Originals Computational Mechanics 20 (1997) 507–519 © Springer-Verlag 1997

Dynamics of 3-D co-rotational beams

M. A. Crisfield, U. Galvanetto, G. Jelenić

Abstract This paper discusses different types of implicit time integration algorithms for the dynamics of spatial beams. The algorithms are based on a form of co-rotational technique which is external to the element. Both end-point and mid-point formulations are presented. The latter can be considered as an 'approximately energy conserving algorithm'. A new method is described for introducing numerical damping. Finally some numerical examples are presented in order to illustrate the differences in performance of the different integration schemes.

1 Introduction

The present paper deals with the dynamic behaviour of spatial beams undergoing large translations and rotations but small strains. The beams are assumed to be homogeneous, isotropic and linearly elastic. One of the first important papers in this field was written by Simo and Vu-Quoc (1988) and described an isoparametric approach. Alternative approaches were proposed by Cardona and Géradin (1988), Iura and Atluri (1988), and Quadrelli and Atluri (1996), among others. Most formulations involved forms of end-point dynamic equilibrium, either directly via the Newmark time integration procedure (Newmark 1959) or indirectly with the aid of the α-method (Hilbert et al. 1977) which introduces a form of numerical damping. An alternative approach was initiated by Simo, Tarnow and Doblare (1995) and involved an algorithm that conserves both energy and the momenta. The method can be considered as a particular form of mid-point procedure.

In the present paper, a co-rotational approach is chosen following the framework suggested in the static field by Rankin and Brogan (1986). All the algorithms presented in this paper are based on an 'element independent' co-rotational technique. A similar approach was also adopted by Crisfield and Shi (1994) and by Galvanetto and Crisfield (1996) in the dynamic case of planar trusses and planar beam structures. The paper is organized as follows: the second section briefly presents some general ideas about the kinematics of a spatial beam and about spatial rotations; in the third section we describe the co-rotational static framework which can easily accept any type of static beam with two nodes and six degrees of freedom per node; the fourth section gives the co-rotational inertial force vector and the dynamic equations for a beam structure, which are discretised in the same section, for the case of end-point algorithms, and in the fifth section, for the case of mid-point algorithms. Finally, some numerical examples compare the different performances of the different integration algorithms.

Communicated by S. N. Atluri, 20 May 1997

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2 Kinematics

For the sake of simplicity the undeformed line of the centroids of the cross-sections is assumed to be a straight line parametrized by a co-ordinate x. A local reference (body attached frame) is associated to any cross-section (see Fig. 1) of which the first unit vector, $\mathbf{i}_1$ is normal to the cross-section and therefore is aligned to the axis of the beam in the undeformed configuration, and the two other unit vectors, $\mathbf{i}_2$ and $\mathbf{i}_3$ are directed along the principal axes of inertia of the cross-section. The shape of the cross-section is constant along the axis of the beam.

In the deformed configuration the line of the centroids is a three dimensional curve described by the relation:

$$\mathbf{r}(x) = \mathbf{r}_0(x) + \mathbf{d}(x) \tag{1}$$

where $\mathbf{r}_0(x)$ is the initial position of the point indicated by the co-ordinate x and $\mathbf{d}(x)$ is the translation of the same point (see Fig. 1).

The shape and the area of the cross-sections do not change after the deformation and the cross-sections remain planar but not necessarily orthogonal to the centroidal line. The orientation of the local reference $\mathbf{i}_i (i = 1, 2, 3)$ in the deformed configuration with respect to an inertial reference system $\mathbf{I}_i (i = 1, 2, 3)$ is defined by an orthogonal transformation $\mathbf{U}$ as:

$$\mathbf{i}_i(x) = \mathbf{U}(x)\mathbf{I}_i \quad (i = 1, 2, 3) \tag{2}$$

We will refer to the matrix $\mathbf{U}$ as a 'rotation matrix' and it possesses the well known properties:

$$\det \mathbf{U} = 1; \quad \mathbf{U}^{-1} = \mathbf{U}^t \tag{3}$$

The deformed configuration of the beam is then fully defined by the position vector of the deformed centroidal line, $\mathbf{r}(x)$ and by the orientation of the local reference with respect to the global reference.

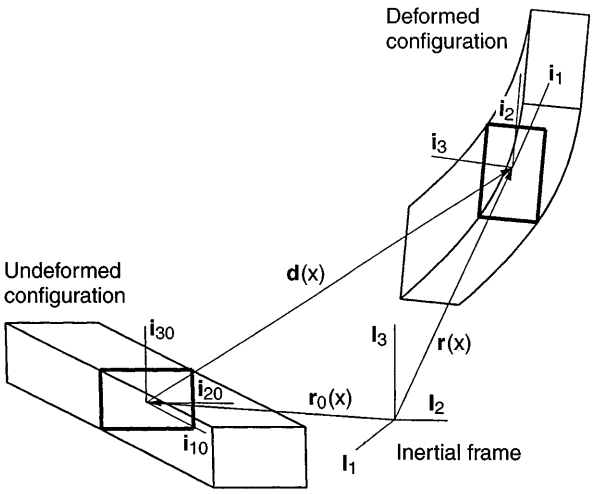

Fig. 1. Beam geometry

It is well known that rotation matrices can be parametrized using only three independent parameters (Crisfield 1997). One way to do so is by introducing the rotational pseudo-vector:

$$\boldsymbol{\psi} = \psi_1 \mathbf{I}_1 + \psi_2 \mathbf{I}_2 + \psi_3 \mathbf{I}_3 = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \quad \psi = |\boldsymbol{\psi}| = [\boldsymbol{\psi}^t \boldsymbol{\psi}]^{1/2} \tag{4}$$

The rotation matrix $\mathbf{U}$, as a function of $\boldsymbol{\psi}$, is then given by:

$$\begin{aligned} \mathbf{U}(\boldsymbol{\psi}) &= \mathbf{I} + \frac{\sin\psi}{\psi}\mathbf{S}(\boldsymbol{\psi}) + \frac{1-\cos\psi}{\psi^2}\mathbf{S}(\boldsymbol{\psi})\mathbf{S}(\boldsymbol{\psi}) \\ &= \exp \mathbf{S}(\boldsymbol{\psi}) \end{aligned} \tag{5}$$

where $\mathbf{I}$ is the identity matrix, and $\mathbf{S}(\boldsymbol{\psi})$ is the skew-symmetric matrix given by:

$$\mathbf{S}(\boldsymbol{\psi}) = \begin{pmatrix} 0 & -\psi_3 & \psi_2 \\ \psi_3 & 0 & -\psi_1 \\ -\psi_2 & \psi_1 & 0 \end{pmatrix} \tag{6}$$

It is also well known (Crisfield 1997) that:

$$\delta\mathbf{U} = \mathbf{S}(\delta\boldsymbol{\vartheta})\mathbf{U} \tag{7}$$

where $\delta\mathbf{U}$ is the variation (infinitesimal change) of the rotation matrix $\mathbf{U}$. Note that $\delta\boldsymbol{\vartheta}$ in (7) is the infinitesimal rotation superposed onto the rotation defined by $\mathbf{U}$ and not the variation of the rotational pseudovector $\boldsymbol{\psi}$. By varying (5) and comparing the result with (7) the following relationship is obtained

$$\delta(\exp \mathbf{S}(\boldsymbol{\psi})) = \mathbf{S}(\delta\boldsymbol{\vartheta})\mathbf{U} \tag{8}$$

which after some algebraic manipulations reduces to (Crisfield 1997, Simo and Vu-Quoc 1988)

$$\delta\boldsymbol{\vartheta} = \mathbf{H}(\boldsymbol{\psi})\delta\boldsymbol{\psi} \tag{9}$$

with

$$\mathbf{H}(\boldsymbol{\psi}) = \frac{1}{\psi^2}\left(1 - \frac{\sin\psi}{\psi}\right)\boldsymbol{\psi}\boldsymbol{\psi}^t + \frac{\sin\psi}{\psi}\mathbf{I} + \frac{1-\cos\psi}{\psi^2}\mathbf{S}(\boldsymbol{\psi}) \tag{10}$$

To avoid confusion, $\delta\boldsymbol{\vartheta}$ is sometimes referred to as a "spin variable" and $\delta\boldsymbol{\psi}$ as an "additive infinitesimal rotational parameter" (Crisfield 1997). See Simo and Vu-Quoc (1986) and Ibrahimbegović, Frey and Kožar (1995) for the typical static formulations based on spin variables or additive infinitesimal rotational parameters, respectively. Further discussion on the relationship between the two can be found in Cardona and Géradin (1988) and Iura and Atluri (1989), while the comparative analysis of the two formulations based on the interpolation of different variables as applied to the dynamics of isoparametric beam elements is given in Jelenić and Crisfield (1997). It is also worth noting that while the additive infinitesimal rotational parameter $\delta\boldsymbol{\psi}$ is the variation of the rotational pseudovector $\boldsymbol{\psi}$, the spin variable $\delta\boldsymbol{\vartheta}$ cannot be understood as a "variation of $\boldsymbol{\vartheta}$", because $\boldsymbol{\vartheta}$ is not defined at all. The formulation presented in this work will be based on the spin variables.

It should also be noted that for any two vectors $\mathbf{v}$ and $\mathbf{w}$:

$$\mathbf{v} \times \mathbf{w} = \mathbf{S}(\mathbf{v})\mathbf{w} = -\mathbf{S}(\mathbf{w})\mathbf{v} = -\mathbf{w} \times \mathbf{v} \tag{11}$$

where $\times$ indicates the cross product between vectors.

In the co-rotational approach the static and the dynamic description are generally reduced to nodal quantities. In particular the deformed configuration in our co-rotational approach is fully defined by the position vectors and the local triads of the two nodes of a finite element.

