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On finite 2-groups all of whose subgroups are mutually isomorphic

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Abstract In this paper we investigate the title groups which we call isomaximal. We give the list of all isomaximal 2-groups with abelian maximal subgroups. Further, we prove some properties of isomaximal 2-groups with nonabelian maximal subgroups. After that, we investigate the structure of isomaximal groups of order less than 64. Finally, in Theorem 14. we show that the minimal nonmetacyclic group of order 32 possesses a unique isomaximal extension of order 64.

Keywords: p-groups, maximal subgroups, minimal nonabelian, minimal nonmetacyclic.

MSC(2000): 20D15

1 Introduction

The groups pointed in the title are interesting enough themselves because of their supposed symmetry. There arise at once two questions:
1) which are these groups, and
2) which groups can appear as maximal subgroups in such groups.

The concrete motivation for this research was however a recent result concerning second-metacyclic finite 2-groups. A group G is *metacyclic* if there exists a cyclic normal subgroup N of G with cyclic factor group G/N. A group with some non-metacyclic maximal subgroup and with all second-maximal subgroups being metacyclic we call a *second-metacyclic group.*

A minimal non-metacyclic group is a non-metacyclic group with all its proper subgroups being metacyclic. By a result of N. Blackburn there are only four such groups:

Theorem 1. (see Janko [2 , Th.7.1]) *Let G be a minimal non-metacyclic group. Then G is one of the following groups:*
(a) The elementary abelian group E_8 of order 8,
(b) The direct product $Q_8 \times Z_2$,
(c) The central product $Q_8 * Z_4$ of order 2^4,
(d) $G = \langle a, b, c \mid a^4 = b^4 = [a, b] = 1,\ c^2 = a^2b^2,\ a^c = a^{-1},\ b^c = a^2b^3\rangle$, where G is special of order 2^5 with exp $G = 4$, $\Omega_1(G) = G' = Z(G) = \Phi(G) = \langle a^2, b^2\rangle \cong E_4$ and $M = \langle a\rangle \times \langle b\rangle \cong Z_4 \times Z_4$ is the unique abelian maximal subgroup of G, the other six all being isomorphic to the semidirect product $Z_4 \cdot Z_4$.

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The second-metacyclic finite 2-groups were determined in [1] (Ćepulić, Ivanković, Kovač Striko). It turned out that there are 17 such groups, four among them being of order 16, ten of order 32 and three of order 64. Each of them contains, of course, some minimal non-metacyclic group as a subgroup of index 2.

By Theorem 1.2 and Remark 2.1 of [2], we get the following result:

Theorem 2. *There is only one nonabelian second-maximal finite 2-group G with all its maximal subgroups being mutually isomorphic, the group:*
(1) $G = \langle a, b, c, d \mid a^4 = b^4 = 1,\ c^2 = a^2b^2,\ d^2 = a^2,\ [a, b] = [c, d] = 1,\ [a, c] = a^2,\ [a, d] = [b, c] = a^2b^2,\ [b, d] = b^2\rangle$

G is of order 64. The maximal subgroups of G are all isomorphic to the minimal non-metacyclic group of order 32 in Theorem 1(d).

The above investigation was suggested by Professor Zvonimir Janko. Recently he made some research in which the group (1) had some importance. This fact was also a motive for the present investigation. Our notation is standard. In addition, $\Omega^*_k(G) = \langle g \in G \mid |g| = p^k\rangle$, for any p-group G. Obviously, $\Omega^*_1(G) = \Omega_1(G)$

For the sake of brevity we introduce the following terms:

Definition 1. A group G with all its maximal subgroups being mutually isomorphic we shall call *isomaximal* group.

Definition 2. We call an abelian group *homocyclic* if it is the direct product of isomorphic cyclic groups.

