Low-Sensitivity, Low-Power 4th-Order Low-Pass Active-RC Allpole Filter Using Impedance Tapering

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Abstract—The analytical design procedure of low-sensitivity, low-power, low-pass (LP) 2nd- and 3rd-order class-4 active-RC allpole filters, using impedance tapering, has already been published [1][2]. In this paper the desensitisation using impedance tapering is applied to the design of LP 4th-order filters. The numerical design procedure was performed by Newton’s iterative method. Analytically designed unity-gain LP 4th-order filters [3] can provide initial values for Newton’s method. The sensitivities of a filter transfer function to passive component tolerances, as well as active gain variations are examined by the Schoeffler sensitivity and Monte Carlo SPICE simulation. Butterworth and Chebyshev 0.5dB filter examples illustrate the design method.

I. INTRODUCTION

A procedure for the analytical design of low-sensitivity class-4 2nd- and 3rd-order Sallen-and-Key [4] active resistance-capacitance (RC) low-pass (LP) allpole filters was presented in [1], with the realizability constraints in [2]. It was shown in [1] that by the use of “impedance tapering”, in which L-sections of the RC network are successively impedance scaled upwards, from the driving source to the positive amplifier input, the sensitivity of the filter characteristics to passive component tolerances can be significantly decreased.

In this paper, the design method based on “impedance tapering” is extended to the design of 4th-order LP active-RC filters each with a single operational amplifier (opamp). It is also demonstrated that obtaining an analytical solution is possible only for lower-than-4th-order filters and for a special case of 4th-order unity-gain filter (β=1) [3]. The examples of Butterworth and Chebyshev filters with 0.5dB pass-band ripple illustrate optimal filter design with minimum passive and active sensitivities.

II. FOURTH-ORDER ALLPOLE FILTER

Consider the 4th-order single-amplifier LP filter shown in Fig. 1. It is a low-power circuit, insofar as it uses only one opamp. Its voltage transfer function \( T(s) \) is given by:

\[
T(s) = \frac{V_2}{V_1} = \frac{\beta a_0}{s^4 + a_4 s^3 + a_3 s^2 + a_2 s + a_1}, \tag{1}
\]

where coefficients \( a_i \) (i=0,...,3) as a function of components of the circuit are given by (2):

\[
a_0 = (R_2 R_3 R_4 C_1 C_2 C_3 C_4)^{-1},
\]

\[
a_1 = a_0 \left\{ R_1 (C_1 + C_2 + C_3 + C_4) + R_2 (C_2 + C_3 + C_4) + R_3 (C_3 + C_4) + R_4 (C_4) \right\}
\]

\[
a_2 = a_0 \left\{ R_1 R_2 C_1 (C_2 + C_3 + C_4) + R_2 R_3 C_2 (C_3 + C_4) + R_3 R_4 C_3 (C_4) + R_4 (C_4) \right\}
\]

\[
a_3 = a_0 \left\{ R_1 R_2 R_3 C_1 (C_2 + C_3 + C_4) + R_2 R_3 R_4 C_2 (C_3 + C_4) + R_3 R_4 (C_3 + C_4) + R_4 (C_4) \right\}
\]

Transfer function \( T(s) \) in (1), can be written in terms of pole Q-factors, \( q_i \), and pole frequencies \( \omega_i \) (i=1, 2) as

\[
T(s) = \frac{\beta_0^2 \omega_1^2}{s^4 + (\omega_0 / q_0) s + \omega_1^2} \frac{\omega_2^2}{s^4 + (\omega_0 / q_0) s + \omega_2^2}, \tag{3}
\]

Note that the gain for the class-4 circuit is given by:

\[
\beta = 1 + R_2 / R_1 \geq 1. \tag{4}
\]