3 Statics

In the present paper the static behaviour of spatial beams is described via a co-rotational formulation in which the co-rotational technique is taken outside the algorithm for the element computations (see Crisfield 1997, Rankin and Brogan 1986). In this way the method should be applicable to any beam element which has two nodes and six degrees of freedom at each node. We assume that the 'internal element behaviour' is linear whereas all of the non-linearity is introduced via the co-rotational technique. We define as *local* displacements the displacement components which induce any state of deformation in the beam. They are expressed in a local element reference system (to be defined later) as:

$$\mathbf{p}_l^t = (\mathbf{d}_{l1}^t, \boldsymbol{\theta}_{l1}^t, \mathbf{d}_{l2}^t, \boldsymbol{\theta}_{l2}^t) \tag{12}$$

where

$$\mathbf{d}_{l1}^t = (u_{l1}, v_{l1}, w_{l1}) \tag{13}$$

$$\boldsymbol{\theta}_{l1}^t = (\theta_{l1,1}, \theta_{l1,2}, \theta_{l1,3}) \tag{14}$$

The first axis of the local element reference system lies along the element between nodes 1 and 2 so that its unit vector $\mathbf{u}_{e1}$ is defined as:

$$\mathbf{u}_{e1} = (\mathbf{r}_{0,21} + \mathbf{d}_{21})/l_n \tag{15}$$

where $\mathbf{r}_{0,21}$ is the difference $\mathbf{r}_{0,2} - \mathbf{r}_{0,1}$ between the initial co-ordinates of the nodes, $\mathbf{d}_{21}$ is the difference $\mathbf{d}_2 - \mathbf{d}_1$ between the current translational displacements of the

nodes and l_n is the current length of the element. The origin of the local element co-rotating frame is chosen to coincide with node 1 and as its axis 1 passes through node 2, we have:

$$\mathbf{d}_{l1} = \mathbf{0}$$
$$\mathbf{d}_{l2}^t = (u_l, 0, 0) \tag{16}$$

where u_l completely defines the axial deformation according to the following equation:

$$u_l = l_n - l = ((\mathbf{r}_{0,21} + \mathbf{d}_{21})^t(\mathbf{r}_{0,21} + \mathbf{d}_{21}))^{1/2} - (\mathbf{r}_{0,21}^t\mathbf{r}_{0,21})^{1/2} \tag{17}$$

where $l = l_0$ is the initial length of the element. Figure 2 shows a deformed element with two nodal triads $\mathbf{U}_1$ and $\mathbf{U}_2$, and the element triad $\mathbf{U}_e$. $\mathbf{U}_1$ and $\mathbf{U}_2$ are known from the initial conditions or from the current solution of the problem. In the present paper the triad $\mathbf{U}_e$ is defined as follows: let $\boldsymbol{\gamma}$ be the pseudo-vector associated with the rotation from $\mathbf{U}_1$ to $\mathbf{U}_2$. Since $\boldsymbol{\gamma}$ will be only moderately large, the rotation matrix:

$$\mathbf{R}\left(\frac{\boldsymbol{\gamma}}{2}\right) = \mathbf{I} + \frac{\sin(\gamma/2)}{\gamma/2}\mathbf{S}(\gamma/2) + \frac{1-\cos(\gamma/2)}{(\gamma/2)^2} \times \mathbf{S}(\gamma/2)\mathbf{S}(\gamma/2) \tag{18}$$

can be considered as a reasonable representation of the rotation from $\mathbf{U}_1$ to a 'mean configuration' triad $\mathbf{R}_m$:

$$\mathbf{R}_m = \mathbf{R}\left(\frac{\boldsymbol{\gamma}}{2}\right)\mathbf{U}_1 \tag{19}$$

The 'mean configuration' triad $\mathbf{R}_m$ is then rotated in such a way that its first vector $\mathbf{r}_{m1}$ coincides with $\mathbf{u}_{e1}$. In this manner it is possible to define $\mathbf{u}_{e2}$ and $\mathbf{u}_{e3}$ as:

$$\mathbf{u}_{e2} = \mathbf{r}_{m2} - \frac{\mathbf{r}_{m2}^t\mathbf{u}_{e1}}{2}(\mathbf{u}_{e1} + \mathbf{r}_{m1}) \tag{20}$$

$$\mathbf{u}_{e3} = \mathbf{r}_{m3} - \frac{\mathbf{r}_{m3}^t\mathbf{u}_{e1}}{2}(\mathbf{u}_{e1} + \mathbf{r}_{m1}) \tag{21}$$

Note that such a local element reference frame is only approximately orthogonal (more details can be found in chapter 17 of Crisfield 1997 or in Crisfield 1990). An alternative way to define $\mathbf{U}_e$ is given in Nour-Omid and Rankin (1991) or in chapter 18 of Crisfield (1997). In the case of small strains the local rotations can be defined as (chapter 17 of Crisfield 1997):

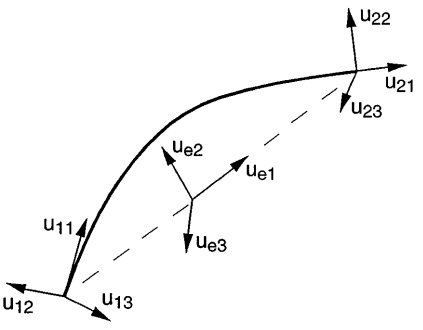

Fig. 2. Two noded deformed beam element

$$2\theta_{l1,1} = -\mathbf{u}_{13}^t\mathbf{u}_{e2} + \mathbf{u}_{12}^t\mathbf{u}_{e3}$$
$$2\theta_{l1,2} = -\mathbf{u}_{11}^t\mathbf{u}_{e3} + \mathbf{u}_{13}^t\mathbf{u}_{e1}$$
$$2\theta_{l1,3} = -\mathbf{u}_{12}^t\mathbf{u}_{e1} + \mathbf{u}_{11}^t\mathbf{u}_{e2}$$
$$2\theta_{l2,1} = -\mathbf{u}_{23}^t\mathbf{u}_{e2} + \mathbf{u}_{22}^t\mathbf{u}_{e3}$$
$$2\theta_{l2,2} = -\mathbf{u}_{21}^t\mathbf{u}_{e3} + \mathbf{u}_{23}^t\mathbf{u}_{e1}$$
$$2\theta_{l2,3} = -\mathbf{u}_{22}^t\mathbf{u}_{e1} + \mathbf{u}_{21}^t\mathbf{u}_{e2} \tag{22}$$

where $\mathbf{u}_{ij}$, $j = 1, 2, 3$ are the components of the rotation matrix $\mathbf{U}_i$, $i = 1, 2$ and $\mathbf{u}_{ej}$ those of $\mathbf{U}_e$. Once we have defined the local displacements, the application of the element-independent co-rotational procedure consists of the following points:

1) The local displacements $\mathbf{p}_l$ are computed from the global nodal displacements $\mathbf{p}$ using relations (17) and (22). Their (infinitesimal) changes are obtained in the way which is standard in the co-rotational applications:

$$\delta\mathbf{p}_l = \mathbf{T}\delta\mathbf{p} \tag{23}$$

where the matrix $\mathbf{T}$ can be derived by differentiation of relations (17) and (22). Details are given in Crisfield (1997) and Crisfield (1990).

2) Since the *local* element is linear, the *local* nodal generalized forces $\mathbf{q}_{il}$ are computed from the local displacements with the standard relation:

$$\mathbf{q}_{il} = \mathbf{K}_l\mathbf{p}_l \tag{24}$$

where $\mathbf{K}_l$ is the usual linear stiffness matrix.

3) The *global* nodal generalized forces $\mathbf{q}_i$ can be expressed by means of the corresponding local quantities in the following way:

$$\mathbf{q}_i = \mathbf{T}^t\mathbf{q}_{il} = \mathbf{T}^t\mathbf{K}_l\mathbf{p}_l \tag{25}$$

This expression can be derived by equating the virtual work in the local and in the global systems: $\mathbf{q}_{il}^t\delta\mathbf{p}_{lv} = \mathbf{q}_i^t\delta\mathbf{p}_v$, employing (23) and noting the equivalence between the infinitesimal changes $\delta\mathbf{p}_l$ and the virtual changes $\delta\mathbf{p}_{lv}$.

4) The tangent stiffness equation can be finally written as:

$$\delta\mathbf{q}_i = \mathbf{T}^t\delta\mathbf{q}_{il} + \delta\mathbf{T}^t\mathbf{q}_{il} = \mathbf{T}^t\mathbf{K}_l\mathbf{T}\delta\mathbf{p} + \mathbf{K}_{t\sigma}(\mathbf{q}_{il})\delta\mathbf{p} = (\mathbf{T}^t\mathbf{K}_l\mathbf{T} + \mathbf{K}_{t\sigma}(\mathbf{q}_{il}))\delta\mathbf{p} = \mathbf{K}_t\delta\mathbf{p} \tag{26}$$

where $\mathbf{T}^t\mathbf{K}_l\mathbf{T}$ is generally called material stiffness matrix, $\mathbf{K}_{t\sigma}$, the 'geometric' stiffness matrix, is a function of $\mathbf{q}_{il}$ and $\mathbf{K}_t$ is the static tangent stiffness matrix. The full expressions for such matrices are given in Crisfield (1990) and in chapter 17 of Crisfield (1997) for a particular co-rotational formulation. The subsequent work on dynamics could equally be applied to alternative co-rotational formulations such as that in Nour-Omid and Rankin (1991) since we are not here concerned with the detail of $\mathbf{T}$ or its variation.

4
End-point algorithm

The strong form of the equations of motion assume the following form (Simo 1985):

$$\dot{\mathbf{k}} = \mathbf{n} + (\mathbf{UN})' \tag{27}$$

$$\dot{\boldsymbol{\pi}} = \mathbf{m} + \mathbf{r}' \times (\mathbf{UN}) + (\mathbf{UM})' \tag{28}$$

where a dot indicates the derivative with respect to time, a prime $'$ indicates the derivative with respect to the axial coordinate, $\mathbf{k}$ is the linear momentum per unit of length, $\boldsymbol{\pi}$ the angular momentum per unit of length with respect to the centroid of the cross-section, $\mathbf{n}$ and $\mathbf{m}$ are external force and moment resultants per unit of length, $\mathbf{N}$ and $\mathbf{M}$ are the vectors of internal force and moment resultants, $\mathbf{U}$ and $\mathbf{r}$ are previously defined rotation matrix and position vector. $\mathbf{k}$ and $\boldsymbol{\pi}$ are defined by:

$$\mathbf{k} = A\rho\dot{\mathbf{d}} \tag{29}$$

$$\boldsymbol{\pi} = \mathbf{U}\mathbf{I}_\rho\mathbf{U}^t\dot{\boldsymbol{\vartheta}} \tag{30}$$

where A is the area of the cross-section of the beam, ρ the mass density, $\dot{\mathbf{d}}$ the velocity of the center of mass of the section, $\mathbf{I}_\rho$ the mass moment of inertia tensor, defined as:

$$\mathbf{I}_\rho = \rho \begin{pmatrix} J_1 + J_2 & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_2 \end{pmatrix} \tag{31}$$

where J_1 and J_2 are the principal moments of inertia of the cross-section and $\dot{\boldsymbol{\vartheta}}$ is the angular velocity, the components of which are expressed in the inertial frame

$$\dot{\boldsymbol{\vartheta}} = \frac{\mathrm{d}\boldsymbol{\vartheta}}{\mathrm{d}t} \tag{32}$$

where the differential $\mathrm{d}\boldsymbol{\vartheta}$ is related to the differential $\mathrm{d}\mathbf{U}$ via $\mathrm{d}\mathbf{U} = \mathbf{S}(\mathrm{d}\boldsymbol{\vartheta})\mathbf{U}$ (see also Eq. (7)). Further to the discussion in Sect. 2, it must be noted that $\dot{\boldsymbol{\vartheta}} \neq \dot{\boldsymbol{\psi}}$, i.e. the angular velocity is not equal to the time derivative of the rotational pseudovector extracted from the rotation matrix $\mathbf{U}$.