2 Abelian isomaximal groups

The case of abelian isomaximal groups is simple. We have:

Theorem 3. *If G is an abelian isomaximal group, then G is is a homocyclic p-group, for some prime p.*

Proof. As known, abelian groups are direct products of cyclic groups of prime power orders. Thus, if G is not itself cyclic of prime power order, then for any two components $B \cong Z_{p^n}, C \cong Z_{q^r}$ of prime power orders of the direct product, $G \cong Z_{p^n} \times Z_{q^r} \times A$ for some abelian group A. Now, there exist two maximal subgroups M_1 and M_2 of G such that $M_1 \cong Z_{p^{n-1}} \times Z_{q^r} \times A$ and $M_2 \cong Z_{p^n} \times Z_{q^{r-1}} \times A$, which should be isomorphic according our assumption. Therefore $p^{n-1} = q^{r-1}$, implying $p = q$ and $n = r$. Thus, all factors are isomorphic to the same group Z_{p^n} and G is homocyclic.

3 Nonabelian isomaximal 2-groups with abelian maximal subgroups

The groups in question are obviously minimal nonabelian. We use the following known result:

Theorem 4. (Miller - Moreno) *A minimal nonabelian finite 2-group is isomorphic to one of the groups:*
(a) $G = \langle a, b \mid a^{2^\mu} = b^{2^\nu} = 1,\ a^b = a^{1+2^{\mu-1}}\rangle,\ \mu \geqslant 2,\ \nu \geqslant 1,\ |G| = 2^{\mu+\nu}$
(b) $G = \langle a, b, c \mid a^{2^\mu} = b^{2^\nu} = c^2 = 1,\ c = [a, b],\ [a, c] = [b, c] = 1\rangle$,
$\mu, \nu \geqslant 1,\ \mu + \nu \geqslant 2,\ |G| = 2^{\mu+\nu+1}$
(c) $G \cong Q_8$

We have to examine which of these groups are isomaximal. The solution of this problem is given by the following:

Theorem 5. *A nonabelian isomaximal 2-group with abelian maximal subgroups is isomorphic to one of the groups:*
(a) $G = \langle a, b \mid a^{2^\mu} = b^{2^\mu} = 1,\ a^b = a^{1+2^{\mu-1}}\rangle,\ \mu \geqslant 2,\ |G| = 2^{2\mu}$, *maximal subgroups being isomorphic to* $Z_{2^\mu} \times Z_{2^{\mu-1}}$,
(b) $G = \langle a, b, c \mid a^{2^\mu} = b^{2^\mu} = c^2 = 1,\ c = [a, b],\ [a, c] = [b, c] = 1\rangle,\ \mu \geqslant 2,\ |G| = 2^{2\mu+1}$, *maximal subgroups being isomorphic to* $Z_{2^\mu} \times Z_{2^{\mu-1}} \times Z_2$,
(c) $G \cong Q_8$

Proof. By the introductory remark such a group is isomorphic to one of the groups stated in Theorem 4.(a),(b),(c), which we treat separately.

