcal{V}_\lambda \to \mathcal{W}_\lambda$ *defined in* (5.1), *via the correspondence* (2.9), *is the operator* $A_{B_\lambda} = A_{\lambda,1}|_{\operatorname{dom} A_{B_\lambda}}$ *on the Hilbert space* L, *whose domain is given by*

$$\begin{aligned} \operatorname{dom} A_{B_\lambda} = \Big\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H^1(\mathbb{R}\backslash\{0\}) \oplus H^1(\mathbb{R}) \,: \\ u_1(0^+) - u_1(0^-) = \frac{2\lambda}{z+\lambda} u_2(0) \Big\}\,, \end{aligned} \tag{5.2}$$

as well as the operator $A^*_{B_\lambda} = A_{B^*_\lambda} = A'_{\lambda,1}|_{\operatorname{dom} A^*_{B_\lambda}}$ *on* L, *whose domain is given by*

$$\begin{aligned} \operatorname{dom} A^*_{B_\lambda} = \Big\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H^1(\mathbb{R}\backslash\{0\}) \oplus H^1(\mathbb{R}) \,: \\ u_1(0^+) - u_1(0^-) = -\frac{2\lambda}{\overline{z}+\lambda} u_2(0) \Big\}\,. \end{aligned} \tag{5.3}$$

(ii) *The pair* $(A_{B_\lambda}, A^*_{B_\lambda})$ *is an adjoint pair of bijective realisations of the abstract Friedrichs operators* $(A_{\lambda,0}, A'_{\lambda,0})$ *if an only if* $z \neq 0$ *in* (5.1).

(iii) *The pair* $(A_{B_\lambda}, A_{B_\lambda}^*)$ *has signed boundary map if and only if* $\mathfrak{Re}\, z \leqslant -\lambda$ *in* (5.1).

Remark 5.2. In the case $z = -\lambda$ the conditions in (5.2) and (5.3) are understood as $u_2(0) = 0$, implying $\operatorname{dom} A_{B_\lambda} = \operatorname{dom} A_{B_\lambda}^* = H^1(\mathbb{R}\backslash\{0\}) \oplus H_0^1(\mathbb{R}\backslash\{0\})$. The same applies to (5.4) and (5.5) below, which in this special case $z = -\lambda$ involve the space $H^2(\mathbb{R}\backslash\{0\}) \cap H_0^1(\mathbb{R}\backslash\{0\})$.

Proof of Proposition 5.1. Let $\mathsf{u} \equiv \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \operatorname{dom} A_{B_\lambda} \subseteq W$. In particular, $u_1, u_2 \in H^1(\mathbb{R}\backslash\{0\})$. Owing to (2.9), $p_{\lambda,\mathrm{k}}\mathsf{u} \in \operatorname{dom} B_\lambda = \mathcal{V}_\lambda = \operatorname{span}\{\boldsymbol{\nu}_{\lambda,1} + \boldsymbol{\nu}_{\lambda,2}\}$: thus, (4.10) reads $C_1(\mathsf{u}) = C_2(\mathsf{u})$ which is by (4.11) equivalent to $u_2(0^-) = u_2(0^+) =: u_2(0)$, that is, u_2 is continuous at the origin and hence belongs to $H^1(\mathbb{R})$, which in turn is the same as to say that $C_1(\mathsf{u}) = \frac{1}{2}(u_1(0^+) - u_1(0^-))$. As a consequence,

$$B_\lambda(p_{\lambda,\mathrm{k}}\mathsf{u}) \;=\; B_\lambda(C_1(\mathsf{u})\boldsymbol{\nu}_{\lambda,1} + C_1(\mathsf{u})\boldsymbol{\nu}_{\lambda,2}) \;=\; \tfrac{1}{2}\,(u_1(0^+) - u_1(0^-))\, z\, (\boldsymbol{\nu}'_{\lambda,1} + \boldsymbol{\nu}'_{\lambda,2})\,.$$

Moreover,

$$\begin{aligned} P_{\mathcal{W}_\lambda}(A_{\lambda,1}\mathsf{u}) \;&=\; \|\boldsymbol{\nu}'_{\lambda,1} + \boldsymbol{\nu}'_{\lambda,2}\|^{-2} \langle\, A_{\lambda,1}\mathsf{u} \mid \boldsymbol{\nu}'_{\lambda,1} + \boldsymbol{\nu}'_{\lambda,2} \,\rangle_L \,(\boldsymbol{\nu}'_{\lambda,1} + \boldsymbol{\nu}'_{\lambda,2}) \\ &=\; \lambda\big(u_2(0) - \tfrac{1}{2}(u_1(0^+) - u_1(0^-))\big)\,(\boldsymbol{\nu}'_{\lambda,1} + \boldsymbol{\nu}'_{\lambda,2}) \end{aligned}$$

because $\|\boldsymbol{\nu}'_{\lambda,1} + \boldsymbol{\nu}'_{\lambda,2}\|^2 = \frac{2}{\lambda}$ by orthogonality, and because

$$\begin{aligned} \langle\, A_{\lambda,1}\mathsf{u} \mid \boldsymbol{\nu}'_{\lambda,1} \,\rangle_L \;&=\; \Big\langle \begin{pmatrix} \dot{u}_2 + \lambda u_1 \\ \dot{u}_1 + \lambda u_2 \end{pmatrix} \;\Big|\; \mathbf{1}_{(-\infty,0)} \begin{pmatrix} e^{\lambda x} \\ e^{\lambda x} \end{pmatrix} \Big\rangle_L \\ &=\; \int_{-\infty}^{0} \big(\dot{u}_2(x) + \lambda u_1(x) + \dot{u}_1(x) + \lambda u_2(x)\big)\, e^{\lambda x}\, \mathrm{d}x \\ &=\; u_2(0^-) + u_1(0^-) \end{aligned}$$

and analogously $\langle\, A_{\lambda,1}\mathsf{u} \mid \boldsymbol{\nu}'_{\lambda,2} \,\rangle_L = u_2(0^+) - u_1(0^+)$. Therefore, with the identity $u_2(0^-) = u_2(0^+) =: u_2(0)$, the property $P_{\mathcal{W}_\lambda}(A_{\lambda,1}\mathsf{u}) = B_\lambda(p_{\lambda,\mathrm{k}}\mathsf{u})$ prescribed by (2.9) is equivalent to

$$u_2(0) \;=\; \tfrac{z+\lambda}{2\lambda}\,\big(u_1(0^+) - u_1(0^-)\big)\,. \tag{*}$$

When $z = -\lambda$, (*) implies $u_2(0) = 0$, and hence $u_2 \in \{f \in H^1(\mathbb{R}) : f(0) = 0\} = H_0^1(\mathbb{R}\backslash\{0\})$, whereas it does not add further restrictions to the condition $u_1 \in H^1(\mathbb{R}\setminus\{0\})$, and the conclusion is (5.2) for the case $z = -\lambda$ (see Remark 5.2). When instead $z \neq -\lambda$, then (*) together with the properties $u_1 \in H^1(\mathbb{R}\backslash\{0\})$ and $u_2 \in H^1(\mathbb{R})$ implies (5.2) for the case $z \neq -\lambda$. Formula (5.3) can be established through a completely analogous reasoning, or also by determining A_B^* directly from A_B.

This completes the proof of part (i). Part (ii) is an immediate application of Theorem 2.8(iii). Part (iii) is a direct consequence of Lemma 4.2. □

Next, let us lift the pairs $(A_{B_\lambda}, A_{B_\lambda}^*)$ of the class fixed by (5.1) to pairs $(\Phi(A_{B_\lambda}), \Phi(A_{B_\lambda}^*))$ of operators on $L^2(\mathbb{R})$, by means of the map Φ introduced in (4.3).