The definition of the problem is completed by the relevant boundary and initial conditions. The conventional finite element approach to solve the problem defined by Eqs. (27)–(28) is based on the transformation of the problem into its weak form, transformation which is usually achieved by means of the virtual work principle. It is therefore necessary to introduce some test functions, the virtual displacements, which multiply the terms of Eqs. (27)–(28) and to integrate such a product over the length of the rod. The weak form of the equilibrium obtained in this way is established at a particular point t_n in time.

Restricting our attention to the dynamic terms, we obtain the expression (Simo and Vu-Quoc 1988):

$$\int_0^l (\delta\mathbf{d}_v^t, \delta\boldsymbol{\vartheta}_v^t) \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} A\rho\dot{\mathbf{d}} \\ \mathbf{U}\mathbf{I}_\rho\mathbf{U}^t\dot{\boldsymbol{\vartheta}} \end{pmatrix} \mathrm{d}x = \int_0^l (\delta\mathbf{d}_v^t, \delta\boldsymbol{\vartheta}_v^t) \begin{pmatrix} A\rho\ddot{\mathbf{d}} \\ \dot{\boldsymbol{\vartheta}} \times \mathbf{U}\mathbf{I}_\rho\mathbf{U}^t\dot{\boldsymbol{\vartheta}} + \mathbf{U}\mathbf{I}_\rho\mathbf{U}^t\ddot{\boldsymbol{\vartheta}} \end{pmatrix} \mathrm{d}x \tag{33}$$

with $\ddot{\boldsymbol{\vartheta}} = \frac{\mathrm{d}}{\mathrm{d}t}\dot{\boldsymbol{\vartheta}}$. At this point we have to introduce the interpolation functions for the fields $\delta\mathbf{d}_v, \delta\boldsymbol{\vartheta}_v$. We take into consideration beam elements with two nodes and therefore we obtain:

$$\begin{pmatrix} \delta\mathbf{d}_v \\ \delta\boldsymbol{\vartheta}_v \end{pmatrix} = \begin{pmatrix} N_1(x)\mathbf{I} & \mathbf{0} & N_2(x)\mathbf{I} & \mathbf{0} \\ \mathbf{0} & N_1(x)\mathbf{I} & \mathbf{0} & N_2(x)\mathbf{I} \end{pmatrix} \times \begin{pmatrix} \delta\mathbf{d}_{1v} \\ \delta\boldsymbol{\vartheta}_{1v} \\ \delta\mathbf{d}_{2v} \\ \delta\boldsymbol{\vartheta}_{2v} \end{pmatrix} \tag{34}$$

where $N_1(x)$ and $N_2(x)$ are the shape functions of the two nodes and $\delta\mathbf{d}_{iv}, \delta\boldsymbol{\vartheta}_{iv}$ are the virtual nodal displacements.

It will be noted that in (34), we have provided shape functions to interpolate global quantities. This concept departs from the usual approach adopted for static corotational formulations in which local quantities are interpolated (different orders of polynomials could thus be used for interpolation of different displacement and rotation components). The latter avenue can lead to considerable difficulties (Laskin et al. 1983, Levinson and Kane 1981, Fraeijs de Veubeke 1976). These issues will be discussed further when we address the issue of the mass matrix.

After inserting (34) into (33) and dropping the nodal virtual displacements, we obtain the inertial load vector $\mathbf{q}_{\mathrm{mas}}$ as:

$$\mathbf{q}_{\mathrm{mas}} = \int_0^1 \begin{pmatrix} N_1(x)\mathbf{I} & \mathbf{0} \\ \mathbf{0} & N_1(x)\mathbf{I} \\ N_2(x)\mathbf{I} & \mathbf{0} \\ \mathbf{0} & N_2(x)\mathbf{I} \end{pmatrix} \times \begin{pmatrix} A_\rho\ddot{\mathbf{d}} \\ \dot{\boldsymbol{\vartheta}} \times \mathbf{U}\mathbf{I}_\rho\mathbf{U}^t\dot{\boldsymbol{\vartheta}} + \mathbf{U}\mathbf{I}_\rho\mathbf{U}^t\ddot{\boldsymbol{\vartheta}} \end{pmatrix} \mathrm{d}x \tag{35}$$

The term $\dot{\boldsymbol{\vartheta}} \times \mathbf{U}\mathbf{I}_\rho\mathbf{U}^T\dot{\boldsymbol{\vartheta}}$ is called gyroscopic term and does not appear in the 2-D case. Equation (35) defines the inertial force vector in the case of end-point algorithms.

Standard algebraic manipulations transform Eq. (35) into:

$$\mathbf{q}_{\mathrm{mas}} = \int_0^1 \begin{pmatrix} N_1 A\rho\ddot{\mathbf{d}} \\ N_1\mathbf{U}(\mathbf{S}(\mathbf{w})\mathbf{I}_\rho\mathbf{w} + \mathbf{I}_\rho\mathbf{A}) \\ N_2 A\rho\ddot{\mathbf{d}} \\ N_2\mathbf{U}(\mathbf{S}(\mathbf{w})\mathbf{I}_\rho\mathbf{w} + \mathbf{I}_\rho\mathbf{A}) \end{pmatrix} \mathrm{d}x \tag{36}$$

where $\mathbf{w}$ is the vector of angular velocities in the body-attached frame which is given by Crisfield (1997) or Simo (1985):

$$\mathbf{w} = \mathbf{U}^t\dot{\boldsymbol{\vartheta}} \tag{37}$$

Note that the angular velocity with the components being given in the inertial frame $\dot{\boldsymbol{\vartheta}}$ can be related to the time derivative of the rotational pseudovector $\boldsymbol{\psi}$ (see Eq. (5)) by using (9) and (10) as

$$\dot{\boldsymbol{\vartheta}} = \mathbf{H}(\boldsymbol{\psi})\dot{\boldsymbol{\psi}} \tag{38}$$

as also noted by Atluri and Cazzani (1995). $\mathbf{S}(\mathbf{w})$ is a (3×3) skew symmetric matrix defined by

$$\mathbf{S}(\mathbf{w}) = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} \tag{39}$$

where w_i are the components of the vector $\mathbf{w}$ and $\mathbf{A}$ is the vector of angular accelerations in the body attached frame:

$$\mathbf{A} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{w} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{U}^t\dot{\boldsymbol{\vartheta}}) = \frac{1}{\mathrm{d}t}(\mathrm{d}\mathbf{U}^t\dot{\boldsymbol{\vartheta}} + \mathbf{U}^t\mathrm{d}\dot{\boldsymbol{\vartheta}})$$
$$= \frac{1}{\mathrm{d}t}(\mathbf{U}^t\mathbf{S}^t(\mathrm{d}\boldsymbol{\vartheta})\dot{\boldsymbol{\vartheta}} + \mathbf{U}^t\mathrm{d}\dot{\boldsymbol{\vartheta}}) = \mathbf{U}^t\mathbf{S}^t(\dot{\boldsymbol{\vartheta}})\dot{\boldsymbol{\vartheta}} + \mathbf{U}^t\ddot{\boldsymbol{\vartheta}} = \mathbf{U}^t\ddot{\boldsymbol{\vartheta}} \tag{40}$$

In the co-rotational approach all the quantities in the integral (36) have to be expressed as functions of nodal variables. Therefore to be consistent with the co-rotational technique, we assume that the matrix $\mathbf{U}$ is constant along the element and given by $\mathbf{U}_e$, the element triad shown in Fig. 2. In this way the inertial vector becomes:

$$\mathbf{q}_{\mathrm{mas}} = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_e & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{U}_e \end{pmatrix} \int_0^l \begin{pmatrix} N_1 A\rho\ddot{\mathbf{d}} \\ N_1(\mathbf{S}(\mathbf{w})\mathbf{I}_\rho\mathbf{w} + \mathbf{I}_\rho\mathbf{A}) \\ N_2 A\rho\ddot{\mathbf{d}} \\ N_2(\mathbf{S}(\mathbf{w})\mathbf{I}_\rho\mathbf{w} + \mathbf{I}_\rho\mathbf{A}) \end{pmatrix} \mathrm{d}x \tag{41}$$

Introducing the notation:

$$[\mathbf{U}_e] = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_e & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{U}_e \end{pmatrix} \tag{42}$$

and the space discretisation of the variables $\mathbf{d}$, $\mathbf{w}$ and $\mathbf{A}$ as:

$$\mathbf{d} = N_1\mathbf{d}_1 + N_2\mathbf{d}_2$$
$$\mathbf{w} = N_1\mathbf{w}_1 + N_2\mathbf{w}_2$$
$$\mathbf{A} = N_1\mathbf{A}_1 + N_2\mathbf{A}_2 \tag{43}$$

where

$$N_1 = (l - x)/l$$
$$N_2 = x/l \tag{44}$$

we finally obtain, after some lengthy but straightforward manipulations, the inertial vector as:

$$\mathbf{q}_{\mathrm{mas}} = [\mathbf{U}_e]\left(\mathbf{M}\ddot{\tilde{\mathbf{p}}} + \tfrac{l}{12}\begin{pmatrix} \mathbf{0} \\ 3\mathbf{S}(\mathbf{w}_1)\mathbf{I}_\rho\mathbf{w}_1 + \mathbf{S}(\mathbf{w}_1)\mathbf{I}_\rho\mathbf{w}_2 + \mathbf{S}(\mathbf{w}_2)\mathbf{I}_\rho\mathbf{w}_1 + \mathbf{S}(\mathbf{w}_2)\mathbf{I}_\rho\mathbf{w}_2 \\ \mathbf{0} \\ \mathbf{S}(\mathbf{w}_1)\mathbf{I}_\rho\mathbf{w}_1 + \mathbf{S}(\mathbf{w}_1)\mathbf{I}_\rho\mathbf{w}_2 + \mathbf{S}(\mathbf{w}_2)\mathbf{I}_\rho\mathbf{w}_1 + 3\mathbf{S}(\mathbf{w}_2)\mathbf{I}_\rho\mathbf{w}_2 \end{pmatrix} \right) = [\mathbf{U}_e]\mathbf{f}_{\mathrm{in}} \tag{45}$$

where $\mathbf{M}$ is the standard consistent mass matrix of the linear isoparametric formulation for two noded elements with linear shape functions:

$$\mathbf{M} = \begin{pmatrix} 2\mathbf{M}_t & \mathbf{0} & \mathbf{M}_t & \mathbf{0} \\ \mathbf{0} & 2\mathbf{M}_r & \mathbf{0} & \mathbf{M}_r \\ \mathbf{M}_t & \mathbf{0} & 2\mathbf{M}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_r & \mathbf{0} & 2\mathbf{M}_r \end{pmatrix} \tag{46}$$

where

$$\mathbf{M}_t = \begin{pmatrix} A\rho\frac{l}{6} & 0 & 0 \\ 0 & A\rho\frac{l}{6} & 0 \\ 0 & 0 & A\rho\frac{l}{6} \end{pmatrix} \tag{47}$$

$$\mathbf{M}_r = \frac{l}{6}\mathbf{I}_\rho \tag{48}$$

and l is the length of the beam. The vector $\ddot{\tilde{\mathbf{p}}}$ in (45) is the vector of nodal accelerations; the translational components are defined in the global reference system whereas the rotational ones are given in the body attached frame.

$$\ddot{\tilde{\mathbf{p}}}^t = (\ddot{\mathbf{d}}_1^t, \mathbf{A}_1^t, \ddot{\mathbf{d}}_2^t, \mathbf{A}_2^t) \tag{49}$$

It will be noted that the present formulation has led to the inclusion of terms relating to the conventional mass matrix for a linear Timoshenko formulation. As noted earlier in this section, where the shape functions were defined, there could be considerable difficulties in attempting to incorporate different order shape functions related to an interpolation with respect to a rotating local system. Hence we will here follow an earlier 2-D work (Galvanetto and Crisfield 1996) and will adopt the formulation of (45) (which incorporates the linear Timoshenko mass matrix) even when Bernoulli or engineering formulations are being used for the static terms (details later). However, even with a Timoshenko formulation for the static terms we cannot generally claim a fully consistent formulation, because some quantities are locally and some other globally interpolated (though in 2-D it can be shown that both the interpolations of local and global rotations lead to the same mass matrix). It would seem that Quadrelli and Atluri (1996) also apply such an approach in an alternative co-rotational dynamic formulation.