Introducing the design frequency \( \omega_0 \) and impedance scaling factors \( r_i \) and \( \rho_i \) as in Fig. 1, defined by

\[
\omega_0 = \frac{(R_1 C_1)^{-1}}, \quad R = R_1 R_i, C = C_1 / \rho_i; \quad i=2, 3, 4; \tag{5}
\]

into (2), and using

\[
\alpha_i = a_i / \omega^2; \quad i=0, 1, 2, 3, 4; \tag{6}
\]

we obtain a system of four equations with eight unknowns. To design the 4th-order LP filter we have to solve this system, and therefore we must choose four variables, and then calculate the remaining four. For example, we can calculate the resistive tapering factors \( r_i \) (i=2, 3, 4) and gain \( \beta \) from given coefficients \( a_i \) (i=0, ..., 3), chosen capacitive factors \( \rho_i \) (i=2, 3, 4) and the design frequency \( \omega_0 \). Capacitive scaling factors \( \rho_i \) should geometrically progress, providing “capacitive tapering” in the filter design. Note that, alternatively, we could have started by choosing resistive scaling factors \( r_i \), thus providing a “resistive tapering” design procedure. We can express values of \( r_2 \) and \( \beta \) explicitly, but we obtain a nonlinear relation between \( r_2 \) and \( r_3 \). The new system of four nonlinear equations is given by:

\[
r_4 = r_3 r_2 / (r_2 r_3 a_0) ;
\]

\[
a \cdot r_2^2 r_3^2 + b \cdot r_2 r_3 r_4 + c \cdot r_2 r_4 + d \cdot r_3^2 + e \cdot r_4 + f \cdot r_3 = 0; \tag{7}
\]

\[
g \cdot r_2^2 r_3^2 + h \cdot r_2^2 r_4 + i \cdot r_2 r_3^2 + j \cdot r_3 r_4 + k \cdot r_3^2 + l \cdot r_3 r_4 + m \cdot r_4 = 0;
\]

\[
\beta = 1 + \rho_1 / \rho_2 \cdot (r_2 / r_1) \tag{8}
\]

where the constants a to m in the 2nd and 3rd equations can readily be calculated from eqs. (2) to (6). The next step, applying the Newton-Raphson method, is to solve this system of eight nonlinear equations.
which is the only one that seems to be possible in trying to find an analytical solution, is to merge and factorise the 2nd and 3rd equations of the system (7) in the form given by:

\[(r_3^2 + n r_2^2 + o r_1 + p) \cdot (r_4^2 + q r_3 + t) = 0.\]  

(8)

The form (8) is obviously impossible for representation and we can’t solve the system (7) analytically. Instead, the second and third equations in (7) can be solved for the values of \(r_2\) and \(r_3\) only numerically, using, for example Newton’s iterative method. Once we have the values of \(r_2\) and \(r_3\), the remaining values of \(r_4\) and \(\beta\) readily follow from the first and last equations of the system, respectively.

In the following examples, a 4th-order LP active-RC filter will be solved using the program “Mathematica”.

III. EXAMPLE

Consider the Butterworth and Chebyshev filter with a pass-band ripple \(R_p=0.5\text{dB}\) having the coefficients shown in Table I. The corresponding amplitude responses \(\alpha(\omega)\) in [dB] are shown in Fig. 2.

<table>
<thead>
<tr>
<th>No. Specif.</th>
<th>(R_p)</th>
<th>(a_0)</th>
<th>(a_1)</th>
<th>(a_2)</th>
<th>(a_3)</th>
<th>(a_4)</th>
<th>(\beta)</th>
<th>(\sigma_\alpha(\omega))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 1-80kHz</td>
<td>0.0</td>
<td>6.09</td>
<td>3.20</td>
<td>8.42</td>
<td>13.0</td>
<td>4.97</td>
<td>4.97</td>
<td>0.54</td>
</tr>
<tr>
<td>2. 0.5-3kHz</td>
<td>0.5</td>
<td>2.42</td>
<td>1.13</td>
<td>4.34</td>
<td>6.02</td>
<td>3.00</td>
<td>5.18</td>
<td>0.71</td>
</tr>
</tbody>
</table>

We start with the Butterworth transfer function coefficients in line 1) of Table I. In the design process, various ways of impedance tapering have been applied, i.e. capacitive and resistive, and the resulting component values are presented in Table II with resistors in \([\text{k}\Omega]\) and capacitors in \([\text{pF}]\). A sensitivity analysis was performed. The standard deviations \(\sigma_\alpha(\omega)\) in [dB] (related to the Shoefller sensitivities) of the variation of the log gain \(\Delta \alpha=8.68588\text{[dB]}\) with respect to zero mean and 1% standard deviation of passive components, was calculated at the dominant pole frequency \(\omega_p=\omega_0/2\). This is the frequency vicinity in which the magnitude spread is the highest; it is shown in the last column of Table II.