(a) In this case $\Phi(G) = \mho_1(G) = \langle a^2, b^2\rangle$. If $\nu = 1$, then $G = \langle a, b \mid a^{2^\mu} = b^2 = 1,\ a^b = a^{1+2^{\mu-1}}\rangle$, with nonisomorphic maximal subgroups $M_1 = \langle a\rangle \cong Z_{2^\mu}$ and $M_2 = \langle a^2, b\rangle \cong Z_{2^{\mu-1}} \times Z_2$. Thus $\mu,\ \nu \geqslant 2$, both. One can easily check that $[a^2, b] = [a, b^2] = 1$, so $\Phi(G) = Z(G)$. As $G/\Phi(G) \cong E_4$ there are three maximal subgroups in G: $M_1 = \langle \Phi(G), a\rangle = \langle a, b^2\rangle \cong Z_{2^\mu} \times Z_{2^{\nu-1}}$, $M_2 = \langle \Phi(G), b\rangle = \langle a^2, b\rangle \cong Z_{2^{\mu-1}} \times Z_{2^\nu}$ and $M_3 = \langle \Phi(G), ab\rangle$. By assumption, $M_1 \cong M_2$, implying $\mu = \nu$. As for M_3, $(ab)^2 = ab^2a^b = a^2b^2a^{2^{\mu-1}}$, $(ab)^4 = a^4b^4$, and so $M_3 = \langle a^2, ab\rangle \cong Z_{2^{\mu-1}} \times Z_{2^\mu}$ also.
(b) Now $\Phi(G) = \mho_1(G) = \langle a^2, b^2, c\rangle$. If $\nu = 1$, there are nonisomorphic maximal subgroups $M_1 = \langle a, c\rangle \cong Z_{2^\mu} \times Z_2$ and $M_2 = \langle a^2, b, c\rangle \cong Z_{2^{\mu-1}} \times E_4$ in G. So $\nu \geqslant 2$ and, by symmetry, $\mu \geqslant 2$ as well. We have $[a^2, b] = [a, b^2] = c^2 = 1$, and $\Phi(G) = Z(G)$ again. As $G/\Phi(G) \cong E_4$ there are three maximal groups $M_1 = \langle a, b^2, c\rangle \cong Z_{2^\mu} \times Z_{2^{\nu-1}} \times Z_2$, $M_2 = \langle a^2, b, c\rangle \cong Z_{2^{\mu-1}} \times Z_{2^\nu} \times Z_2$ and $M_3 = \langle \Phi(G), ab\rangle$. Since $M_1 \cong M_2$, so $\mu = \nu$ in this case again. Here $(ab)^2 = a^2b^2c$, $(ab)^4 = a^4b^4$, and $M_3 \cong M_1 \cong Z_{2^\mu} \times Z_{2^{\mu-1}} \times Z_2$.
(c) The quaternion group Q_8 is nonabelian isomaximal group with all maximal subgroups isomorphic to Z_4.

4 Some remarks on isomaximal 2-groups with nonabelian maximal subgroups

Here we state some facts about such groups.

Theorem 6. *If G is an isomaximal 2-group with nonabelian maximal subgroups, then $Z(G) \leqslant \Phi(G)$. Especially, $Z(G) \leqslant Z(M)$ for all maximal subgroups M.*

Proof. Let M be a maximal subgroup of G. If $Z(G) \nleqslant M$, then $G = \langle M, z\rangle$ for some $z \in Z(G)$. Now $\langle M, z\rangle = G = C_G(Z(M))$ and therefore $Z(M) < Z(G)$ implying $Z(G) \cap M = Z(M)$. Let M_1 be a maximal subgroup containing $Z(G)$. Now $Z(M_1) \geqslant Z(G) > Z(M)$, in contradiction with $M_1 \cong M$.

Theorem 7. *In an isomaximal 2-group, which is not cyclic, the exponent of G equals the exponent of its maximal subgroups.*

Proof. Each element of G is contained in some maximal subgroup, which proves the statement.

Theorem 8. *A nonabelian 2-group with a cyclic maximal subgroup cannot be maximal in an isomaximal group.*

Proof. Suppose that G is an isomaximal 2-group. Let M be a nonabelian maximal subgroup of G and let $\langle a\rangle$ be a cyclic maximal subgroup of M. Then M is isomorphic, as known, to one of the following groups:
(i) $\langle a,b \mid a^{2^{n-1}} = b^2 = 1,\ a^b = a^{-1}\rangle \cong D_{2^n},\ n \geqslant 3$,
(ii) $\langle a,b \mid a^{2^{n-1}} = b^4 = 1,\ b^2 = a^{2^{n-2}},\ a^b = a^{-1}\rangle \cong Q_{2^n},\ n \geqslant 3$,
(iii) $\langle a,b \mid a^{2^{n-1}} = b^2 = 1,\ a^b = a^{2^{n-2}-1}\rangle \cong SD_{2^n},\ n \geqslant 4$,
(iv) $\langle a,b \mid a^{2^{n-1}} = b^2 = 1,\ a^b = a^{2^{n-2}+1}\rangle \cong M_{2^n},\ n \geqslant 4$