The discretised dynamic equilibrium equation is given by:

$$\mathbf{q}_{\mathrm{mas},n+1} + \mathbf{q}_{i,n+1} - \mathbf{q}_{e,n+1} = \mathbf{0} \tag{50}$$

where $\mathbf{q}_{i,n+1}$ and $\mathbf{q}_{e,n+1}$ are obtained from Eqs. (27) and (28) in the standard way for static problems; in particular, via (25), for the current co-rotational formulation. In order to solve such a system of algebraic non-linear equations with Newton's method, it is necessary to compute the variation of the vectors $\mathbf{q}_{i,n+1}$ and $\mathbf{q}_{\mathrm{mas}}$ to take into account the contribution of internal forces and inertial forces to the 'generalized' tangent stiffness matrix (details on the derivation of $\delta\mathbf{q}_{i,n+1}$, leading to the static stiffness contribution $\mathbf{K}_t$ of (26) are given in Crisfield 1990 and in Crisfield 1997). From (45), we obtain:

where the first term $\delta[\mathbf{U}_e]\mathbf{f}_{\mathrm{in}}$ represents the variation of the inertial forces due to the variation of global displacements

$$\delta\mathbf{q}_{\text{mas}} = \delta[\mathbf{U}_e]\mathbf{f}_{\text{in}} + [\mathbf{U}_e]\mathbf{M}\delta\ddot{\mathbf{p}} + \frac{l}{12}[\mathbf{U}_e]\begin{pmatrix} \mathbf{0} \\ 3\mathbf{S}(\delta\mathbf{w}_1)\mathbf{I}_\rho\mathbf{w}_1 + 3\mathbf{S}(\mathbf{w}_1)\mathbf{I}_\rho\delta\mathbf{w}_1 + \mathbf{S}(\delta\mathbf{w}_1)\mathbf{I}_\rho\mathbf{w}_2 + \mathbf{S}(\mathbf{w}_1)\mathbf{I}_\rho\delta\mathbf{w}_2 \\ \quad +\mathbf{S}(\delta\mathbf{w}_2)\mathbf{I}_\rho\mathbf{w}_1 + \mathbf{S}(\mathbf{w}_2)\mathbf{I}_\rho\delta\mathbf{w}_1 + \mathbf{S}(\delta\mathbf{w}_2)\mathbf{I}_\rho\mathbf{w}_2 + \mathbf{S}(\mathbf{w}_2)\mathbf{I}_\rho\delta\mathbf{w}_2 \\ \mathbf{0} \\ \mathbf{S}(\delta\mathbf{w}_1)\mathbf{I}_\rho\mathbf{w}_1 + \mathbf{S}(\mathbf{w}_1)\mathbf{I}_\rho\delta\mathbf{w}_1 + \mathbf{S}(\delta\mathbf{w}_1)\mathbf{I}_\rho\mathbf{w}_2 + \mathbf{S}(\mathbf{w}_1)\mathbf{I}_\rho\delta\mathbf{w}_2 \\ \quad +\mathbf{S}(\delta\mathbf{w}_2)\mathbf{I}_\rho\mathbf{w}_1 + \mathbf{S}(\mathbf{w}_2)\mathbf{I}_\rho\delta\mathbf{w}_1 + 3\mathbf{S}(\delta\mathbf{w}_2)\mathbf{I}_\rho\mathbf{w}_2 + 3\mathbf{S}(\mathbf{w}_2)\mathbf{I}_\rho\delta\mathbf{w}_2 \end{pmatrix} \tag{51}$$

and global rotations $\delta\mathbf{p}$, the second term $[\mathbf{U}_e]\mathbf{M}\delta\ddot{\mathbf{p}}$ represents the variation of the inertial forces due to the variation of global translational accelerations and body attached rotational accelerations, the third term $\frac{l}{12}[\mathbf{U}_e](\ldots)$ represents the variation of the inertial forces due to the variation of the angular body velocities. We observe that the definition of the three terms of the variation of $\mathbf{q}_{\text{mas}}$ requires the introduction of a finite difference scheme for the time variable. If we adopt Newmark's interpolation relations (Newmark 1959), we have:

$$\dot{\mathbf{d}}_{n+1} = \frac{\gamma}{\beta\Delta t}(\mathbf{d}_{n+1} - \mathbf{d}_n) + \left(1 - \frac{\gamma}{\beta}\right)\dot{\mathbf{d}}_n + \Delta t\left(1 - \frac{\gamma}{2\beta}\right)\ddot{\mathbf{d}}_n \tag{52}$$

$$\ddot{\mathbf{d}}_{n+1} = \frac{1}{\beta\Delta t^2}((\mathbf{d}_{n+1} - \mathbf{d}_n) - \Delta t\dot{\mathbf{d}}_n - \Delta t^2(0.5 - \beta)\ddot{\mathbf{d}}_n) \tag{53}$$

$$\mathbf{w}_{n+1} = \frac{\gamma}{\beta\Delta t}\Delta\Psi + \left(1 - \frac{\gamma}{\beta}\right)\mathbf{w}_n + \Delta t\left(1 - \frac{\gamma}{2\beta}\right)\mathbf{A}_n \tag{54}$$

$$\mathbf{A}_{n+1} = \frac{1}{\beta\Delta t^2}\left(\Delta\boldsymbol{\Psi} - \Delta t\mathbf{w}_n - \Delta t^2\left(\frac{1}{2} - \beta\right)\mathbf{A}_n\right) \tag{55}$$

where $\mathbf{w}_{n+1}, \mathbf{w}_n, \mathbf{A}_{n+1}, \mathbf{A}_n$ are the values at times t_{n+1} and t_n of the variables that have been defined in Eqs. (37), (40) and (43). The vector $\Delta\boldsymbol{\Psi}$ is the rotational pseudovector which spins $\mathbf{U}_n$ into $\mathbf{U}_{n+1}$, expressed in the body co-ordinates (see chapter 24 of Crisfield 1997), and is linked to its global equivalent $\Delta\boldsymbol{\psi}$ by:

$$\Delta\boldsymbol{\Psi} = \mathbf{U}_{n+1}^t\Delta\boldsymbol{\psi} = \mathbf{U}_n^t\Delta\boldsymbol{\psi} \tag{56}$$

Equation (51) can be expressed in the form:

$$\delta\mathbf{q}_{\text{mas}} = \mathbf{K}_{t,\text{mas}}\delta\mathbf{p} = (\mathbf{K}_{\text{mas1}} + \mathbf{K}_{\text{mas2}} + \mathbf{K}_{\text{mas3}})\delta\mathbf{p} \tag{57}$$

where $\delta\mathbf{p}$ is the variation of the global displacement components. The full expression of the matrices $\mathbf{K}_{\text{mas1}}, \mathbf{K}_{\text{mas2}}$ and $\mathbf{K}_{\text{mas3}}$ is given in Appendix 1.

One of the most commonly used time integration schemes is the so-called α-method (Hilbert et al. 1977) that uses the Newmark's interpolation functions but imposes the dynamic equilibrium according to the equation:

$$(1 + \alpha)(\mathbf{q}_{i,n+1} - \mathbf{q}_{e,n+1}) - \alpha(\mathbf{q}_{i,n} - \mathbf{q}_{e,n}) + \mathbf{q}_{\text{mas},n+1} = \mathbf{0} \tag{58}$$

instead of Eq. (50). In this way the inertia terms $\mathbf{q}_{\text{mas},n+1}$ are related to the 'end-point' t_{n+1} but the internal and external static terms are related to some α-point between t_n and t_{n+1}. The 'effective' stiffness matrix in the α-method (58) is therefore obtained from $(1 + \alpha)\delta\mathbf{q}_{i,n+1} + \delta\mathbf{q}_{\text{mas},n+1} = \overline{\mathbf{K}}_{t,n+1}\delta\mathbf{p}$ as:

$$\overline{\mathbf{K}}_{t,n+1} = (1 + \alpha)\mathbf{K}_{t,n+1} + \mathbf{K}_{t,\text{mas}} \tag{59}$$

where $\mathbf{K}_{t,n+1}$ is the static stiffness matrix defined in Sect. 3 (see also Crisfield 1990 and Crisfield 1997). The time integration parameters β and γ are in the α-method evaluated from $\beta = \frac{(1-\alpha)^2}{4}$ and $\gamma = \frac{1}{2} - \alpha$. In a non-linear environment, analysts typically use the scheme with $\alpha = -0.05$ (Hibbitt and Karlsson 1979). When α is set to zero the conventional implicit Newmark method is recovered.

5
Mid-point algorithm

The co-rotational mid point dynamic algorithms were initially introduced in the study of planar trusses (Crisfield and Shi 1994) and planar beams (Galvanetto and Crisfield 1996) to construct energy-conserving algorithms. The work presented in this section is a partial extension of the work presented in these two papers. Velocities and accelerations at the end of the step can be obtained via the following definitions of mid-point velocities and accelerations:

$$\mathbf{v}_m = \frac{\mathbf{v}_{n+1} + \mathbf{v}_n}{2} = \frac{\Delta\mathbf{d}}{\Delta t} = \frac{\mathbf{d}_{n+1} - \mathbf{d}_n}{\Delta t} \tag{60}$$

$$\mathbf{a}_m = \frac{\mathbf{a}_{n+1} + \mathbf{a}_n}{2} = \frac{\Delta\mathbf{v}}{\Delta t} = \frac{\mathbf{v}_{n+1} - \mathbf{v}_n}{\Delta t} \tag{61}$$

$$\mathbf{w}_m = \frac{\mathbf{w}_{n+1} + \mathbf{w}_n}{2} = \frac{\Delta\boldsymbol{\Psi}}{\Delta t} = \frac{1}{\Delta t}\mathbf{U}_n^t\Delta\boldsymbol{\psi} \tag{62}$$

$$\mathbf{A}_m = \frac{\mathbf{A}_{n+1} + \mathbf{A}_n}{2} = \frac{\Delta\mathbf{w}}{\Delta t} = \frac{\mathbf{w}_{n+1} - \mathbf{w}_n}{\Delta t} \tag{63}$$

We observe that Eqs. (60) and (62) coincide with Eqs. (52) and (54) in the case of trapezoidal rule ($\gamma = 0.5, \beta = 0.25$) and that Eqs. (61) and (63) are given only for completeness since they are not needed in the remainder.