### COMPONENT VALUES OF IMPEDANCE-TAPERED 4th-ORDER LP FILTER

<table>
<thead>
<tr>
<th>No. Tap.</th>
<th>(R_1)</th>
<th>(C_1)</th>
<th>(r_2)</th>
<th>(r_3)</th>
<th>(r_4)</th>
<th>(p_2)</th>
<th>(p_3)</th>
<th>(p_4)</th>
<th>(\beta)</th>
<th>(\sigma_\alpha(\omega))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1)</td>
<td>15.6</td>
<td>100</td>
<td>3.20</td>
<td>5.03</td>
<td>0.17</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2.0</td>
<td>2.227</td>
</tr>
<tr>
<td>2) Cap.</td>
<td>1.59</td>
<td>800</td>
<td>12.6</td>
<td>5.47</td>
<td>5.84</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>2.0</td>
<td>0.765</td>
</tr>
<tr>
<td>3)</td>
<td>0.46</td>
<td>2700</td>
<td>18.7</td>
<td>10.8</td>
<td>25.1</td>
<td>3</td>
<td>9</td>
<td>27</td>
<td>2.0</td>
<td>0.638</td>
</tr>
<tr>
<td>1)</td>
<td>10</td>
<td>193</td>
<td>1</td>
<td>1</td>
<td>0.28</td>
<td>1.12</td>
<td>1.43</td>
<td>4.0</td>
<td>1.605</td>
<td></td>
</tr>
<tr>
<td>2)</td>
<td>10</td>
<td>149</td>
<td>2</td>
<td>4</td>
<td>0.41</td>
<td>7.02</td>
<td>6.68</td>
<td>4.0</td>
<td>1.198</td>
<td></td>
</tr>
<tr>
<td>3)</td>
<td>10</td>
<td>134</td>
<td>3</td>
<td>9</td>
<td>0.27</td>
<td>0.54</td>
<td>1.47</td>
<td>18.1</td>
<td>4.0</td>
<td>1.079</td>
</tr>
</tbody>
</table>

Beside Shoefller’s sensitivity, Monte Carlo runs (MC) are also performed as a double-check and presented in Figs. 4 and 6. The corresponding PSpice circuit model of a 4th-order LP filter with 1% Gaussian distributed, zero-mean resistors and capacitors is shown in Fig. 3. Note that the opamp is simulated by a voltage-controlled-voltage-source with high and constant gain value of \(A=10^{10}\).

Using the Chebyshev transfer function coefficients in line 2) of Table I, we designed the minimum sensitivity circuit, with component values in line 7) of Table III. The min. sensitivity filters are marked by a bold rectangle. The corresponding curves of the Butterworth and Chebyshev filter design parameters in Table III, vs. \(\omega_0\), are shown in Fig. 5 (a) and (b), respectively.
In Fig. 7 we introduced a model of the open-loop gain $A$ device (inside dashed rectangle) to use with PSpice for active component gain variation in the 4th-order LP filter. $R_7$ and $R_9$ are supposed to be tracking resistors, as well as $R_8$ and $R_{10}$. $A$-variations are performed by variations of both $R_7$ and $R_8$ and theirs tracking counterparts. The gain $A$ values are log distributed in the range from $1.3 \cdot 10^2$ to $4.5 \cdot 10^5$. In practice, the gain $A$ changes with temperature and at higher frequencies the gain $A$ becomes smaller.

Note that all passive components take their nominal values. The corresponding MC runs of an active component are presented in Figs. 8 and 9.

From the Chebyshev example in Fig. 10 we conclude that for higher pole Qs, $q_p$, both the active and passive sensitivities are increased.

### A. Design of 4th-order Filter Starting from $\beta=1$ Filter

We use $\beta=1$ filter elements in [3] for the starting values in the Newton’s iterative method, when numerically solving the set of non-linear eqs. (7). We obtain the filters in Table IV and the corresponding MC runs shown in Figs. 11 and 12.