Proposition 5.3. *Let* $B_{\lambda,z} : \mathcal{V}_\lambda \to \mathcal{W}_\lambda$ *be an operator of the type* (5.1) *for some* $z \in \mathbb{C}$*, and let* $A_{B_{\lambda,z}}$ *and* $A^*_{B_{\lambda,z}}$ *be the corresponding operators described in Proposition* 5.1*.*

(i) *Via the map* Φ *introduced in* (4.3) *one has*

$$\begin{aligned} \operatorname{dom}\Phi(A_{B_{\lambda,z}}) &= \Big\{ u \in H^2(\mathbb{R}\backslash\{0\}) \cap H^1(\mathbb{R}) : \\ &\qquad\qquad \dot u(0^+) - \dot u(0^-) = -\frac{2\lambda^2}{z+\lambda} u(0) \Big\} \\ \Phi(A_{B_{\lambda,z}})\, u &= \lambda^{-1}(-\ddot u + \lambda^2 u)\,, \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} \operatorname{dom}\Phi(A^*_{B_{\lambda,z}}) &= \Big\{ u \in H^2(\mathbb{R}\backslash\{0\}) \cap H^1(\mathbb{R}) : \\ &\qquad\qquad \dot u(0^+) - \dot u(0^-) = -\frac{2\lambda^2}{\overline{z}+\lambda} u(0) \Big\} \\ \Phi(A^*_{B_{\lambda,z}})\, u &= \lambda^{-1}(-\ddot u + \lambda^2 u)\,. \end{aligned} \tag{5.5}$$

(ii) $\Phi(A_{B_{\lambda,z}}) = \Phi(A^*_{B_{\lambda,z}})$ *on* $L^2(\mathbb{R})$ *if and only if* $z \in \mathbb{R}$*.*

(iii) *If* $z \neq 0$*, then the operators* $\Phi(A_{B_{\lambda,z}})$ *and* $\Phi(A^*_{B_{\lambda,z}})$ *are mutually adjoint on* $L^2(\mathbb{R})$*.*

(iv) *The two conditions*
- $z \neq 0$ *and* $\Phi(A_{B_{\lambda,z}})$ *is self-adjoint,*
- $z \in \mathbb{R} \setminus \{0\}$*,*

are equivalent. The same equivalence holds replacing $\Phi(A_{B_{\lambda,z}})$ *by* $\Phi(A^*_{B_{\lambda,z}})$*.*

Proof. (i) Let $u \in L^2(\mathbb{R})$. It follows from (4.3) that $u \in \operatorname{dom}\Phi(A_{B_{\lambda,z}})$ if and only if $\begin{pmatrix} -\frac{1}{\lambda}\dot u \\ u \end{pmatrix} \in \operatorname{dom} A_{B_{\lambda,z}}$, and $u \in \operatorname{dom}\Phi(A^*_{B_{\lambda,z}})$ if and only if $\begin{pmatrix} \frac{1}{\lambda}\dot u \\ u \end{pmatrix} \in \operatorname{dom} A^*_{B_{\lambda,z}}$, in which case $\Phi(A_{B_{\lambda,z}})u = \Phi(A^*_{B_{\lambda,z}})u = \frac{1}{\lambda}(-\ddot u + \lambda^2 u)$. Matching this with (5.2) and (5.3) produces at once (5.4) and (5.5).

(ii) By part (i) the action of $\Phi(A_{B_{\lambda,z}})$ and $\Phi(A^*_{B_{\lambda,z}})$ coincide, while $\operatorname{dom}\Phi(A_{B_{\lambda,z}}) = \operatorname{dom}\Phi(A^*_{B_{\lambda,z}})$ if and only if $z \in \mathbb{R}$, so the claim follows.

(iii) $\Phi(A^*_{B_{\lambda,z}}) \subseteq \Phi(A_{B_{\lambda,z}})^*$ because, if $v \in \operatorname{dom}\Phi(A^*_{B_{\lambda,z}})$, then for any $u \in \operatorname{dom}\Phi(A_{B_{\lambda,z}})$ one has

$$\begin{aligned} \langle \Phi(A_{B_{\lambda,z}})u \mid v \rangle_{L^2(\mathbb{R})} &= \Big\langle P_2 A_{B_{\lambda,z}} \begin{pmatrix} -\frac{1}{\lambda}\dot u \\ u \end{pmatrix} \mid v \Big\rangle_{L^2(\mathbb{R})} \\ &= \Big\langle A_{B_{\lambda,z}} \begin{pmatrix} -\frac{1}{\lambda}\dot u \\ u \end{pmatrix} \mid \begin{pmatrix} \frac{1}{\lambda}\dot v \\ v \end{pmatrix} \Big\rangle_L \\ &= \Big\langle \begin{pmatrix} -\frac{1}{\lambda}\dot u \\ u \end{pmatrix} \mid A^*_{B_{\lambda,z}} \begin{pmatrix} \frac{1}{\lambda}\dot v \\ v \end{pmatrix} \Big\rangle_L \\ &= \Big\langle u \mid P_2 A^*_{B_{\lambda,z}} \begin{pmatrix} \frac{1}{\lambda}\dot v \\ v \end{pmatrix} \Big\rangle_{L^2(\mathbb{R})} = \langle u \mid \Phi(A^*_{B_{\lambda,z}})v \rangle_{L^2(\mathbb{R})}, \end{aligned}$$

where in the first and the last equality we used the definition of Φ (4.3), in the second and the fourth identities $P_1 A_{B_{\lambda,z}} \begin{pmatrix} -\frac{1}{\lambda}\dot u \\ u \end{pmatrix} = P_1 A^*_{B_{\lambda,z}} \begin{pmatrix} \frac{1}{\lambda}\dot v \\ v \end{pmatrix} = 0$, and in the third $\begin{pmatrix} \frac{1}{\lambda}\dot v \\ v \end{pmatrix} \in \operatorname{dom} A^*_{B_{\lambda,z}}$. Conversely, let $v \in \operatorname{dom} \Phi(A_{B_{\lambda,z}})^*$ and let $z \neq 0$. Then there exists $\eta \in L^2(\mathbb{R})$ such that

$$\langle\, \Phi(A_{B_{\lambda,z}})u \mid v \,\rangle_{L^2(\mathbb{R})} = \langle\, u \mid \eta \,\rangle_{L^2(\mathbb{R})}$$

for any $u \in \operatorname{dom} \Phi(A_{B_{\lambda,z}})$. Since $z \neq 0$, $A^*_{B_{\lambda,z}}$ is bijection. Hence, $A^*_{B_{\lambda,z}} \mathsf{w} = \begin{pmatrix} 0 \\ \eta \end{pmatrix}$ for some $\mathsf{w} \in \operatorname{dom} A^*_{B_{\lambda,z}}$. In fact, necessarily $\mathsf{w} = \begin{pmatrix} \frac{1}{\lambda}\dot w \\ w \end{pmatrix}$ for some $w \in \operatorname{dom} \Phi(A^*_{B_{\lambda,z}})$, because $P_1 A^*_{B_{\lambda,z}} \mathsf{w} = 0$. Therefore,

$$\begin{aligned}
\langle\, u \mid \eta \,\rangle_{L^2(\mathbb{R})} &= \Big\langle \begin{pmatrix} -\frac{1}{\lambda}\dot u \\ u \end{pmatrix} \mid \begin{pmatrix} 0 \\ \eta \end{pmatrix} \Big\rangle_L = \Big\langle \begin{pmatrix} -\frac{1}{\lambda}\dot u \\ u \end{pmatrix} \mid A^*_{B_{\lambda,z}} \begin{pmatrix} \frac{1}{\lambda}\dot w \\ w \end{pmatrix} \Big\rangle_L \\
&= \Big\langle A_{B_{\lambda,z}} \begin{pmatrix} -\frac{1}{\lambda}\dot u \\ u \end{pmatrix} \mid \begin{pmatrix} \frac{1}{\lambda}\dot w \\ w \end{pmatrix} \Big\rangle_L = \Big\langle P_2 A_{B_{\lambda,z}} \begin{pmatrix} -\frac{1}{\lambda}\dot u \\ u \end{pmatrix} \mid w \Big\rangle_{L^2(\mathbb{R})} \\
&= \langle\, \Phi(A_{B_{\lambda,z}})u \mid w \,\rangle_{L^2(\mathbb{R})}
\end{aligned}$$

for arbitrary $u \in \operatorname{dom} \Phi(A_{B_{\lambda,z}})$. As a consequence, $v - w$ is orthogonal to the range of $\Phi(A_{B_{\lambda,z}})$. Obviously, the surjectivity of $A_{B_{\lambda,z}}$ implies the surjectivity of $\Phi(A_{B_{\lambda,z}})$, whence $v = w \in \operatorname{dom} \Phi(A^*_{B_{\lambda,z}})$, that is, $\Phi(A^*_{B_{\lambda,z}}) \supseteq \Phi(A_{B_{\lambda,z}})^*$. The conclusion is $\Phi(A^*_{B_{\lambda,z}}) = \Phi(A_{B_{\lambda,z}})^*$. An analogous argument, exchanging the two operators, shows that $\Phi(A_{B_{\lambda,z}}) = \Phi(A^*_{B_{\lambda,z}})^*$.