The main difference between the end-point algorithm and its mid-point counterpart consists in the fact that, while the former imposes the dynamic equilibrium in a discrete set of temporal instants, the latter effectively equates the change of the total momentum of the system to the impulse of the internal and external forces acting on the system during a time step (see e.g. Simo et al. 1995).

The equilibrium equation which is satisfied by the mid point algorithm is:

$$\mathbf{q}_{\text{mas},m} + \mathbf{q}_{i,m} = \mathbf{q}_{e,m} \tag{64}$$

where $\mathbf{q}_{\text{mas},m}, \mathbf{q}_{i,m}$ and $\mathbf{q}_{e,m}$ are vectors of 'mid-point' inertial, internal and external nodal element forces. The vector of mid-point inertial forces will be derived from the request that its scalar product with the element vector of incremental displacements and rotations be equal to the change of kinetic energy over a time step, i.e.:

$$\Delta K = K_{n+1} - K_n = \mathbf{q}^t_{\text{mas},m}\Delta\mathbf{p} \tag{65}$$

where the kinetic energy at any time is given by

$$K = \frac{1}{2}\int_0^l (\dot{\mathbf{d}}^t A\rho\dot{\mathbf{d}} + \mathbf{w}^t\mathbf{I}_\rho\mathbf{w})\mathrm{d}x \tag{66}$$

By inserting $(43)_1$ and $(43)_2$ into (66) and (66) into (65) we obtain:

$$\begin{aligned}\mathbf{q}^t_{\text{mas},m}\Delta\mathbf{p} &= \tfrac{1}{2}(\dot{\mathbf{p}}^t_{n+1}[\mathbf{U}^*_{e,n+1}]\mathbf{M}[\mathbf{U}^*_{e,n+1}]^t\dot{\mathbf{p}}_{n+1} \\ &\quad - \dot{\mathbf{p}}^t_n[\mathbf{U}^*_{e,n}]\mathbf{M}[\mathbf{U}^*_{e,n}]^t\dot{\mathbf{p}}_n) \\ &= \tfrac{1}{2}([\mathbf{U}^*_{e,n+1}]^t\dot{\mathbf{p}}_{n+1} - [\mathbf{U}^*_{e,n}]^t\dot{\mathbf{p}}_n)^t\mathbf{M} \\ &\quad \times([\mathbf{U}^*_{e,n+1}]^t\dot{\mathbf{p}}_{n+1} + [\mathbf{U}^*_{e,n}]^t\dot{\mathbf{p}}_n)\end{aligned} \tag{67}$$

where $\mathbf{M}$ is the linear mass matrix defined by (46)–(48),

$$[\mathbf{U}^*_{e,n+1}] = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{1,n+1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{U}_{2,n+1} \end{pmatrix} \tag{68}$$

and $[\mathbf{U}^*_{e,n}]$ is defined analogously. By using (60), (62), $(56)_1$ and $(56)_2$, Eq. (67) becomes:

$$\begin{aligned}\mathbf{q}^t_{\text{mas},m}\Delta\mathbf{p} &= \frac{1}{\Delta t}([\mathbf{U}^*_{e,n+1}]\mathbf{M}[\mathbf{U}^*_{e,n+1}]^t\dot{\mathbf{p}}_{n+1} \\ &\quad - [\mathbf{U}^*_{e,n}]\mathbf{M}[\mathbf{U}^*_{e,n}]^t\dot{\mathbf{p}}_n)^t\Delta\mathbf{p}\end{aligned} \tag{69}$$

Therefore the mid-point inertial force vector is defined as:

$$\mathbf{q}_{\text{mas},m} = \frac{1}{\Delta t}([\mathbf{U}^*_{e,n+1}]\mathbf{M}\dot{\tilde{\mathbf{p}}}_{n+1} - [\mathbf{U}^*_{e,n}]\mathbf{M}\dot{\tilde{\mathbf{p}}}_n) \tag{70}$$

where

$$\dot{\tilde{\mathbf{p}}}^t = \dot{\mathbf{p}}^t[\mathbf{U}^*_e] = (\dot{\mathbf{d}}^t_1, \mathbf{w}^t_1, \dot{\mathbf{d}}^t_2, \mathbf{w}^t_2) \tag{71}$$

The nodal internal and external forces acting during the time step are represented by their 'mid-point' values:

$$\mathbf{q}_{e,m} = \frac{\mathbf{q}_{e,n+1} + \mathbf{q}_{e,n}}{2} \tag{72}$$

$$\mathbf{q}_{i,m} = \left(\frac{\mathbf{T}_{n+1} + \mathbf{T}_n}{2}\right)^t\frac{\mathbf{q}_{il,n+1} + \mathbf{q}_{il,n}}{2} \tag{73}$$

where $\mathbf{q}_{e,n}$ and $\mathbf{q}_{e,n+1}$ are the nodal external forces at time n and $n+1$ respectively while $\mathbf{q}_{il,n}$ and $\mathbf{q}_{il,n+1}$ are the corresponding local nodal internal forces, and the matrices $\mathbf{T}_n$ and $\mathbf{T}_{n+1}$ are defined in Sect. 3. Note that these 'mid-point internal forces' are not the average of the internal forces at times t_n and t_{n+1}, nor are they directly computed from stresses devised at a mid-point configuration.

The dynamic equilibrium equation is given by:

$$\begin{aligned}\mathbf{g}^*_m &= \frac{(\mathbf{T}_{n+1} + \mathbf{T}_n)^t}{2}\frac{\mathbf{q}_{il,n+1} + \mathbf{q}_{il,n}}{2} - \mathbf{q}_{e,m} + \mathbf{q}_{\text{mas},m} \\ &= \mathbf{q}_{i,m} - \mathbf{q}_{e,m} + \mathbf{q}_{\text{mas},m} = \mathbf{0}\end{aligned} \tag{74}$$

The inertial contribution to the effective stiffness matrix is obtained from the variation of Eq. (70):

$$\begin{aligned}\delta\mathbf{q}_{\text{mas},m} &= \frac{1}{\Delta t}(\delta[\mathbf{U}^*_{e,n+1}]\mathbf{M}\dot{\tilde{\mathbf{p}}}_{n+1} + [\mathbf{U}^*_{e,n+1}]\mathbf{M}\delta\dot{\tilde{\mathbf{p}}}_{n+1}) \\ &= \mathbf{K}_{t,\text{mas}}\delta\mathbf{p}\end{aligned} \tag{75}$$

Matrix $\mathbf{K}_{t,\text{mas}}$ is derived in Appendix 2.

The static mid-point stiffness matrix $\mathbf{K}_{\text{static}}$ is given by:

$$\begin{aligned}\delta\mathbf{q}_{i,m} &= \mathbf{K}_{\text{static}}\delta\mathbf{p} \\ &= \frac{(\mathbf{T}_{n+1} + \mathbf{T}_n)^t}{2}\frac{\delta\mathbf{q}_{il,n+1}}{2} + \frac{\delta\mathbf{T}^t_{n+1}}{2}\frac{\mathbf{q}_{il,n+1} + \mathbf{q}_{il,n}}{2} \\ &= \frac{(\mathbf{T}_{n+1} + \mathbf{T}_n)^t}{2}\frac{\mathbf{K}_l}{2}\mathbf{T}_{n+1}\delta\mathbf{p} \\ &\quad + \frac{1}{2}\mathbf{K}_{t\sigma}\left(\frac{\mathbf{q}_{il,n+1} + \mathbf{q}_{il,n}}{2}\right)\delta\mathbf{p}\end{aligned} \tag{76}$$

and comes from the variation of Eq. (73) in a way analogous to that followed in Sect. 3. The 'geometric stiffness matrix' takes an identical form to that of the end-point formulation (although now with a factor of 1/2 and with different local internal forces) but the material stiffness matrix now becomes non-symmetric. However, in the dynamics of 3-D beams, even the conventional end-point equilibrium formulation leads to a non-symmetric stiffness matrix (via the mass term), so this does not lead to any new problems. The 'effective' stiffness matrix of the mid-point method is then expressed by the sum of $\mathbf{K}_{\text{static}}$ with the dynamic term $\mathbf{K}_{t,\text{mas}}$.

We observe that the mid-point algorithm is in some way simpler than the end-point algorithm, in fact in the mid-point algorithm there is no gyroscopic term and the linearization on the inertial terms is simpler.

5.1 Approximate energy conservation

The co-rotational mid-point type algorithms were first introduced in order to conserve the full energy of the mechanical system. It is in general necessary to introduce some sophistications to achieve the full energy conservation (Crisfield and Shi 1994, Galvanetto and Crisfield 1996) but it is possible to show that even the plain mid-point algorithm (74) approximately conserves the energy. A first order forward Taylor expansion about $\mathbf{p}_{l,n}$ would give

$$\mathbf{p}_{l,n+1} = \mathbf{p}_{l,n} + \left.\frac{\partial\mathbf{p}_l}{\partial\mathbf{p}}\right|_n\Delta\mathbf{p} = \mathbf{p}_{l,n} + \mathbf{T}_n\Delta\mathbf{p} \tag{77}$$

while a backward expansion about $\mathbf{p}_{l,n+1}$ would give

$$\mathbf{p}_{l,n} = \mathbf{p}_{l,n+1} - \left.\frac{\partial\mathbf{p}_l}{\partial\mathbf{p}}\right|_{n+1}\Delta\mathbf{p} = \mathbf{p}_{l,n+1} - \mathbf{T}_{n+1}\Delta\mathbf{p} \tag{78}$$

A significantly better approximation than either of the above can be obtained by taking their average so that

$$\Delta \mathbf{p}_l = \mathbf{p}_{l,n+1} - \mathbf{p}_{l,n} \simeq \frac{(\mathbf{T}_{n+1} + \mathbf{T}_n)}{2} \Delta \mathbf{p} \tag{79}$$

If we were to assume that the above approximation is identically satisfied, i.e.