In the cases (i), (ii) and (iii), $\langle a\rangle$ is the unique cyclic maximal subgroup of M and so $\langle a\rangle$ char M, $\langle a\rangle \lhd G$. Moreover, $Z(M) = \langle a^{2^{n-2}}\rangle \cong Z_2$, and by Theorem 6. it is $Z(G) = Z(M)$. For each $g = a^\alpha b \in M\backslash\langle a\rangle$, we have $(a^\alpha b)^2 = a^\alpha b^2 (a^b)^\alpha \in \langle a^{2^{n-2}}\rangle = Z(G)$.

We see that all elements of $M\backslash\langle a\rangle$ have order 2 or 4 and their squares are in $Z(G)$. Let M_1 be another maximal subgroup of G. The intersection $M_1 \cap M$ does not contain all elements of order 2 and order 4 of M_1, because such elements generate M_1. Thus, there exists some $c \in M_1\backslash M,\ c^2 \in Z(G) < \langle a\rangle$, so that $G = \langle M,c\rangle = \langle a,b,c\rangle$. Now, $G/\langle a\rangle \cong E_4$, and $\langle a,b\rangle,\ \langle a,c\rangle,\ \langle a,bc\rangle$ are maximal in G containing $\langle a\rangle$ as cyclic maximal subgroup. Hence $a^b = a^c = a^\alpha$ where $\alpha = -1$ in the cases (i) and (ii), and $\alpha = 2^{n-2}-1$ in the case (iii). It follows, $a^{bc} = a^{\alpha^2} = a$, as $\alpha^2 \equiv 1(\mathrm{mod}\ 2^{n-1})$ also in the case (iii) as $n \geqslant 3$. But now, the maximal subgroup $\langle a,bc\rangle$ would be abelian, a contradiction.

The case (iv) is somewhat different. The group M_{2^n} is minimal nonabelian. It has three maximal subgroups, two of them - $\langle a\rangle$ and $\langle ab\rangle$ being cyclic and the third - $\langle a^2,b\rangle$ is isomorphic to $Z_{2^{n-2}} \times Z_2$. Therefore $K = \langle a^2,b\rangle$ char M, $K \lhd G$. Elements of $M\backslash K$ are all of order 2^{n-1}: $(a^\alpha b)^2 = a^\alpha b^2 (a^b)^\alpha = a^{\alpha+\alpha(1+2^{n-2})} = a^{2\alpha(1+2^{n-3})}$, where $2 \nmid \alpha$ and $n \geqslant 4$.

If $G/K \cong Z_4$, then $c \in G\backslash M$, $c^2 \in M\backslash K$ and so $|c| = 2^n$, and $\exp G > \exp M$, contradictioning Theorem 7. Thus $G/K \cong E_4$. Let $M_1 = \langle K,c\rangle$ be another maximal subgroup of G. Then $M_1 \cong M$ and $|c| = 2^{n-1}$, $c^{2^{n-2}} = a^{2^{n-2}} = \tau$, as $\langle\tau\rangle = \Omega_1(\mho_1(K))$. Now $[a,b] = [c,b] = \tau$, so $b^{ac} = (\tau b)^c = \tau\tau b = b$. Thus, $M_2 = \langle ac,b\rangle$ would be maximal and abelian, in contradiction with $M_2 \cong M$.

The theorem is proved.
As an obvious corrollary of Theorem 8. we have:

Theorem 9. *The exponent of an isomaximal 2-group G with a nonabelian maximal subgroup is at most $|G| : 8$.*

Proof. Let M be a maximal group of G. By Theorem 7 we have $\exp G = \exp M$ and by Theorem 8, $\exp M \leqslant |M| : 4$, so $\exp G \leqslant |G| : 8$.

Also the following holds true.