(iv) If $z \neq 0$ and $\Phi(A_{B_{\lambda,z}})$ is self-adjoint on $L^2(\mathbb{R})$, then by part (iii)

$$\Phi(A_{B_{\lambda,z}}) \;=\; \Phi(A_{B_{\lambda,z}})^* \;=\; \Phi(A^*_{B_{\lambda,z}})\,,$$

so by part (ii) $z \in \mathbb{R}$. Thus, the overall conclusion is $z \in \mathbb{R} \setminus \{0\}$. Conversely, for $z \in \mathbb{R} \setminus \{0\}$ by parts (ii) and (iii) follows

$$\Phi(A_{B_{\lambda,z}}) \;=\; \Phi(A^*_{B_{\lambda,z}}) \;=\; \Phi(A_{B_{\lambda,z}})^*\,,$$

obtaining that $\Phi(A_{B_{\lambda,z}})$ is self-adjoint on $L^2(\mathbb{R})$. For $\Phi(A_{B_{\lambda,z}})$ the reasoning is completely analogous. □

With Proposition 5.3, we have identified operators of the form $A_{B_{\lambda,z}}$ for generic $z \in \mathbb{C}$, and $\lambda > 0$.

In fact, the double parametrisation in z and λ is somewhat redundant, but this is precisely the advantage we wanted to benefit from with carrying on an arbitrary λ. Indeed, it is easy to see from (5.4) that the boundary condition remains unaltered if one replaces the pair (λ, z) with $(1, \frac{z+\lambda}{\lambda^2} - 1)$, and that, correspondingly, one has the identity

$$\Phi\big(A_{B_{1,\frac{z+\lambda}{\lambda^2}-1}}\big) = \lambda\, \Phi(A_{B_{\lambda,z}}) - (\lambda^2 - 1)\mathbb{1}\,. \tag{5.6}$$

This allows us to overcome the (only apparent) restriction that it was needed to insert in the statement of Proposition 5.3(iii)-(iv), in the proof of which we had to use the bijectivity of $A_{B_{\lambda,z}}$, which is only available for $z \neq 0$. If, for concreteness, we focus on the two families

$$\{\Phi(A_{B_{1,z}})\,|\,z\in\mathbb{C}\} \qquad \text{and} \qquad \{\Phi(A^*_{B_{1,z}})\,|\,z\in\mathbb{C}\}\,,$$

then we deduce from Proposition 5.3 the following relevant properties.

Corollary 5.4. *Under the assumptions of Proposition* 5.3*,*

(i) *one has*

$$\begin{aligned}\operatorname{dom}\Phi(A_{B_{1,z}}) &= \Big\{u\in H^2(\mathbb{R}\backslash\{0\})\cap H^1(\mathbb{R}) : \\ &\qquad \dot u(0^+)-\dot u(0^-) = -\frac{2}{z+1}u(0)\Big\} \\ \Phi(A_{B_{1,z}})\,u &= -\ddot u + u\,,\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}\operatorname{dom}\Phi(A^*_{B_{1,z}}) &= \Big\{u\in H^2(\mathbb{R}\backslash\{0\})\cap H^1(\mathbb{R}) : \\ &\qquad \dot u(0^+)-\dot u(0^-) = -\frac{2}{\overline{z}+1}u(0)\Big\} \\ \Phi(A^*_{B_{1,z}})\,u &= -\ddot u + u\,;\end{aligned} \tag{5.8}$$

(ii) $\Phi(A_{B_{1,z}}) = \Phi(A^*_{B_{1,z}})$ *on* $L^2(\mathbb{R})$ *if and only if* $z\in\mathbb{R}$*;*

(iii) $\Phi(A_{B_{1,z}})$ *and* $\Phi(A^*_{B_{1,z}})$ *are mutually adjoint on* $L^2(\mathbb{R})$*;*

(iv) $\Phi(A_{B_{1,z}})$ *is self-adjoint on* $L^2(\mathbb{R})$ *if and only if* $z\in\mathbb{R}$*. The same holds for* $\Phi(A^*_{B_{1,z}})$*.*

Proof. Parts (i) and (ii) follow at once from Proposition 5.3(i)-(ii), as well as parts (iii) and (iv) when $z\neq 0$. It remains to discuss the case $z=0$, i.e., to prove that $\Phi(A_{B_{1,0}})$ is self-adjoint. By (5.6), $\Phi(A_{B_{1,0}}) = 2\Phi(A_{B_{2,2}}) - 3\mathbb{1}$, and applying Proposition 5.3(iv), we come to the conclusion. □

Let us collect together all the arguments and results of this Section. The following Summary makes the outline stated at the beginning of Section 4 explicit for the δ-type extensions.

Summary 5.5 (δ-type closed extensions of $\mathring{H}$ realised as Friedrichs systems).

- There exists a collection of operators $\Phi(A_{B_{1,z}})$ on $L^2(\mathbb{R})$ of the form (5.7), where $B_{1,z}$ runs in the class (5.1) for $z\in\mathbb{C}$ and $\lambda=1$, such that $\Phi(A_{B_{1,z}})$ is closed (Corollary 5.4(iii)) and $\mathring{H}+\mathbb{1}\subseteq\Phi(A_{B_{1,z}})$ (Corollary 5.4(i)). Thus,

  $$\mathscr{C}_\delta \;:=\; \big\{\Phi(A_{B_{1,z}})-\mathbb{1} \;:\; z\in\mathbb{C}\big\}$$

  is a collection of *closed extensions of* $\mathring{H}$.
- All the $\Phi(A_{B_{1,z}})$'s arising in $\mathscr{C}_\delta$, but the exceptional one corresponding to $z=0$ (i.e., to $B_{1,0}=\mathbb{O}$), are associated to pairs $(A_{B_{1,z}}, A^*_{B_{1,z}})$ that, by Proposition 5.1(ii), are pairs of *bijective realisations* of the abstract Friedrichs operators $(A_{1,0}, A'_{1,0})$ obtained by reducing the second order

differential operator $\mathring{H} + \mathbb{1}$ to the first order vector-valued differential operator T_1 of formula (4.2).

- In fact, $\mathscr{C}_\delta$ has two noticeable sub-families. The sub-family of $\mathscr{C}_\delta$ selected by $\mathfrak{Re}\, z \leqslant -1$ corresponds to the only pairs $(A_{B_{1,z}}, A^*_{B_{1,z}})$ that are *bijective realisations with signed boundary map* relative to $(A_{1,0}, A'_{1,0})$, as follows from Proposition 5.1(iii).
- In turn, the sub-family of $\mathscr{C}_\delta$ selected by $z \in \mathbb{R}$ is the maximal sub-family of self-adjoint operators (Corollary 5.4(iv)). It is precisely the one-parameter family $\{-\Delta_\alpha \,|\, \alpha \in (-\infty, +\infty]\}$ of operators introduced in (3.2), since comparing (3.2) and (5.7) gives
$$\Phi(A_{B_{1,z}}) - \mathbb{1} \;=\; -\Delta_\alpha\,, \qquad \alpha := -\tfrac{2}{z+1}\,, \tag{5.9}$$
where $\alpha = +\infty$ is understood to correspond to $z = -1$.
- We have thus recovered the family of δ-type *self-adjoint extensions* of $\mathring{H}$, together with the larger family $\mathscr{C}_\delta$ of closed extensions of $\mathring{H}$ characterised by the same form $\dot{u}(0^+) - \dot{u}(0^-) = \alpha u(0)$ of boundary condition at $x = 0$ for possibly non-real α's, and we have done it through a novel conceptual path, as compared to the traditional methods that had let to Theorem 3.1(ii), that is, *understanding each such extension as an abstract Friedrichs system.*
- Each self-adjoint extension $-\Delta_{\alpha=-2/(z+1)} = \Phi(A_{B_{1,z}}) - \mathbb{1}$ of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}|_{C^\infty_c(\mathbb{R}\setminus\{0\})}$ is indeed understood as a pair $(A_{B_{1,z}}, A^*_{B_{1,z}})$ of closed bijective realisations of the underlying Friedrichs pair $(T_1, \widetilde{T}_1)$ naturally associated with $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}|_{C^\infty_c(\mathbb{R}\setminus\{0\})}$. Most noticeably, the extensions with $\alpha \geqslant 0$ ($z \leqslant -1$), namely the repulsive (non-confining) contact interaction Hamiltonians of δ-type, correspond to bijective realisations with signed boundary map.