$$\Delta \mathbf{p}_l = \frac{(\mathbf{T}_{n+1} + \mathbf{T}_n)}{2} \Delta \mathbf{p} \tag{80}$$

we would write the change in strain energy over the increment as

$$\begin{aligned} \Delta\phi &= \frac{(\mathbf{q}_{il,n+1} + \mathbf{q}_{il,n})^t}{2} \Delta \mathbf{p}_l \\ &= \left(\frac{1}{2} (\mathbf{T}_{n+1} + \mathbf{T}_n)^t \frac{\mathbf{q}_{il,n+1} + \mathbf{q}_{il,n}}{2} \right)^t \Delta \mathbf{p} \end{aligned} \tag{81}$$

while the kinetic energy change is given by (65) and the external potential energy increment, in case of constant external forces, is given by:

$$\Delta P = -\mathbf{q}_e^t \Delta \mathbf{p} \tag{82}$$

Therefore if (80) was to hold, the total change of energy over the increment would be written using (74) as

$$\Delta E_{\text{tot}} = \Delta K + \Delta P + \Delta \phi = \mathbf{g}_m^{*t} \Delta \mathbf{p} \tag{83}$$

and when the numerical integration reaches convergence, so that $\mathbf{g}_m^* \to \mathbf{0}$, the total energy of the system would be conserved. In reality Eq. (80) is only approximately satisfied via (79) so the total energy is only approximately conserved in the sense

$$\Delta E_{\text{tot}} = \Delta K + \Delta P + \Delta \phi \simeq \mathbf{g}_m^{*t} \Delta \mathbf{p} \tag{84}$$

5.2
Numerical damping

In some cases, as we shall see in the examples, the approximate conservation of energy of the mid-point scheme is not sufficient to ensure the numerical stability of the algorithm which undergoes an uncontrolled growth of energy and finally is not able to reach convergence. To avoid such behaviour, it is common to introduce some kind of numerical damping which prevents, or at least limits, the possibility of having uncontrolled energy growth. It is a desirable feature of the numerical damping algorithm to depend on only one parameter and to allow for an easy recovery of the original undamped algorithm. The α-method presented above satisfies these requirements and in case of linear conservative systems makes the total energy continually decrease in time. We will now introduce an alternative damping correction as applied to the mid-point algorithm, which was inspired by Armero and Petőcz (1996), and will be shown to be beneficial to the stability of the numerical integration. In this formulation, in place of (73), the mid-point internal forces are now defined as:

$$\begin{aligned} \mathbf{q}_{im} = &\frac{(\mathbf{T}_{n+1} + \mathbf{T}_n)^t}{2} \frac{\mathbf{q}_{il,n+1} + \mathbf{q}_{il,n}}{2} \\ &+ \xi \frac{(\mathbf{T}_{n+1} + \mathbf{T}_n)^t}{2} \frac{\mathbf{q}_{il,n+1} - \mathbf{q}_{il,n}}{2} \end{aligned} \tag{85}$$

If we now consider the internal product $\mathbf{g}_{mc}^{*t} \Delta \mathbf{p}$, where, in $\mathbf{g}_{mc}^*$, the $\xi \frac{(\mathbf{T}_{n+1}+\mathbf{T}_n)^t}{2} \frac{\mathbf{q}_{il,n+1}-\mathbf{q}_{il,n}}{2}$ term has been added to $\mathbf{g}_m^*$ of (74), we obtain:

$$\mathbf{g}_{mc}^{*t} \Delta \mathbf{p} = \mathbf{g}_m^{*t} \Delta \mathbf{p} + \Delta \mathbf{p}^t \xi \frac{(\mathbf{T}_{n+1} + \mathbf{T}_n)^t}{2} \frac{\mathbf{q}_{il,n+1} - \mathbf{q}_{il,n}}{2} = 0 \tag{86}$$

Therefore, since $\Delta E_{\text{tot}} \simeq \mathbf{g}_m^{*t} \Delta \mathbf{p}$ (see (84)), we have:

$$\Delta E_{\text{tot}} + \Delta \mathbf{p}^t \xi \frac{(\mathbf{T}_{n+1} + \mathbf{T}_n)^t}{2} \frac{\mathbf{q}_{il,n+1} - \mathbf{q}_{il,n}}{2} \simeq 0 \tag{87}$$

from which it is possible to deduce:

$$\Delta E_{\text{tot}} \simeq -\Delta \mathbf{p}^t \xi \frac{(\mathbf{T}_{n+1} + \mathbf{T}_n)^t}{2} \frac{\mathbf{q}_{il,n+1} - \mathbf{q}_{il,n}}{2} \tag{88}$$

It is now necessary to show that the energy change ΔE_{tot} is negative to show that the correction damps the energy of the system. To do so let us consider the term:

$$\begin{aligned} &\Delta \mathbf{p}^t \frac{(\mathbf{T}_{n+1} + \mathbf{T}_n)^t}{2} \frac{\mathbf{q}_{il,n+1} - \mathbf{q}_{il,n}}{2} \\ &\quad \simeq \frac{1}{2} \Delta \mathbf{p}^t \frac{(\mathbf{T}_{n+1} + \mathbf{T}_n)^t}{2} \mathbf{K}_l \frac{\mathbf{T}_{n+1} + \mathbf{T}_n}{2} \Delta \mathbf{p} > 0 \end{aligned} \tag{89}$$

where $\mathbf{K}_l$ is the local stiffness matrix coming from (24) and the approximate equality stems from (79). The middle term in (89) is a quadratic form and therefore is positive, i.e. for any positive value of ξ the right-hand side of (88) is negative. There are several approximations which make our proof 'approximate', but the examples will show that the energy effectively decreases, so that the above mathematical relations can be thought of as a way to better understand the behaviour of the numerical scheme, rather than a formal mathematical demonstration. The parameter ξ, which controls the numerical damping, has to be small in order to avoid excessive energy losses. We also observe that the dissipative properties of the present algorithm do not depend on the linear or non-linear nature of the problem, whereas the dissipative characteristics of the α-method were demonstrated (to the best of our knowledge) for the linear case only.

6
Numerical examples

In this section we compare the performances of four different time integration algorithms: Newmark, α-method, mid-point and mid-point with numerical damping. These four algorithms were introduced in a research version of the LUSAS finite element program equipped with a time step size having facility: if convergence is not reached after a chosen number of iterations (20) the time step will be halved and then gradually increased in consecutive steps by the factor of 1.2 until the original time step size is restored. The numerical integration will be stopped if the time step size is reduced by the halving process to less than 1/10 of the original one. In all the examples the mass matrix is defined by (46).

For the static terms, we will sometimes use a linear Timoshenko formulation and sometimes an 'engineering formulation' (see Sect. 5.6 of Przemieniecki 1968). In the

latter case, as the shear modulus tends to infinity, a Bernoulli beam is recovered. In the current co-rotational formulation the local displacements/rotations are related to the local internal force vector via a constant local stiffness matrix $\mathbf{K}_l$. In these circumstances the engineering element is preferable as it enables a linear change in curvature along the axis of the beam, whereas in the linear Timoshenko element it is constant. On the other hand, the Timoshenko mass matrix is applied in both cases so it is more consistent to use it in conjunction with a Timoshenko formulation for the stiffness part. See also the discussion in Sect. 4.

Example 1

This example was already presented in Simo and Vu-Quoc (1988) and it consists of a right-angle cantilever beam with geometric and material properties shown in Fig. 3. An out-of-plane external force is applied for the first two seconds according to the function shown in the same figure. After the first two seconds, the cantilever beam undergoes free vibrations with the combined presence of bending and torsional modes. Following Simo and Vu-Quoc (1988) the initial step size is chosen as $\Delta t = 0.25$ for all the algorithms and the integration time is set to 2000 seconds. Therefore, if no halving takes place, the whole integration will require 8000 steps. The finite element discretization is composed of ten engineering elements. Figure 4 shows that the Newmark integration experiences a sudden energy growth which eventually prevents the convergence of the

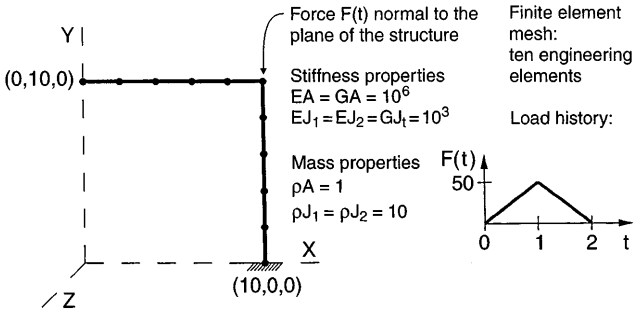

Fig. 3. Example 1

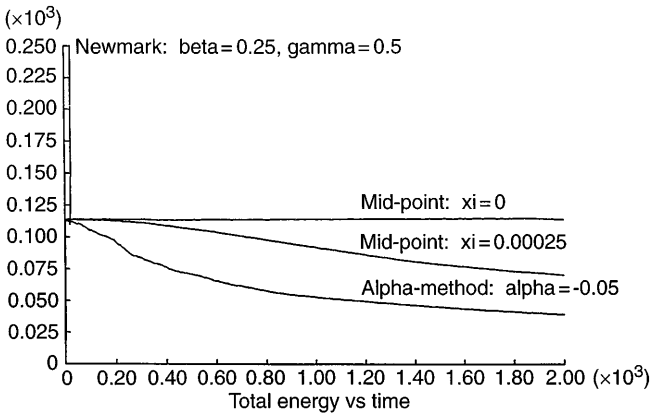

Fig. 4. Example 1, total energy vs time for the analysed algorithms

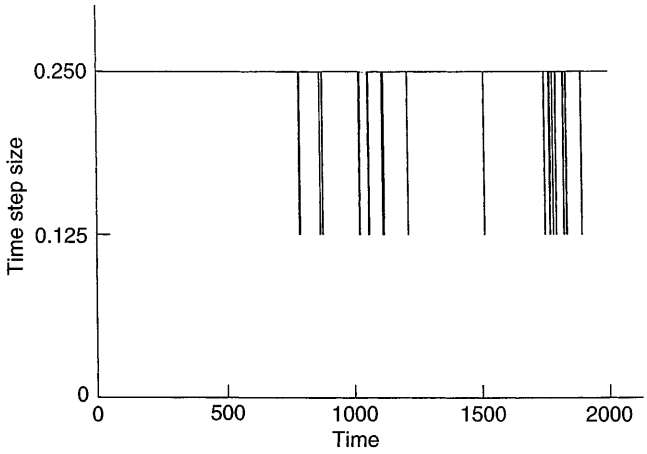

Fig. 5. Example 1, time step halving for the mid-point algorithm

time integration whereas the mid-point algorithm covers the whole integration time with small variations of total energy. However, Fig. 5 indicates that in the latter case there are some significant numerical difficulties in reaching convergence since several time step size halvings are necessary.

Such instabilities/numerical difficulties can be overcome by the introduction of the numerical damping (Fig. 4). Both the α-method and the mid-point algorithm with numerical damping ($\xi = 0.00025$) do not experience any time step cuts, but they cause significant energy losses.