Theorem 10. *Let M be a maximal subgroup of an isomaximal 2-group G. Then:*
(a) If $\Omega_k^*(M) = M$, then $\Omega_k^*(G) = G$
(b) If $\Omega_k^*(M) < M$, then either $\Omega_k^*(M) = \Omega_k^*(G) \leqslant \Phi(G)$, the same being true for all maximal subgroups of G, or there exists some $g \in G\backslash M$ with $|g| = 2^k$ and for each such g it is $G = \Omega_k^*(M) \cdot \langle g\rangle$. Especially, if $k = 1$, then $\Omega_1(M) = \Omega_1(G) \leqslant \Phi(G)$. If moreover exp M = exp G = 4, then $\Omega_1(G) = \Phi(G)$.

Proof. (a) Let M_1 be an other maximal subgroup of G. Since G is isomaximal, we have $\Omega_k^*(M_1) = M_1$ also, and so $\Omega_k^*(G) \geqslant \langle \Omega_k^*(M), \Omega_k^*(M_1)\rangle = \langle M, M_1\rangle = G$, implying $\Omega_k^*(G) = G$.
(b) If there is none element of order 2^k in $G\backslash M$, then $\Omega_k^*(G) = \Omega_k^*(M) < M$. For any other maximal subgroup M_1 of G it is also $\Omega_k^*(M_1) = \Omega_k^*(G)$, as $\Omega_k^*(M_1) \leqslant \Omega_k^*(G)$ and $|\Omega_k^*(M_1)| = |\Omega_k^*(M)|$. Therefore $\Omega_k^*(M) = \Omega_k^*(G) \leqslant \Phi(G)$ for all maximal subgroups M of G.

If, on the contrary, there is some $g \in G\backslash M, |g| = 2^k$, then $\Omega_k^*(G) \geqslant \langle \Omega_k^*(M), g\rangle = \Omega_k^*(M)\cdot\langle g\rangle$ because of $\Omega_k^*(M) \lhd G$. If $\Omega_k^*(M) \cdot \langle g\rangle < G$, then there exists some maximal subgroup M_1 containing $\Omega_k^*(M) \cdot \langle g\rangle$. But now $\Omega_k^*(M_1) \geqslant \langle \Omega_k^*(M), g\rangle > \Omega_k^*(M)$, a contradiction. Thus $\Omega_k^*(M) \cdot \langle g\rangle = G$. If $k = 1$ then for any $g \in G\backslash M$, $|g| = 2$, it would be $\Omega_1(M) \cdot \langle g\rangle < G$, and so $\Omega_1(M) = \Omega_1(G) \leqslant \Phi(G)$. If moreover $\exp G = 4$, then $\mho_1(G) \leqslant \Omega_1(G)$, and so $\Phi(G) = \mho_1(G) \leqslant \Omega_1(G) \leqslant \Phi(G)$, implying $\Omega_1(G) = \Phi(G)$.

5 Isomaximal 2-groups of order $\leqslant$ 32

A. Groups of order $\leqslant$ 16

By inspection we immediately see that the following holds:

Theorem 11. *Let G be an isomaximal 2-group of order $\leqslant 16$.*
Then G is isomorphic to one of the following groups:
(i) any group of order $\leqslant 4$,
(ii) $Z_8,\ E_8$ *or* Q_8*, all of order* 8,
(iii) $Z_{16},\ Z_4 \times Z_4,\ E_{16}$*, and the semidirect product* $Z_4 \cdot Z_4$.

Remark There are fourteen groups of order 16:
five abelian - $Z_{16},\ Z_8 \times Z_2,\ Z_4 \times Z_4,\ Z_4 \times E_4,\ E_{16}$,
four nonabelian of exponent 8 - $D_{16},\ Q_{16},\ SD_{16},\ M_{16}$,
five nonabelian of exponent 4 - $D_8 \times Z_2,\ Q_8 \times Z_2$, the central product $Q_8 * Z_4$, and semidirect products $E_4 \cdot Z_4,\ Z_4 \cdot Z_4$.