6. Bijective realisations of Friedrichs operators and 1D 'δ'-extensions'

In this Section we discuss a second relevant special case, in complete analogy to the previous Section (and for this reason we only sketch the analogous proofs): instead of (5.1) we choose now $a_1 = -a_2$ in Lemma 4.2, and hence we consider the case
$$\begin{aligned} \mathcal{V}_\lambda \;&=\; \mathrm{span}\{\boldsymbol{\nu}_{\lambda,1} - \boldsymbol{\nu}_{\lambda,2}\} \\ \mathcal{W}_\lambda \;&=\; \mathrm{span}\{\boldsymbol{\nu}'_{\lambda,1} - \boldsymbol{\nu}'_{\lambda,2}\} \\ B_\lambda \,&:\, \mathcal{V}_\lambda \to \mathcal{W}_\lambda \\ B_\lambda(\boldsymbol{\nu}_{\lambda,1} - \boldsymbol{\nu}_{\lambda,2}) \;&=\; -z\,(\boldsymbol{\nu}'_{\lambda,1} - \boldsymbol{\nu}'_{\lambda,2}) \qquad \text{for a fixed } z \in \mathbb{C}\,. \end{aligned} \tag{6.1}$$
We have the following.

Proposition 6.1.

(i) *Associated with the operator* $B_\lambda : \mathcal{V}_\lambda \to \mathcal{W}_\lambda$ *defined in* (6.1), *via the correspondence* (2.9), *is the operator* $A_{B_\lambda} = A_{\lambda,1}|_{\operatorname{dom} A_{B_\lambda}}$ *on the Hilbert space* L, *whose domain is given by*

$$\operatorname{dom} A_{B_\lambda} = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H^1(\mathbb{R}) \oplus H^1(\mathbb{R}\setminus\{0\}) : u_2(0^+) - u_2(0^-) = \frac{2\lambda}{z+\lambda} u_1(0) \right\}, \tag{6.2}$$

as well as the operator $A^*_{B_\lambda} = A_{B^*_\lambda} = A'_{\lambda,1}|_{\operatorname{dom} A^*_{B_\lambda}}$ *on* L, *whose domain is given by*

$$\operatorname{dom} A^*_{B_\lambda} = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H^1(\mathbb{R}) \oplus H^1(\mathbb{R}\setminus\{0\}) : u_2(0^+) - u_2(0^-) = -\frac{2\lambda}{\overline{z}+\lambda} u_1(0) \right\}. \tag{6.3}$$

(ii) *The pair* $(A_{B_\lambda}, A^*_{B_\lambda})$ *is an adjoint pair of bijective realisations of the abstract Friedrichs operators* $(A_{\lambda,0}, A'_{\lambda,0})$ *if an only if* $z \neq 0$ *in* (6.1).

(iii) *The pair* $(A_{B_\lambda}, A^*_{B_\lambda})$ *has signed boundary map if and only if* $\mathfrak{Re}\, z \leqslant -\lambda$ *in* (6.1).

Remark 6.2. In the case $z = -\lambda$ the conditions in (6.2) and (6.3) are understood as $u_1(0) = 0$, implying $\operatorname{dom} A_{B_\lambda} = \operatorname{dom} A^*_{B_\lambda} = H^1_0(\mathbb{R}\setminus\{0\}) \oplus H^1(\mathbb{R}\setminus\{0\})$. The same applies to (6.4) and (6.5) below.

Proof of Proposition 6.1. Let $\mathsf{u} \equiv \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \operatorname{dom} A_{B_\lambda} \subseteq W$. In particular, $u_1, u_2 \in H^1(\mathbb{R}\setminus\{0\})$. Owing to (2.9), $p_{\lambda,\mathrm{k}}\mathsf{u} \in \operatorname{dom} B_\lambda = \mathcal{V}_\lambda = \operatorname{span}\{\boldsymbol{\nu}_{\lambda,1} - \boldsymbol{\nu}_{\lambda,2}\}$: thus, (4.10) reads $C_2(\mathsf{u}) = -C_1(\mathsf{u})$, which is by (4.11) equivalent to $u_1(0^-) = u_1(0^+) =: u_1(0)$, that is, u_1 is continuous at the origin and hence belongs to $H^1(\mathbb{R})$. As a consequence,

$$B_\lambda(p_{\lambda,\mathrm{k}}\mathsf{u}) = B_\lambda(C_1(\mathsf{u})\boldsymbol{\nu}_{\lambda,1} - C_1(\mathsf{u})\boldsymbol{\nu}_{\lambda,2}) = \tfrac{1}{2}\left(u_2(0^+) - u_2(0^-)\right) z \,(\boldsymbol{\nu}'_{\lambda,1} - \boldsymbol{\nu}'_{\lambda,2}),$$

and arguing as in the proof of Proposition 5.1(i) we find

$$\begin{aligned} P_{\mathcal{W}_\lambda}(A_{\lambda,1}\mathsf{u}) &= \|\boldsymbol{\nu}'_{\lambda,1} - \boldsymbol{\nu}'_{\lambda,2}\|^{-2} \langle A_{\lambda,1}\mathsf{u} \mid \boldsymbol{\nu}'_{\lambda,1} - \boldsymbol{\nu}'_{\lambda,2} \rangle_L \,(\boldsymbol{\nu}'_{\lambda,1} - \boldsymbol{\nu}'_{\lambda,2}) \\ &= \lambda\big(u_1(0) - \tfrac{1}{2}(u_2(0^+) - u_2(0^-))\big)\,(\boldsymbol{\nu}'_{\lambda,1} - \boldsymbol{\nu}'_{\lambda,2}). \end{aligned}$$

Therefore, the property $P_{\mathcal{W}_\lambda}(A_{\lambda,1}\mathsf{u}) = B_\lambda(p_{\lambda,\mathrm{k}}\mathsf{u})$ prescribed by (2.9) is equivalent to

$$u_1(0) = \tfrac{z+\lambda}{2\lambda}\big(u_2(0^+) - u_2(0^-)\big), \tag{**}$$

thus obtaining (6.2). The statement for $\operatorname{dom} A^*_{B_\lambda}$ follows in the same manner. This completes the proof of part (i). Part (ii) is an immediate application of Theorem 2.8(iii), while part (iii) is a direct consequence of Lemma 4.2. $\square$

Next, we lift the pairs $(A_{B_\lambda}, A^*_{B_\lambda})$ of the class fixed by (6.1) to pairs $(\Phi(A_{B_\lambda}), \Phi(A^*_{B_\lambda}))$ of operators on $L^2(\mathbb{R})$, by means of the map Φ introduced in (4.3). The result is the following and its proof is completely analogous to the proof of Proposition 5.3.