Example 2

In this second example (Jelenić and Crisfield 1997) the free flight of an unrestrained flexible beam is examined. The initial configuration, the geometric and material properties are given in Fig. 6. The motion of the beam is determined by the initial conditions which are also given in the same figure. The initial velocities along the x axis will generate a bending deformation, a translational motion along the x axis and a rotation around the z axis while the initial velocities along the z axis will generate a rotation around an axis which is normal to the centroidal line and to the z axis. The initial time step is $\Delta t = 0.008$ to provide an acceptable time integration of the motion induced by the rigid body

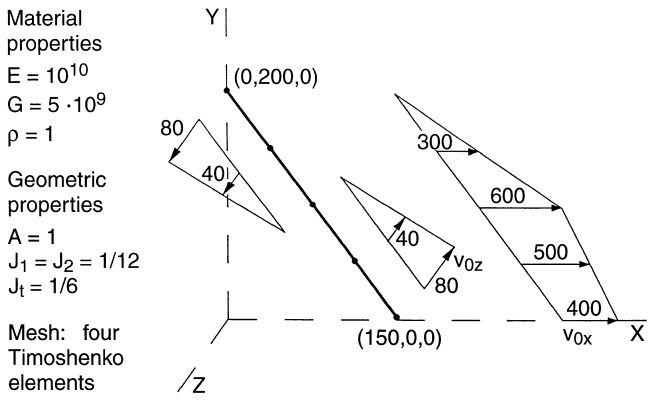

Fig. 6. Example 2

modes and by the six lowest bending modes (Jelenić and Crisfield 1997). The time integration interval is of 200 seconds. The beam is modelled using four equal linear Timoshenko elements. In this case, both the Newmark and the mid-point algorithms are not able to cover the whole 200 seconds whereas the α-method and the mid-point algorithm with numerical damping ($\xi = 0.00025$) are more or less equivalent with similar final energy losses (see Fig. 7).

The marked detail in Fig. 7 is blown-up in Fig. 8 and shows an important difference between the numerical damping of the mid-point scheme and that of the α-method: in fact while the mid-point damped algorithm exhibits a quasi-monotonic decay of energy (recall the approximate character of the proof in Sect. 6.2) the α-method experiences some limited increases which could in some circumstances lead to possible numerical instability. This behaviour has been confirmed by several examples.

Example 3

In the third example we consider the free flight of a beam which is affected by very small strains so that the motion is similar to a rigid body motion. The geometry, initial

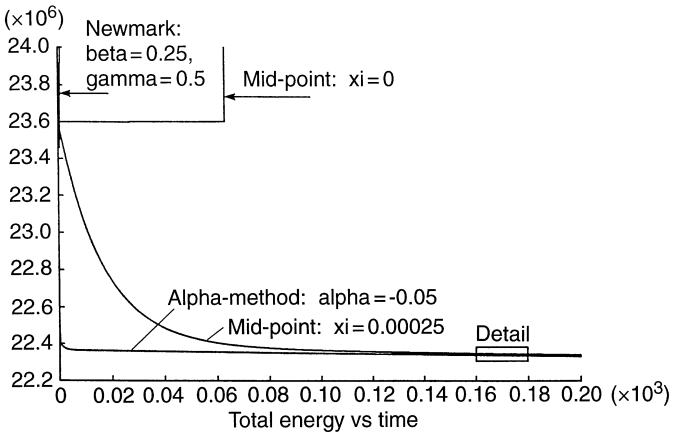

Fig. 7. Example 2, total energy vs time for the analysed algorithms

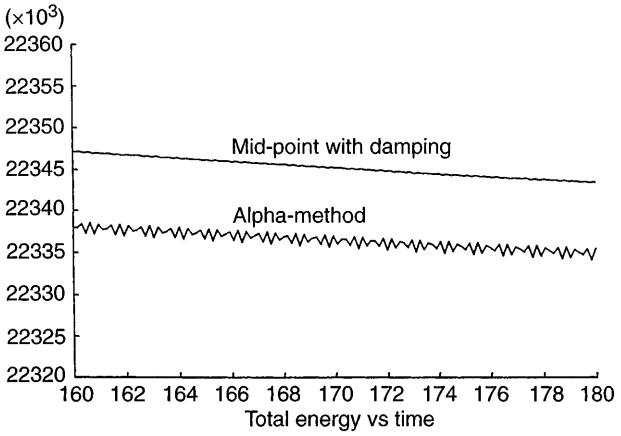

Fig. 8. Example 2, differences in the damping characteristics between the mid-point scheme and the α-method

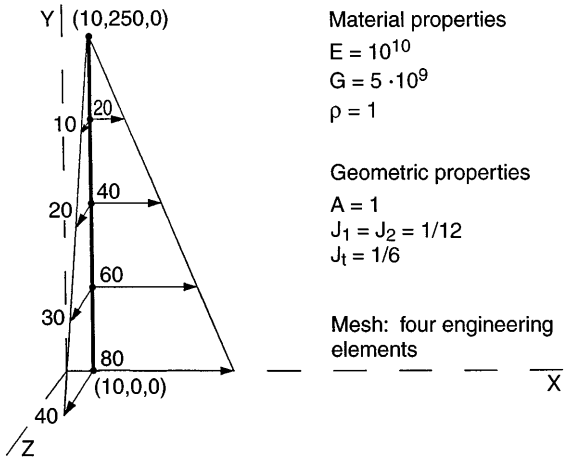

Fig. 9. Example 3

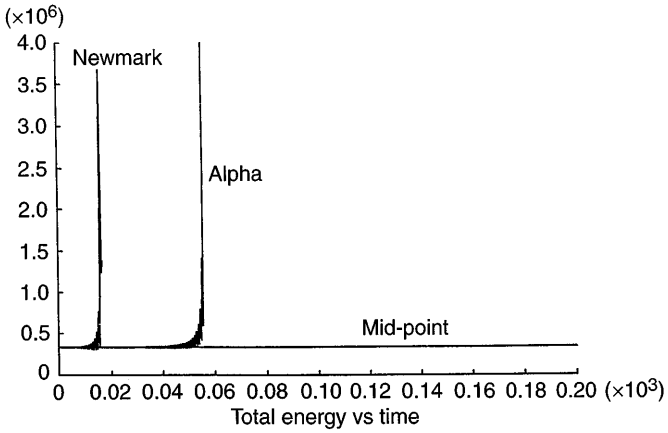

Fig. 10. Example 3, total energy vs time for Newmark, α and mid-point (with no damping) schemes

conditions and mechanical properties are given in Fig. 9. The principal axes of inertia of the cross-section are initially aligned with global co-ordinate axes X and Z. In particular, the initial conditions are given in such a way that only small strains are induced in the beam. The mesh is composed of four engineering elements and the initial time step size is $\Delta t = 0.4$ which gives approximately 44 integration points per revolution of the beam. In Fig. 10 we show that the Newmark and the α-method undergo an uncontrolled energy growth whereas the mid-point algorithm successfully integrates the motion for 200 seconds with no halving of the time step size.

7 Conclusions

Some integration algorithms for the dynamics of three-dimensional co-rotational beams have been presented. They allow the use of any linear static beam element, provided it has two nodes and six degrees of freedom per node. The dynamic term is more delicate and it requires the definition of a 'global' mass matrix which is obtained with linear interpolation functions for both the translational and the rotational displacements. The numerical simulations suggest that the Newmark algorithm is the

least robust among those analysed and that it is greatly improved by the additional numerical damping introduced by the α-method. The mid-point scheme is generally better than the Newmark method and it may also give much better results than the α-method in the case of quasi-rigid body motions. The proposed mid-point algorithm only approximately conserves energy and, in some cases, it can experience uncontrolled energy growth. This can be avoided by the introduction of the numerical dissipation presented in this paper.

Appendix 1
Linearization of end-point inertia terms

The equilibrium Eq. (58) is solved by means of the Newton's linearization which requires the definition of the term $\delta\mathbf{q}_{\text{mas}}$ given in Eq. (51). To give the full expression of $\delta\mathbf{q}_{\text{mas}}$ we have to perform the linearization of $\mathbf{U}_{n+1}, \mathbf{d}_{n+1}, \mathbf{w}_{n+1}$, and $\mathbf{A}_{n+1}$. From the theory of 3D rotations it is known that (Crisfield 1997):

$$\delta\mathbf{U}_{n+1} = \mathbf{S}(\delta\boldsymbol{\vartheta}_{n+1})\mathbf{U}_{n+1} \tag{90}$$

from Eq. (53)–(56) and from chapter 16 of Crisfield (1997) it is possible to obtain:

$$\delta\ddot{\mathbf{d}}_{n+1} = \frac{1}{\beta\Delta t^2}\delta\mathbf{d}_{n+1} \tag{91}$$

$$\begin{aligned}\delta\mathbf{w}_{n+1} &= \frac{\gamma}{\beta\Delta t}\delta(\Delta\boldsymbol{\Psi}) = \frac{\gamma}{\beta\Delta t}\mathbf{U}_n^t\delta(\Delta\boldsymbol{\psi}) \\ &= \frac{\gamma}{\beta\Delta t}\mathbf{U}_n^t\mathbf{H}^{-1}(\Delta\boldsymbol{\psi})\delta\boldsymbol{\vartheta}_{n+1}\end{aligned} \tag{92}$$

where $\delta(\Delta\boldsymbol{\psi})$ is an additive change (in the limit) of the rotational variables and $\delta\boldsymbol{\vartheta}$ is a multiplicative change of the rotational variables (Crisfield 1997), i.e. the spin variable that would be obtained from the iterative solver. $\mathbf{H}^{-1}$ is the matrix which relates the non-additive pseudo-vector changes to the additive pseudo-vector changes and is given by Eq. (16.94) of Crisfield (1997) as

$$\begin{aligned}\mathbf{H}^{-1}(\Delta\boldsymbol{\psi}) = {} & \frac{\frac{\Delta\psi}{2}}{\tan\frac{\Delta\psi}{2}}\mathbf{I} - \mathbf{S}\left(\frac{1}{2}\Delta\boldsymbol{\psi}\right) \\ & + \left(1 - \frac{\frac{\Delta\psi}{2}}{\tan\frac{\Delta\psi}{2}}\right)\frac{\Delta\boldsymbol{\psi}\Delta\boldsymbol{\psi}^t}{\Delta\psi^2}\end{aligned} \tag{94}$$

Therefore, by means of Eq. (90)–(93), we linearize the expression of the inertial forces (45). Using the notation of (51) and (57) we obtain the following results.