#### Table IV

<table>
<thead>
<tr>
<th>No.</th>
<th>Tap.</th>
<th>$R_1$</th>
<th>$C_1$</th>
<th>$r_2$</th>
<th>$r_3$</th>
<th>$r_4$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>$p_4$</th>
<th>$\beta$</th>
<th>$\sigma_0(\omega_0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\beta=1$</td>
<td>0.611</td>
<td>5</td>
<td>1.08</td>
<td>5.24</td>
<td>2.76</td>
<td>1.34</td>
<td>1.0</td>
<td>62.1</td>
<td>1.0</td>
<td>0.168</td>
</tr>
<tr>
<td>2</td>
<td>Res.</td>
<td>10</td>
<td>289</td>
<td>1.08</td>
<td>5.24</td>
<td>2.76</td>
<td>1.17</td>
<td>2.47</td>
<td>23.1</td>
<td>1.1</td>
<td>0.258</td>
</tr>
<tr>
<td>3</td>
<td>Res.</td>
<td>10</td>
<td>267</td>
<td>1.08</td>
<td>5.24</td>
<td>2.76</td>
<td>0.96</td>
<td>3.60</td>
<td>14.0</td>
<td>1.3</td>
<td>0.406</td>
</tr>
<tr>
<td>4</td>
<td>Res.</td>
<td>10</td>
<td>228</td>
<td>1.08</td>
<td>5.24</td>
<td>2.76</td>
<td>0.63</td>
<td>5.47</td>
<td>7.44</td>
<td>2.0</td>
<td>0.780</td>
</tr>
<tr>
<td>5</td>
<td>Cap.</td>
<td>1.288</td>
<td>5</td>
<td>0.40</td>
<td>0.70</td>
<td>2.83</td>
<td>1.34</td>
<td>1.0</td>
<td>62.1</td>
<td>1.1</td>
<td>0.342</td>
</tr>
<tr>
<td>6</td>
<td>Cap.</td>
<td>5.780</td>
<td>5</td>
<td>0.06</td>
<td>0.07</td>
<td>0.48</td>
<td>1.34</td>
<td>1.0</td>
<td>62.1</td>
<td>1.3</td>
<td>1.433</td>
</tr>
</tbody>
</table>

We chose resistors as starting values and then increased the values of $\beta$ by changing the design frequency $\omega_0$, thus obtaining the circuits no. 1)-4). Our goal is to reach, for example, the min. active sensitivity (filter no. 3 in Table IV, see Fig. 12). Note that for the 4th-order filter, we do not have any analytical expression by which we can calculate the min. GSP, as is possible for the 2nd-order case [5].
Fig. 11. Passive components Monte Carlo runs of amplitude response of capacitively-tapered 4th-order LP filter as a function of $\beta$ (Table IV).

Fig. 12. Active component (opamp) MC runs of amplitude response of capacitively-tapered 4th-order LP filter as a function of $\beta$ (Table IV).

It follows from Table IV that we must be careful not to choose capacitors as starting values, because we will then obtain bad circuits: no. 5)-6). The corresponding curves for design parameters in Table IV, vs. $\omega_0$, which can be represented by $\beta$, are shown in Fig. 13. As follows from [3] it is not possible to design the Chebyshev filter starting from the $\beta=1$ filter.

B. Comparison of 4th-order Filter with 2-Biquad CAS

In this section we compare the performance of the single opamp, (1-OA) 4th-order LP filter with a cascade (CAS) of two 2nd-order LP filter “biquads” as in [1]. We apply capacitive impedance tapering with resistor values selected for GSP-minimization, to minimize both passive and active sensitivity, to the two biquads in cascade [5]. The component values are given in Table V.

![Component Values of 2nd-Order LP Sections in CAS](image)

IV. CONCLUSION

In this paper we presented the optimal design of single opamp 4th-order LP filters. Unfortunately, the design equations for higher-than-third order filters defy any analytical solution, thus we presented a numerical design procedure. In one example we used unity gain filter elements as the starting values. It is shown in numerous examples that impedance tapering reduces both passive and active sensitivities. The unity gain tapered filter is shown to have min. passive sensitivity, but the active sensitivity is still too high. Therefore, an optimal design procedure is to apply capacitive tapering, and then to change the design frequency $\omega_0$ to minimize active sensitivity, and by that, to keep the passive sensitivity reduced, as well. Finally, we compared the tapered single opamp 4th-order LP filter to a cascade of two 2nd-order tapered LP “biquads”. The latter has substantially reduced sensitivities, and the design equations for the min. GSP are available in closed form. Thus, the decision on which approach to take is typically one of tradeoffs: low power and element count vs. low sensitivity and design simplicity.

REFERENCES