Proposition 6.3. *Let $B_{\lambda,z} : \mathcal{V}_\lambda \to \mathcal{W}_\lambda$ be an operator of the type* (6.1) *for some $z \in \mathbb{C}$, and let $A_{B_{\lambda,z}}$ and $A^*_{B_{\lambda,z}}$ be the corresponding operators described in Proposition* 6.1*.*

(i) *Via the map Φ introduced in* (4.3) *one has*

$$\begin{aligned} \mathrm{dom}\,\Phi(A_{B_{\lambda,z}}) \;&=\; \left\{ \begin{array}{c} u \in H^2(\mathbb{R}\backslash\{0\}) \textit{ such that} \\ \dot{u}(0^+) = \dot{u}(0^-) =: \dot{u}(0)\,, \\ u(0^+) - u(0^-) = -\frac{2}{z+\lambda}\dot{u}(0) \end{array} \right\} \\ \Phi(A_{B_{\lambda,z}})\,u \;&=\; \lambda^{-1}(-\ddot{u} + \lambda^2 u)\,, \end{aligned} \tag{6.4}$$

and

$$\begin{aligned} \mathrm{dom}\,\Phi(A^*_{B_{\lambda,z}}) \;&=\; \left\{ \begin{array}{c} u \in H^2(\mathbb{R}\backslash\{0\}) \textit{ such that} \\ \dot{u}(0^+) = \dot{u}(0^-) =: \dot{u}(0)\,, \\ u(0^+) - u(0^-) = -\frac{2}{\overline{z}+\lambda}\dot{u}(0) \end{array} \right\} \\ \Phi(A^*_{B_{\lambda,z}})\,u \;&=\; \lambda^{-1}(-\ddot{u} + \lambda^2 u)\,. \end{aligned} \tag{6.5}$$

(ii) $\Phi(A_{B_{\lambda,z}}) = \Phi(A^*_{B_{\lambda,z}})$ *on $L^2(\mathbb{R})$ if and only if $z \in \mathbb{R}$.*

(iii) *If $z \neq 0$, then the operators $\Phi(A_{B_\lambda})$ and $\Phi(A^*_{B_\lambda})$ are mutually adjoint on $L^2(\mathbb{R})$.*

(iv) *The two conditions*
- *$z \neq 0$ and $\Phi(A_{B_{\lambda,z}})$ is self-adjoint;*
- *$z \in \mathbb{R} \setminus \{0\}$,*

are equivalent. The same equivalence holds replacing $\Phi(A_{B_{\lambda,z}})$ by $\Phi(A^*_{B_{\lambda,z}})$.

As in the previous Section, we exploit the redundancy of the double parametrisation of the operators $A_{B_{\lambda,z}}$, and from (6.4) we deduce the operator identity

$$\Phi\big(A_{B_{1,z+\lambda-1}}\big) \;=\; \lambda\,\Phi(A_{B_{\lambda,z}}) - (\lambda^2 - 1)\mathbb{1}\,. \tag{6.6}$$

We then focus, for concreteness, on the two families

$$\{\Phi(A_{B_{1,z}})\,|\,z \in \mathbb{C}\} \qquad \text{and} \qquad \{\Phi(A^*_{B_{1,z}})\,|\,z \in \mathbb{C}\}$$

and we deduce from Proposition 6.3 the following relevant properties.

Corollary 6.4. *Under the assumptions of Proposition* 6.3*,*

(i) *one has*

$$\begin{aligned} \mathrm{dom}\,\Phi(A_{B_{1,z}}) \;&=\; \left\{ \begin{array}{c} u \in H^2(\mathbb{R}\backslash\{0\}) \textit{ such that} \\ \dot{u}(0^+) = \dot{u}(0^-) =: \dot{u}(0)\,, \\ u(0^+) - u(0^-) = -\frac{2}{z+1}\dot{u}(0) \end{array} \right\} \\ \Phi(A_{B_{1,z}})\,u \;&=\; -\ddot{u} + u\,, \end{aligned} \tag{6.7}$$

and

$$\begin{aligned} \operatorname{dom}\Phi(A_{B_{1,z}}) \;&=\; \left\{ \begin{array}{c} u\in H^2(\mathbb{R}\backslash\{0\}) \textit{ such that} \\ \dot u(0^+)=\dot u(0^-)=:\dot u(0)\,, \\ u(0^+)-u(0^-)=-\frac{2}{\overline{z}+1}\dot u(0) \end{array} \right\} \\ \Phi(A^*_{B_{1,z}})\,u \;&=\; -\ddot u+u\,; \end{aligned} \tag{6.8}$$

(ii) $\Phi(A_{B_{1,z}})=\Phi(A^*_{B_{1,z}})$ *on* $L^2(\mathbb{R})$ *if and only if* $z\in\mathbb{R}$*;*

(iii) $\Phi(A_{B_{1,z}})$ *and* $\Phi(A^*_{B_{1,z}})$ *are mutually adjoint on* $L^2(\mathbb{R})$*;*

(iv) $\Phi(A_{B_{1,z}})$ *is self-adjoint on* $L^2(\mathbb{R})$ *if and only if* $z\in\mathbb{R}$*. The same holds for* $\Phi(A^*_{B_{1,z}})$*.*

Proof. Parts (i) and (ii) follow at once from Proposition 6.3(i)-(ii), as well as parts (iii) and (iv) when $z\neq 0$. It remains to discuss the case $z=0$, i.e., to prove that $\Phi(A_{B_{1,0}})$ is self-adjoint. By (6.6), $\Phi(A_{B_{1,0}})=2\Phi(A_{B_{2,-1}})-3\mathbb{1}$, and applying Proposition 6.3(iv), we come to the conclusion. □

Let us collect together all the arguments and results of this Section. The following Summary makes the outline stated at the beginning of Section 4 explicit for the δ'-type extensions.

Summary 6.5 (δ'-type closed extensions of $\mathring{H}$ realised as Friedrichs systems).

- There exists a collection of operators $\Phi(A_{B_{1,z}})$ on $L^2(\mathbb{R})$ of the form (6.7), where $B_{1,z}$ runs in the class (6.1) for $z\in\mathbb{C}$ and $\lambda=1$, such that $\Phi(A_{B_{1,z}})$ is closed (Corollary 6.4(iii)) and $\mathring{H}+\mathbb{1}\subseteq\Phi(A_{B_{1,z}})$ (Corollary 6.4(i)). Thus,
$$\mathscr{C}_{\delta'}\;:=\;\big\{\Phi(A_{B_{1,z}})-\mathbb{1}\;:\;z\in\mathbb{C}\big\}$$
is a collection of *closed extensions of* $\mathring{H}$.
- All the $\Phi(A_{B_{1,z}})$'s arising in $\mathscr{C}_{\delta'}$, but the exceptional one corresponding to $z=0$ (i.e., to $B_{1,0}=\mathbb{O}$), are associated to pairs $(A_{B_{1,z}},A^*_{B_{1,z}})$ that, by Proposition 6.1(ii), are pairs of *bijective realisations* of the abstract Friedrichs operators $(A_{1,0},A'_{1,0})$ obtained by reducing the second order differential operator $\mathring{H}+\mathbb{1}$ to the first order vector-valued differential operator T_1 of formula (4.2).
- In fact, $\mathscr{C}_{\delta'}$ has two noticeable sub-families. The sub-family of $\mathscr{C}_{\delta'}$ selected by $\mathfrak{Re}\,z\leqslant -1$ corresponds to the only pairs $(A_{B_{1,z}},A^*_{B_{1,z}})$ that are *bijective realisations with signed boundary map* relative to $(A_{1,0},A'_{1,0})$, as follows from Proposition 6.1(iii).
- In turn, the sub-family of $\mathscr{C}_{\delta'}$ selected by $z\in\mathbb{R}$ is the maximal sub-family of self-adjoint operators (Corollary 6.4(iv)). It is precisely the one-parameter family $\{\Xi_\beta\,|\,\beta\in(-\infty,+\infty]\}$ of operators introduced in (3.3), since comparing (3.3) and (6.7) gives
$$\Phi(A_{B_{1,z}})-\mathbb{1}\;=\;\Xi_\beta\,,\qquad \beta:=-\tfrac{2}{z+1}\,, \tag{6.9}$$
where $\beta=+\infty$ is understood to correspond to $z=-1$.

- We have thus recovered the family of δ'-type *self-adjoint extensions* of $\mathring{H}$, together with the larger family $\mathscr{C}_{\delta'}$ of closed extensions of $\mathring{H}$ characterised by the same form $u(0^+) - u(0^-) = \beta\dot{u}(0)$ of boundary condition at $x = 0$ for possibly non-real β's, and we have done it through a novel conceptual path, as compared to the traditional methods that had let to Theorem 3.1(iii), that is, *understanding each such extension as an abstract Friedrichs system.*
- Each self-adjoint extension $\Xi_{\beta=-2/(z+1)} = \Phi(A_{B_{1,z}}) - \mathbb{1}$ of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}|_{C_c^\infty(\mathbb{R}\backslash\{0\})}$ is indeed understood as a pair $(A_{B_{1,z}}, A^*_{B_{1,z}})$ of closed bijective realisations of the underlying Friedrichs pair $(T_1, \widetilde{T}_1)$ naturally associated with $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}|_{C_c^\infty(\mathbb{R}\backslash\{0\})}$. Most noticeably, the extensions with $\beta \geqslant 0$ $(z \leqslant -1)$, namely the repulsive (non-confining) contact interaction Hamiltonians of δ'-type, correspond to bijective realisations with signed boundary map.