B. Groups of order 32 with abelian maximal subgroups

There are three such subgroups:

Theorem 12. *Let G be an isomaximal group of order 32 with abelian maximal subgroups. Then:*
(i) G is abelian and either $G \cong Z_{32}$ or $G \cong E_{32}$, or
(ii) G is nonabelian and $G = \langle a, b, c \mid a^4 = b^4 = c^2 = 1,\ [a, b] = c,\ [a, c] = [b, c] = 1\rangle$, the maximal subgroups being isomorphic to $Z_4 \times Z_2 \times Z_2$.
Proof. This follows immediately from the statements of Theorem 2 and Theorem 4.

C. Groups of order 32 with nonabelian maximal subgroups

These groups would be extensions of nonabelian groups in the Remark 1. But such groups do not exist. We have

Theorem 13. *Nonabelian groups of order 16 cannot be extended to isomaximal groups of order 32.*
Proof. By Theorem 8, the four nonabelian groups of order 16 and exponent 8, that is possessing cyclic maximal subgroups - $D_{16},\ Q_{16},\ SD_{16}$ and M_{16} cannot have such extensions.

The remaining five cases we should examine separately. In the following we denote by M the supposed maximal subgroup and by G its extension that should be isomaximal.

1) $M \cong D_8 \times Z_2,\ M = \langle a, b, c \mid a^4 = b^2 = c^2 = 1,\ a^b = a^3,\ [a, c] = [b, c] = 1\rangle$

Let M_1 be another maximal subgroup of G. The group M is generated by its involutions and so is M_1. Thus there is some $d \in M_1 \backslash M,\ d^2 = 1$ and $G = \langle M, d\rangle$. The elements of order 4 in M generate $K = \langle a, c\rangle \cong Z_4 \times Z_2$, implying $\langle a^2\rangle$, K char M and so $\langle a^2\rangle$, $K \lhd G$. As the elements in $M \backslash K$ invert the elements of order 4 in K and centralize the involutions in K, the element $d \in \langle a, c, d\rangle$ does the same. So we have $a^d = a^3,\ c^d = c$. Now, $a^{bd} = (a^3)^3 = a$ and the maximal subgroup $\langle a, c, bd\rangle$ would be abelian, a contradiction.

2) $M \cong Q_8 \times Z_2,\ M = \langle a, b, c \mid a^4 = c^2 = 1,\ b^2 = a^2,\ a^b = a^3,\ [a, c] = [b, c] = 1\rangle$

The group M has only three involutions and $K = \langle a^2, c\rangle = \Omega_1(M)$ char M, $\langle a^2, c\rangle \lhd G$. Also $\langle a^2\rangle = \mho_1(M)$ char G. None of elements $d \in G \backslash M$ is an involution since otherwise the maximal subgroups containing $\langle a^2, c, d\rangle$ would contain more than three involutions, a contradiction. Because of exp $G =$ exp M, by Theorem 7, each $d \in G \backslash M$ is of order 4 and $\langle a^2, c\rangle$ is contained in every maximal subgroup. Thus $\langle a^2, c\rangle = \Phi(G) = \Omega_1(G)$. Since $\mho_1(G) = \Phi(G)$, generally, we can assume without loss that $d^2 = c$. But now $\langle a^2, c, a, d\rangle = \langle a, d\rangle$ is 2 - generated, contradicting the fact that it should be isomorphic to M.

3) $M \cong Q_8 * Z_4,\ M = \langle a, b, c \mid a^4 = 1,\ b^2 = c^2 = a^2,\ a^b = a^3,\ [a, c] = [b, c] = 1\rangle$