First term $\mathbf{K}_{\text{mas1}}$: By using (37) we obtain

$$\delta[\mathbf{U}_e]\mathbf{f}_{\text{in}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \delta\mathbf{u}_{e1}, \delta\mathbf{u}_{e2}, \delta\mathbf{u}_{e3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \delta\mathbf{u}_{e1}, \delta\mathbf{u}_{e2}, \delta\mathbf{u}_{e3} \end{pmatrix} \begin{pmatrix} f_{\text{in},1} \\ f_{\text{in},2} \\ \vdots \\ f_{\text{in},12} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ f_{\text{in},4}\delta\mathbf{u}_{e1} + f_{\text{in},5}\delta\mathbf{u}_{e2} + f_{\text{in},6}\delta\mathbf{u}_{e3} \\ \mathbf{0} \\ f_{\text{in},10}\delta\mathbf{u}_{e1} + f_{\text{in},11}\delta\mathbf{u}_{e2} + f_{\text{in},12}\delta\mathbf{u}_{e3} \end{pmatrix} \tag{95}$$

In contrast to the other $\mathbf{K}_{\text{mas}}$ terms, this matrix depends on the particular definition of the co-rotational element frame (see (12), (17) and (18)). Here we follow Crisfield (1997) and Crisfield (1990) and in this case from Eqs. (17.21) and (17.32) of Crisfield (1997) (or (43) and (49) of Crisfield (1990) we obtain:

$$\delta\mathbf{u}_{e1} = (-\mathbf{A}, \mathbf{0}, \mathbf{A}, \mathbf{0})\delta\mathbf{p} \tag{96}$$

$$\delta\mathbf{u}_{e2} = [\mathbf{L}(\mathbf{r}_2)]^t\delta\mathbf{p} \tag{97}$$

$$\delta\mathbf{u}_{e3} = [\mathbf{L}(\mathbf{r}_3)]^t\delta\mathbf{p} \tag{98}$$

where matrices $\mathbf{A}$, $\mathbf{L}(\mathbf{r}_2)$ and $\mathbf{L}(\mathbf{r}_3)$ are given in Crisfield (1990) and Crisfield (1997). Finally we obtain:

$$\delta[\mathbf{U}_e]\mathbf{f}_{\text{in}} = \mathbf{K}_{\text{mas1}}\delta\mathbf{p} = \begin{pmatrix} \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0} \\ f_{\text{in},4}[-\mathbf{A}, \mathbf{0}, \mathbf{A}, \mathbf{0}] + f_{\text{in},5}[\mathbf{L}(\mathbf{r}_2)]^t + f_{in,6}[\mathbf{L}(\mathbf{r}_3)]^t \\ \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0} \\ f_{\text{in},10}[-\mathbf{A}, \mathbf{0}, \mathbf{A}, \mathbf{0}] + f_{\text{in},11}[\mathbf{L}(\mathbf{r}_2)]^t + f_{\text{in},12}[\mathbf{L}(\mathbf{r}_3)]^t \end{pmatrix}\delta\mathbf{p} \tag{99}$$

$$\begin{aligned}\delta\mathbf{A}_{n+1} &= \frac{1}{\beta\Delta t^2}\delta(\Delta\boldsymbol{\Psi}) = \frac{1}{\beta\Delta t^2}\mathbf{U}_n^t\delta(\Delta\boldsymbol{\psi}) \\ &= \frac{1}{\beta\Delta t^2}\mathbf{U}_n^t\mathbf{H}^{-1}(\Delta\boldsymbol{\psi})\delta\boldsymbol{\vartheta}_{n+1}\end{aligned} \tag{93}$$

Second term $\mathbf{K}_{\text{mas2}}$: Recalling relations (91) and (93) it is possible to obtain the second term as:

$$[\mathbf{U}_e]\mathbf{M}\delta\ddot{\tilde{\mathbf{p}}} = \mathbf{K}_{\text{mas2}}\delta\mathbf{p} = \frac{1}{\beta\Delta t^2}[\mathbf{U}_e]\mathbf{M}\begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}^t_{n,1}\mathbf{H}^{-1}(\Delta\boldsymbol{\psi}_1) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{U}^t_{n,2}\mathbf{H}^{-1}(\Delta\boldsymbol{\psi}_2) \end{pmatrix}\delta\mathbf{p} \tag{100}$$

Third term $\mathbf{K}_{\text{mas3}}$: The third term in (51) can be rewritten as:

$$\begin{aligned}\frac{l}{12}[\mathbf{U}_e](\ldots) = \frac{l}{12}[\mathbf{U}_e]&\begin{pmatrix} \mathbf{0} \\ -3\mathbf{S}(\mathbf{I}_\rho\mathbf{w}_1)\delta\mathbf{w}_1 + 3\mathbf{S}(\mathbf{w}_1)\mathbf{I}_\rho\delta\mathbf{w}_1 - \mathbf{S}(\mathbf{I}_\rho\mathbf{w}_2)\delta\mathbf{w}_1 + \mathbf{S}(\mathbf{w}_2)\mathbf{I}_\rho\delta\mathbf{w}_1 \\ \mathbf{0} \\ -\mathbf{S}(\mathbf{I}_\rho\mathbf{w}_1)\delta\mathbf{w}_1 + \mathbf{S}(\mathbf{w}_1)\mathbf{I}_\rho\delta\mathbf{w}_1 - \mathbf{S}(\mathbf{I}_\rho\mathbf{w}_2)\delta\mathbf{w}_1 + \mathbf{S}(\mathbf{w}_2)\mathbf{I}_\rho\delta\mathbf{w}_1 \end{pmatrix} \\ + \frac{l}{12}[\mathbf{U}_e]&\begin{pmatrix} \mathbf{0} \\ \mathbf{S}(\mathbf{w}_1)\mathbf{I}_\rho\delta\mathbf{w}_2 - \mathbf{S}(\mathbf{I}_\rho\mathbf{w}_1)\delta\mathbf{w}_2 - \mathbf{S}(\mathbf{I}_\rho\mathbf{w}_2)\delta\mathbf{w}_2 + \mathbf{S}(\mathbf{w}_2)\mathbf{I}_\rho\delta\mathbf{w}_2 \\ \mathbf{0} \\ \mathbf{S}(\mathbf{w}_1)\mathbf{I}_\rho\delta\mathbf{w}_2 - \mathbf{S}(\mathbf{I}_\rho\mathbf{w}_1)\delta\mathbf{w}_2 - 3\mathbf{S}(\mathbf{I}_\rho\mathbf{w}_2)\delta\mathbf{w}_2 + 3\mathbf{S}(\mathbf{w}_2)\mathbf{I}_\rho\delta\mathbf{w}_2 \end{pmatrix}\end{aligned} \tag{101}$$

Equation (101) therefore becomes:

$$\begin{aligned}\frac{l}{12}[\mathbf{U}_e](\ldots) = \frac{l}{12}[\mathbf{U}_e]&\begin{pmatrix} \mathbf{0} \\ 3(\mathbf{S}(\mathbf{w}_1)\mathbf{I}_\rho - \mathbf{S}(\mathbf{I}_\rho\mathbf{w}_1)) + \mathbf{S}(\mathbf{w}_2)\mathbf{I}_\rho - \mathbf{S}(\mathbf{I}_\rho\mathbf{w}_2) \\ \mathbf{0} \\ \mathbf{S}(\mathbf{w}_1)\mathbf{I}_\rho - \mathbf{S}(\mathbf{I}_\rho\mathbf{w}_1) + \mathbf{S}(\mathbf{w}_2)\mathbf{I}_\rho - \mathbf{S}(\mathbf{I}_\rho\mathbf{w}_2) \end{pmatrix}\delta\mathbf{w}_1 \\ + \frac{l}{12}[\mathbf{U}_e]&\begin{pmatrix} \mathbf{0} \\ \mathbf{S}(\mathbf{w}_1)\mathbf{I}_\rho - \mathbf{S}(\mathbf{I}_\rho\mathbf{w}_1) + \mathbf{S}(\mathbf{w}_2)\mathbf{I}_\rho - \mathbf{S}(\mathbf{I}_\rho\mathbf{w}_2) \\ \mathbf{0} \\ \mathbf{S}(\mathbf{w}_1)\mathbf{I}_\rho - \mathbf{S}(\mathbf{I}_\rho\mathbf{w}_1) + 3(\mathbf{S}(\mathbf{w}_2)\mathbf{I}_\rho - \mathbf{S}(\mathbf{I}_\rho\mathbf{w}_2)) \end{pmatrix}\delta\mathbf{w}_2\end{aligned} \tag{102}$$

Remembering expression (92) and (57) we finally obtain:

$$\mathbf{K}_{\text{mas3}}\delta\mathbf{p} = \frac{l}{12}\frac{\gamma}{\beta\Delta t}[\mathbf{U}_e]\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 3\mathbf{K}_1 + \mathbf{K}_2 & \mathbf{0} & \mathbf{K}_1 + \mathbf{K}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_1 + \mathbf{K}_2 & \mathbf{0} & \mathbf{K}_1 + 3\mathbf{K}_2 \end{pmatrix}\begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}^t_{n,1}\mathbf{H}^{-1}(\Delta\boldsymbol{\psi}_1) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{U}^t_{n,2}\mathbf{H}^{-1}(\Delta\boldsymbol{\psi}_2) \end{pmatrix}\delta\mathbf{p} \tag{103}$$

where

$$\mathbf{K}_i = \mathbf{S}(\mathbf{w}_i)\mathbf{I}_\rho - \mathbf{S}(\mathbf{I}_\rho\mathbf{w}_i) \tag{104}$$

Appendix 2
Linearization of mid-point inertia terms

The inertial contribution to the 'effective stiffness matrix' is derived by varying Eq. (70) as shown in Eq. (75).

First term : $\delta[\mathbf{U}^*_{e,n+1}]\mathbf{M}\dot{\tilde{\mathbf{p}}}_{n+1}$

By using (68), (46), (71) and (90) we obtain:

$$\delta[\mathbf{U}^*_{e,n+1}]\mathbf{M}\dot{\tilde{\mathbf{p}}}_{n+1} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{S}(\mathbf{U}_{1,n+1}\mathbf{M}_r(2\mathbf{w}_{1,n+1} + \mathbf{w}_{2,n+1})) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{S}(\mathbf{U}_{2,n+1}\mathbf{M}_r(\mathbf{w}_{1,n+1} + 2\mathbf{w}_{2,n+1})) \end{pmatrix}\delta\mathbf{p} \tag{105}$$

Second term : $[\mathbf{U}^*_{e,n+1}]\mathbf{M}\delta\dot{\tilde{\mathbf{p}}}_{n+1}$

Following a procedure similar to that used for the second term of Appendix 1 and using (62) it is possible to show that:

$$\delta\mathbf{w}_{n+1} = \frac{2}{\Delta t}\mathbf{U}^t_n\delta(\Delta\boldsymbol{\psi}) = \frac{2}{\Delta t}\mathbf{U}^t_n\mathbf{H}^{-1}(\Delta\boldsymbol{\psi})\delta\boldsymbol{\vartheta}_{n+1} \qquad (106)$$

and therefore

$$[\mathbf{U}^*_{e,n+1}]\mathbf{M}\delta\dot{\tilde{\mathbf{p}}}_{n+1} = \frac{2}{\Delta t}[\mathbf{U}^*_{e,n+1}\mathbf{M}]\begin{pmatrix}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{U}^t_{1,n}\mathbf{H}^{-1}(\Delta\boldsymbol{\psi}_1) & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{U}^t_{2,n}\mathbf{H}^{-1}(\Delta\boldsymbol{\psi}_2)\end{pmatrix}\delta\mathbf{p} \qquad (107)$$

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