7. Bijective realisations of Friedrichs operators and 3D point interactions

In view of the canonical decomposition presented in Theorem 3.2, we develop in this Section the analogous discussion of Sections 4, 5, and 6, now applied to the starting operator

$$h_0 = -\frac{\mathrm{d}^2}{\mathrm{d}r^2}\,, \qquad \mathrm{dom}\, h_0 \;=\; H^2_0(\mathbb{R}^+)\,, \tag{7.1}$$

namely the only radial operator which one has to study the self-adjoint extensions of, in order to qualify the self-adjoint realisations of the three-dimensional Hamiltonian of contact interaction centred at the origin.

The analogy is stringent with what discussed already for the one-dimensional case, even if the problem now is on the half line, instead of the whole real line.

Acting on the Hilbert space

$$L \;:=\; L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^+) \tag{7.2}$$

we introduce the densely defined closed operators $A_{\lambda,0} : L \to L$ and $A_{\lambda,0} : L \to L$, for arbitrary $\lambda > 0$, defined by

$$\begin{aligned} A_{\lambda,0} \;&:=\; \sigma\,\frac{\mathrm{d}}{\mathrm{d}x} + \lambda\mathbb{1}\,, \qquad & \mathrm{dom}\, A_{\lambda,0} \;&:=\; \mathrm{dom}\, A'_{\lambda,0} \\ A'_{\lambda,0} \;&:=\; -\sigma\,\frac{\mathrm{d}}{\mathrm{d}x} + \lambda\mathbb{1}\,, & &:=\; H^1_0(\mathbb{R}^+) \oplus H^1_0(\mathbb{R}^+) \;=:\; W_0\,. \end{aligned} \tag{7.3}$$

The pair $(A_{\lambda,0}, A'_{\lambda,0})$ is a joint pair of closed abstract Friedrichs operators (Definition 2.3). The adjoints $A_{\lambda,1} := (A'_{\lambda,0})^*$ and $A'_{\lambda,1} := A^*_{\lambda,0}$ have domain

$$\mathrm{dom}\, A_{\lambda,1} \;=\; \mathrm{dom}\, A'_{\lambda,1} \;=\; H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^+) \;=:\; W\,,$$

and the same weak differential action as $A_{\lambda,0}$ and $A'_{\lambda,0}$. The boundary form associated with $(A_{\lambda,0}, A'_{\lambda,0})$ (Definition 2.4) is obtained by integration by parts: for any $\mathsf{u} \equiv \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\mathsf{v} \equiv \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in W,

$$D[\mathsf{u},\mathsf{v}] \;=\; \langle\, A_{\lambda,1}\mathsf{u} \mid \mathsf{v}\,\rangle_L - \langle\, \mathsf{u} \mid A'_{\lambda,1}\mathsf{v}\,\rangle_L \;=\; -\big(u_2(0^+)\bar{v}_1(0^+) + u_1(0^+)\bar{v}_2(0^+)\big)\,.$$

Now, choosing the subspaces

$$V \;:=\; \widetilde{V} \;:=\; H^1_0(\mathbb{R}^+) \oplus H^1(\mathbb{R}^+)\,, \tag{7.4}$$

it is immediately seen that $W_0 \subseteq V \subseteq W$, $W_0 \subseteq \widetilde{V} \subseteq W$, that correspondingly $A_{\lambda,1}|_V$ and $A'_{\lambda,1}|_V$ are mutually adjoint operators, and that the boundary form D vanishes on V. Therefore,

$$A_{\lambda,\mathrm{r}} \;:=\; A_{\lambda,1}|_V \qquad \text{and} \qquad A^*_{\lambda,\mathrm{r}} \;=\; A'_{\lambda,1}|_V \tag{7.5}$$

form an adjoint pair of bijective realisations with signed boundary map relative to $(A_{\lambda,0}, A'_{\lambda,0})$, with $\operatorname{dom} A_{\lambda,\mathrm{r}} = \operatorname{dom} A^*_{\lambda,\mathrm{r}} = V$.

The relevant kernels and projections needed in order to apply Theorems 2.8 and 2.9 to the present case are determined straightforwardly as follows. One has

$$\ker A_{\lambda,1} \;=\; \operatorname{span}\{\boldsymbol{\nu}_\lambda\} \qquad \text{and} \qquad \ker A'_{\lambda,1} \;=\; \operatorname{span}\{\boldsymbol{\nu}'_\lambda\}\,, \tag{7.6}$$

where

$$\boldsymbol{\nu}_\lambda(x) \;:=\; \begin{pmatrix} e^{-\lambda x} \\ e^{-\lambda x} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\nu}'_\lambda(x) \;:=\; \begin{pmatrix} e^{-\lambda x} \\ -e^{-\lambda x} \end{pmatrix}\,, \tag{7.7}$$

and the vectors $\boldsymbol{\nu}_\lambda$ and $\boldsymbol{\nu}'_\lambda$ are pairwise orthogonal in L and $\|\boldsymbol{\nu}_\lambda\|_L = \|\boldsymbol{\nu}'_\lambda\|_L = \frac{1}{\sqrt{\lambda}}$. Further, with respect to the choice (7.5) for $A_{\lambda,\mathrm{r}}$, the (non-orthogonal) projections $p_{\lambda,\mathrm{k}} : \operatorname{dom} A_{\lambda,1} \to \ker A_{\lambda,1}$ and $p_{\lambda,\mathrm{k}'} : \operatorname{dom} A'_{\lambda,1} \to \ker A'_{\lambda,1}$ defined in (2.6)–(2.7) act in the present case as

$$\begin{aligned} p_{\lambda,\mathrm{k}}\mathsf{u} \;&=\; u_1(0^+)\,\boldsymbol{\nu}_\lambda \\ p_{\lambda,\mathrm{k}'}\mathsf{u} \;&=\; u_1(0^+)\,\boldsymbol{\nu}'_\lambda\,. \end{aligned} \tag{7.8}$$

Next, in view of the extension scheme of Theorems 2.8 and 2.9, we qualify the pairs (B_λ, B^*_λ) of densely defined and mutually adjoint operators $B_\lambda : \mathcal{V}_\lambda \to \mathcal{W}_\lambda$ and $B^*_\lambda : \mathcal{W}_\lambda \to \mathcal{V}_\lambda$, with domains $\operatorname{dom} B_\lambda \subseteq \mathcal{V}_\lambda$ and $\operatorname{dom} B^*_\lambda \subseteq \mathcal{W}_\lambda$, where $\mathcal{V}_\lambda$ and $\mathcal{W}_\lambda$ are closed subspaces of $\ker A_{\lambda,1}$ and $\ker A'_{\lambda,1}$.

Since $\dim\ker A_{\lambda,1} = \dim\ker A'_{\lambda,1} = 1$, such $\mathcal{V}_\lambda$ and $\mathcal{W}_\lambda$ can be zero- or one-dimensional, and B_λ is necessarily bounded. The zero-dimensional case is trivial and yields the operators (7.5). We then consider the case $\dim\mathcal{V}_\lambda = \dim\mathcal{W}_\lambda = 1$, i.e.,

$$\begin{aligned} \mathcal{V}_\lambda \;&=\; \ker A_{\lambda,1} \\ \mathcal{W}_\lambda \;&=\; \ker A'_{\lambda,1} \\ B_\lambda \;&:\; \mathcal{V}_\lambda \to \mathcal{W}_\lambda \\ B_\lambda\boldsymbol{\nu}_\lambda \;&=\; z\,\boldsymbol{\nu}'_\lambda \qquad \text{for a fixed } z \in \mathbb{C}\,. \end{aligned} \tag{7.9}$$

Reasoning as in the proof of Lemma 4.2, we find the following.