This group contains a unique maximal subgroup isomorphic to Q_8, the group $K = \langle a, b\rangle$, the other maximal subgroups being isomorphic to D_8 or $Z_4 \times Z_2$. Thus K char M, $K \lhd G$. Similarly, $Z(M) = \langle c\rangle$ char M and so $\langle c\rangle \lhd G$. The group M has 7 involutions and 8 elements of order four. As it does not contain any subgroup isomorphic to E_8, the group M is generated by its involutions. Let M_1 be another maximal subgroup of G. Then there is some involution $d_1 \in M_1 \backslash M$ and $G = \langle M, d_1\rangle$. Now $\langle K, d_1\rangle = \langle a, b, d_1\rangle$ is maximal in G, $\langle K, d_1\rangle \cong M \cong Q_8 \times Z_4$. Thus there exists some $d \in \langle K, d_1\rangle \backslash K,\ |d| = 4,\ d^2 = a^2$ in $Z(\langle K, d_1\rangle)$. Thus we have $[a, d] = [b, d] = 1$, and also $c^d \in \langle c\rangle$. If $c^d = c$, then $d \in Z(G)$, a contradiction as $Z(G) \leqslant Z(M)$, by Theorem 6, and $d \notin M$. So $c^d = c^3$. It follows that $(bc)^{ad} = (b^3c)^d = b^3c^3 = bc$. Now, $(bc)^2 = b^2c^2 = 1$ and $(ad)^2 = a^2d^2 = 1$, so it would be $\langle a^2, bc, ad\rangle \cong E_8$. But none maximal subgroup of G would contain such a group. We get a contradiction.

4) $M \cong E_4 \cdot Z_4,\ M = \langle a, b, c \mid a^2 = b^2 = c^4 = 1,\ [a, b] = [a, c] = 1,\ b^c = ab\rangle$

Here $\langle a, b, c^2\rangle \cong E_8$ and $(a^\alpha b^\beta c^{2\gamma} c)^2 = a^\alpha b^\beta c^{2\gamma} c^2 a^\alpha a^\beta b^\beta c^{2\gamma} = a^\beta c^2 \neq 1$. We see that the groups $\langle a\rangle,\ K = \langle a, c^2\rangle$, and $L = \langle a, b, c^2\rangle = \Omega_1(M)$ are all characteristic in M, so $\langle a\rangle,\ K,\ L \lhd G$. If $d \in G \backslash M$ is an involution, then $M_1 = \langle L, d\rangle$ contains more than 7 involutions, a contradiction because $M_1 \cong M$ and M has exactly 7 involutions. Thus $|d| = 4$ for all $d \in G \backslash L$ and $L = \Omega_1(G) = \Omega_1(M) = \Phi(G) = \mho_1(G)$, by Theorem 10. As $\mho_1(M) = \Phi(M) = \langle a, c^2\rangle$, there exists some $d \in G \backslash M$ so that $d^2 = a^\alpha c^{2\gamma} b \in L \backslash K$. But now b can be supstituted by d^2, as $(d^2)^c = ad^2$ and we can assume without loss that $d^2 = b$. We see that $a^d = a,\ b^d = b$ and $(c^2)^d = ac^2$ as $K \lhd G$ and $(c^2)^d = c^2$ would imply $\langle a, b, c^2, d\rangle = \langle a, c^2, d\rangle \cong E_4 \times Z_4 \not\cong M$, a contradiction. Obviously $c^d = a^\alpha b^\beta c^{2\gamma} c$ and so $(c^2)^d = (c^d)^2 = (b^\beta c)^2 = b^\beta c^2 (b^\beta)^c = b^\beta c^2 a^\beta b^\beta = a^\beta c^2 = ac^2$. Therefore $\beta = 1$ and $c^d = a^\alpha b c^{2\gamma} c$. From $c^{d^2} = c^b = ac = (c^d)^d = (a^\alpha b c^{2\gamma} c)^d = a^\alpha b a^\gamma c^{2\gamma} a^\alpha b c^{2\gamma} c = a^\gamma c$ it follows $\gamma = 1$, $c^d = a^\alpha b c^3$, and from $(cd)^2 = cd^2c^d = cba^\alpha bc^3 = a^\alpha \neq 1$ that $\alpha = 1,\ c^d = abc^3$. But now $(c^3d)^2 = (c^2cd)^2 = c^2(cd)^2(c^2)^{cd} = c^2aac^2 = 1$, a contradiction as $|c^3d| = 4$.