Lemma 7.1. *Let $\lambda > 0$, and let $\mathcal{V}_\lambda, \mathcal{W}_\lambda$ and B_λ be as in* (7.9)*. Then condition* (2.13) *is satisfied if and only if*

$$\mathfrak{Re}\, z \;\geqslant\; \lambda\,.$$

Arguing in the same way as for Propositions 5.1 and 6.1, for each B_λ we determine the domain of corresponding realisations A_{B_λ} with the choice (7.9).

Proposition 7.2.

(i) *Associated with the operator $B_\lambda : \mathcal{V}_\lambda \to \mathcal{W}_\lambda$ defined in* (7.9)*, via the correspondence* (2.9)*, is the operator $A_{B_\lambda} = A_{\lambda,1}|_{\mathrm{dom}\, A_{B_\lambda}}$ on the Hilbert space L, whose domain is given by*

$$\mathrm{dom}\, A_{B_\lambda} \;=\; \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^+) \;:\; u_1(0^+) = \frac{\lambda}{\lambda - z} u_2(0^+) \right\}\,, \tag{7.10}$$

as well as the operator $A_{B_\lambda}^* = A_{B_\lambda^*} = A'_{\lambda,1}|_{\mathrm{dom}\, A_{B_\lambda}^*}$ on L, whose domain is given by

$$\mathrm{dom}\, A_{B_\lambda}^* \;=\; \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H^1(\mathbb{R}^+) \oplus H^1(\mathbb{R}^+) \;:\; u_1(0^+) = -\frac{\lambda}{\lambda - \overline{z}} u_2(0^+) \right\}\,. \tag{7.11}$$

(ii) *The pair $(A_{B_\lambda}, A_{B_\lambda}^*)$ is an adjoint pair of bijective realisations of the abstract Friedrichs operators $(A_{\lambda,0}, A'_{\lambda,0})$ if an only if $z \neq 0$ in* (7.9)*.*

(iii) *The pair $(A_{B_\lambda}, A_{B_\lambda}^*)$ has signed boundary map if and only if $\mathfrak{Re}\, z \geqslant \lambda$ in* (7.9)*.*

Reasoning as for Propositions 5.3 and 6.3, we lift the pairs $(A_{B_\lambda}, A_{B_\lambda}^*)$ of the class identified by the choice (7.9) to pairs $(\Phi(A_{B_\lambda}), \Phi(A_{B_\lambda}^*))$ of operators on $L^2(\mathbb{R}^+)$. The lift is now defined, in analogy to (4.3), as the linear map

$$\begin{aligned} \Phi \;&:\; \mathfrak{L}(L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^+)) \longrightarrow \mathfrak{L}(L^2(\mathbb{R}^+))\,, \\ \mathrm{dom}\, \Phi(A) \;&:=\; \left\{ u \in L^2(\mathbb{R}^+) : (\exists!\, v_u \in L^2(\mathbb{R}))\;\; \begin{pmatrix} v_u \\ u \end{pmatrix} \in \mathrm{dom}\, A \cap \ker P_1 A \right\}, \\ \Phi(A)\, u \;&:=\; P_2 A \begin{pmatrix} v_u \\ u \end{pmatrix}\,, \end{aligned} \tag{7.12}$$

where $\mathfrak{L}(X)$ is the space of *linear* (not necessarily bounded) maps on the vector space X and $P_j : L^2(\mathbb{R}^+) \oplus L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$, $j \in \{1, 2\}$, is the orthogonal projection onto the j-th component of L.

Proposition 7.3. *Let $B_{\lambda,z} : \mathcal{V}_\lambda \to \mathcal{W}_\lambda$ be an operator of the type* (7.9) *for some $z \in \mathbb{C}$, and let $A_{B_{\lambda,z}}$ and $A_{B_{\lambda,z}}^*$ be the corresponding operators described in Proposition* 7.2*.*

(i) *Via the map* Φ *introduced in* (7.12) *one has*

$$\begin{aligned}\operatorname{dom}\Phi(A_{B_{\lambda,z}}) &= \Big\{u\in H^2(\mathbb{R}^+) : \dot u(0^+) = -\frac{\lambda^2}{\lambda - z}u(0^+)\Big\}\\ \Phi(A_{B_{\lambda,z}})\,u &= \lambda^{-1}(-\ddot u + \lambda^2 u)\,,\end{aligned}\tag{7.13}$$

and

$$\begin{aligned}\operatorname{dom}\Phi(A^*_{B_{\lambda,z}}) &= \Big\{u\in H^2(\mathbb{R}^+) : \dot u(0^+) = -\frac{\lambda^2}{\lambda - \overline{z}}u(0^+)\Big\}\\ \Phi(A^*_{B_{\lambda,z}})\,u &= \lambda^{-1}(-\ddot u + \lambda^2 u)\,.\end{aligned}\tag{7.14}$$

(ii) $\Phi(A_{B_{\lambda,z}}) = \Phi(A^*_{B_{\lambda,z}})$ *on* $L^2(\mathbb{R})$ *if and only if* $z\in\mathbb{R}$.

(iii) *If* $z\neq 0$, *then the operators* $\Phi(A_{B_{\lambda,z}})$ *and* $\Phi(A^*_{B_{\lambda,z}})$ *are mutually adjoint on* $L^2(\mathbb{R})$.

(iv) *The two conditions*
- $z\neq 0$ *and* $\Phi(A_{B_{\lambda,z}})$ *is self-adjoint;*
- $z\in\mathbb{R}\setminus\{0\}$,

are equivalent. The same equivalence holds replacing $\Phi(A_{B_{\lambda,z}})$ *by* $\Phi(A^*_{B_{\lambda,z}})$.

As a consequence, reasoning as for Corollaries 6.4 and 5.4 we find the following relevant properties.

Corollary 7.4. *Under the assumptions of Proposition* 7.3*,*

(i) *one has*

$$\begin{aligned}\operatorname{dom}\Phi(A_{B_{1,z}}) &= \Big\{u\in H^2(\mathbb{R}^+) : \dot u(0^+) = -\frac{1}{1 - z}u(0^+)\Big\}\\ \Phi(A_{B_{1,z}})\,u &= -\ddot u + u\,,\end{aligned}\tag{7.15}$$

and

$$\begin{aligned}\operatorname{dom}\Phi(A^*_{B_{1,z}}) &= \Big\{u\in H^2(\mathbb{R}^+) : \dot u(0^+) = -\frac{1}{1 - \overline{z}}u(0^+)\Big\}\\ \Phi(A^*_{B_{1,z}})\,u &= -\ddot u + u\,;\end{aligned}\tag{7.16}$$

(ii) $\Phi(A_{B_{1,z}}) = \Phi(A^*_{B_{1,z}})$ *on* $L^2(\mathbb{R})$ *if and only if* $z\in\mathbb{R}$*;*

(iii) $\Phi(A_{B_{1,z}})$ *and* $\Phi(A^*_{B_{1,z}})$ *are mutually adjoint on* $L^2(\mathbb{R})$*;*

(iv) $\Phi(A_{B_{1,z}})$ *is self-adjoint on* $L^2(\mathbb{R})$ *if and only if* $z\in\mathbb{R}$. *The same holds for* $\Phi(A^*_{B_{1,z}})$.

Based on the results above, we can repeat the analogous considerations stated in Summary 5.5 and Summary 6.5. In short, we have identified all closed realisations of operator h_0 given by (7.1), and hence also of operator $\mathring{H}$ defined in (3.4), characterised by the same boundary condition $\dot u(0^+) = 4\pi\alpha u(0^+)$ at $x=0$ for possibly complex α's, recovering the self-adjoint extensions for real α's. We have done it through a novel conceptual path, as compared to the traditional methods that had let to Theorem 3.2(iii), that is, *understanding each such realisation as an abstract Friedrichs system.*

Remark 7.5. An alternative, equivalent route for the above three-dimensional analysis, instead of exploiting the construction of $-\Delta_\alpha$ through the partial wave decomposition as in Theorem 3.2, and hence working with the radial Hilbert space $L^2(\mathbb{R}^+)$, is to develop a genuinely three-dimensional discussion as we sketch here below.

On the Hilbert space

$$L \;:=\; L^2(\mathbb{R}^3;\mathbb{C}^3)\oplus L^2(\mathbb{R}^3)\,,$$

for arbitrary $\lambda>0$, we define densely defined operators $T_\lambda, \widetilde{T}_\lambda : L\to L$ by

$$\begin{aligned} T_\lambda \;&:=\; \begin{pmatrix} 0 & \nabla \\ \mathrm{div} & 0\end{pmatrix} + \lambda\mathbb{1} \\ \widetilde{T}_\lambda \;&:=\; -\begin{pmatrix} 0 & \nabla \\ \mathrm{div} & 0\end{pmatrix} + \lambda\mathbb{1} \\ \mathrm{dom}\, T_\lambda \;&:=\; \mathrm{dom}\,\widetilde{T}_\lambda \;:=\; C_c^\infty(\mathbb{R}\backslash\{0\};\mathbb{C}^3)\oplus C_c^\infty(\mathbb{R}\backslash\{0\})\,. \end{aligned} \tag{7.17}$$

It can be easily seen that $(T_\lambda,\widetilde{T}_\lambda)$ is a joint pair of abstract Friedrichs operators and

$$\Phi(T_\lambda)=\lambda^{-1}(\mathring{H}+\lambda^2\mathbb{1})\,,$$

where $\mathring{H}$ is given by (3.4) and $\Phi:\mathfrak{L}(L)\to\mathfrak{L}(L^2(\mathbb{R}^3))$ is defined by analogy with (4.3). The domains of their operator closures and adjoints are given by

$$\begin{aligned} \mathrm{dom}\,\overline{T_\lambda} \;&=\; \mathrm{dom}\,\overline{\widetilde{T}_\lambda} \;=\; L^2_{\mathrm{div},0}(\mathbb{R}^3\backslash\{0\};\mathbb{C}^3)\oplus H^1_0(\mathbb{R}^3\backslash\{0\}) \\ \mathrm{dom}\,T_\lambda^* \;&=\; \mathrm{dom}\,\widetilde{T}_\lambda^* \;=\; L^2_{\mathrm{div}}(\mathbb{R}^3\backslash\{0\};\mathbb{C}^3)\oplus H^1(\mathbb{R}^3\backslash\{0\})\,, \end{aligned}$$

where $L^2_{\mathrm{div}}(\mathbb{R}^3\backslash\{0\};\mathbb{C}^3):=\{\mathsf{u}\in L^2(\mathbb{R}^3;\mathbb{C}^3):\mathrm{div}\,\mathsf{u}\in L^2(\mathbb{R}^3)\}$ (here div is a weak (distributional) differential operator on $\mathbb{R}^3\backslash\{0\}$) and $L^2_{\mathrm{div},0}(\mathbb{R}^3\backslash\{0\};\mathbb{C}^3)$ is the closure of $C_c^\infty(\mathbb{R}^3\backslash\{0\};\mathbb{C}^3)$ in $L^2_{\mathrm{div}}(\mathbb{R}^3\backslash\{0\};\mathbb{C}^3)$. Furthermore, as a weak differential operator, $\widetilde{T}_\lambda^*$, respectively T_λ^*, acts formally as T_λ, respectively $\widetilde{T}_\lambda$.

One can check that for

$$V \;:=\; \widetilde{V} \;:=\; L^2_{\mathrm{div}}(\mathbb{R}^3;\mathbb{C}^3)\oplus H^1(\mathbb{R}^3) \tag{7.18}$$

the pair $(\widetilde{T}_\lambda^*|_V, T_\lambda^*|_V)$ (or $(A_{\lambda,1}|_V, A'_{\lambda,1}|_V)$ with the notation $A_{\lambda,1}:=\widetilde{T}_\lambda^*$, $A'_{\lambda,1}:=T_\lambda^*$ used in this Section) form an adjoint pair of bijective realisations with signed boundary map relative to $(T_\lambda,\widetilde{T}_\lambda)$, which could be taken as the reference one for the application of Theorems 2.8 and 2.9. This choice corresponds to the self-adjoint negative Laplacian $-\Delta_\infty$ on $L^2(\mathbb{R}^3)$ in the sense that $\mathrm{dom}\,\Phi(\widetilde{T}_\lambda^*|_V)=\mathrm{dom}\,\Phi(T_\lambda^*|_V)=H^2(\mathbb{R}^3)$ and

$$\Phi(\widetilde{T}_\lambda^*|_V) \;=\; \Phi(T_\lambda^*|_V) \;=\; \lambda^{-1}(-\Delta_\infty+\lambda^2\mathbb{1})\,,$$

as an identity of operators on L; therefore, this choice differs from (7.5) used in the above computations, which corresponds to $-\Delta_0$, i.e., to the case $\alpha=0$ in (3.5).

The kernels of the adjoints are explicitly given by

$$\ker\widetilde{T}_\lambda^* = \mathrm{span}\left\{\begin{pmatrix} -\frac{1}{\lambda}\nabla\psi_\lambda \\ \psi_\lambda\end{pmatrix}\right\} \qquad \text{and} \qquad \ker T_\lambda^* = \mathrm{span}\left\{\begin{pmatrix} \frac{1}{\lambda}\nabla\psi_\lambda \\ \psi_\lambda\end{pmatrix}\right\}\,,$$

where $\psi_\lambda(x) = \frac{e^{-\lambda|x|}}{|x|}$.

Thus, one is left with expressing the non-orthogonal projections (2.7) with respect to the choice (7.18), and one way to do that is precisely by exploiting the decomposition of $L^2(\mathbb{R}^3)$ into the radial part and spherical harmonics, described in Theorem 3.2(ii), as was done in the above computations.

8. Further examples and concluding remarks

Having recognised that Hamiltonians of contact interactions supported at one point (or, more generally, at a finite number of fixed points) are, in a suitable correspondence scheme, nothing but abstract Friedrichs operators on Hilbert space, and having recognised that a sub-class of them corresponds to the relevant sub-class of bijective realisations of Friedrichs operators with signed boundary map, is a new fact that deserves interest per se and even more for bringing novel examples of abstract Friedrichs systems, as compared to all the previously known examples arising from concrete boundary value problems or Cauchy problems for partial differential equations.

In addition, we also highlighted the intrinsic connection between the well-known extension scheme that yields the family of contact interaction Hamiltonians as self-adjoint realisations of the symmetric free Hamiltonian restricted away from the interaction centre(s), and the general classification [8] for the bijective realisations of a given pair of abstract Friedrichs operators, including those with signed boundary. The latter leads in a natural way to the *closed* (not necessarily self-adjoint) extensions of the free Hamiltonian initially restricted away from the interaction centre and satisfying the same type of contact boundary condition.

Along the same line of reasoning of the present discussion, one could approach even further classes of contact interaction Hamiltonians for which our correspondence scheme is applicable in connection with abstract Friedrichs operators. The most noticeable example, to our view, would be the topical class of Hamiltonians of point interactions supported on a hyper-surface Σ of $\mathbb{R}^d$ [11, 10, 9]. At the core of their construction as self-adjoint operators $H_{\alpha,\Sigma}$ or $H_{\beta,\Sigma}$ on $L^2(\mathbb{R}^d)$ is a boundary condition for each point $x \in \Sigma$, along the normal to Σ, that is actually the one-dimensional δ-type boundary condition (3.2) or the δ'-type boundary condition (3.3), where now the couplings α or β are suitable given functions $\alpha(x)$ or $\beta(x)$ on Σ.

Thus, even if we do not develop this point further, we are confident that a reduction scheme as in Section 4 should be applicable, and proceeding as in Sections 5 and 6 one could recognise $H_{\alpha,\Sigma}$ and $H_{\beta,\Sigma}$ as lifted-to-second-order versions of joint pairs of closed abstract Friedrichs operators on $L^2(\mathbb{R}^d)$. For fairly generic surfaces, hence in the lack of special symmetries, the main difficulty and the key point is to reproduce the reduction scheme pointwise along the surface. This would match the same spirit of the above-mentioned works, where the boundary condition of self-adjointness for the contact interaction Hamiltonian does hold at each point of the surface.

Acknowledgment

We warmly thank N. Antonić and G. Dell'Antonio for enlightening discussions on the subject.

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