5) $M \cong Z_4 \cdot Z_4,\ M = \langle a, b \mid a^4 = b^4 = 1,\ a^b = a^3 \rangle$

Here $Z(M) = \Omega_1(M) = \mho_1(M) = \langle a^2, b^2 \rangle \equiv K$. M has 3 involutions and 12 elements of order four. Among elements of order four there are 8 with square b^2 and 4 with square a^2. Thus $a^2,\ b^2,\ a^2b^2$ are all characteristic in M and so $\langle a^2,\ b^2 \rangle \leqslant Z(G)$. By Theorem 6. this implies that $Z(G) = Z(M) = K$ and by Theorem 10. $Z(G) = \Omega_1(G) = \Phi(G)$. Now $G/\Phi(G) = G/K \cong E_8$ and all elements in $G \backslash K$ are of order 4. For any $c \in G \backslash M$ we have $G = \langle a, b, c \rangle$. It is $[a, b] = a^2$. Denote $[a, c] = z_1,\ [b, c] = z_2,\ c^2 = z_3$. Here $z_1, z_2, z_3 \in K$ and $z_1, z_2, z_3 \neq 1$ because the maximal subgroups $\langle a, c, K \rangle$ and $\langle b, c, K \rangle$ are not abelian and $|c| = 4$. Since all maximal subgroups are nonabelian, we have also $[ab, c] = z_1z_2 \neq 1,\ [a, bc] = a^2z_1 \neq 1,\ [ac, b] = a^2z_2 \neq 1,\ [ab, ac] = a^2z_1z_2 \neq 1$. But now $z_1z_2 \in \{b^2, a^2b^2\},\ z_1 \neq z_2$ and $z_1z_2 \neq a^2$, a contradiction.

The Theorem is proved.

6 The isomaximal extension of the minimal nonmetacyclic subgroup of order 32

We have

Theorem 14. *The minimal nonmetacyclic group of order 32 (see Theorem 1.(d)):*
$M = \langle a, b, c \mid a^4 = b^4 = [a, b] = 1,\ c^2 = a^2b^2,\ a^c = a^3,\ b^c = a^2b^3 \rangle$
has its second - metacyclic extension (see Theorem 2.):
$G = \langle a, b, c, d \mid a^4 = b^4 = 1,\ c^2 = a^2b^2,\ d^2 = a^2,\ [a, b] = [c, d] = 1,\ [a, c] = a^2,\ [b, d] = b^2,\ [b, c] = [a, d] = a^2b^2 \rangle$
as the unique isomaximal extension.

Proof. According Theorem 1.(d), M has $L = \langle a, b \rangle \cong Z_4 \times Z_4$ as unique abelian maximal subgroup, the other 6 being isomorphic to $Z_4 \cdot Z_4$. Here $Z(M) = \langle a^2, b^2 \rangle = \Omega_1(M) = \Phi(M) \equiv K$ and by Theorem 10. $K = Z(G) = \Phi(G)$. Thus $G/K \cong E_{16}$ and G has 15 maximal subgroups. On the other side there are exactly 3 maximal subgroups sharing the same subgroup of type $Z_4 \times Z_4$. Thus there are just 5 different subgroups in G isomorphic to $Z_4 \times Z_4$.

Denote the group L by L_1. For another such subgroup L_2 the intersection $L_1 \cap L_2$ does not contain any element x of order 4, since otherwise $C_G(x) \geqslant \langle L_1, L_2 \rangle$, which is maximal or equals G, in contradiction with exp $Z(M) = 2$. So $L_1 \cap L_2 = K$ and $G = L_1 \cdot L_2$.
Moreover, the five sets $L_i \backslash K$ for different groups $L_i,\ i = 1, 2, 3, 4, 5$, isomorphic to $Z_4 \times Z_4$ form a partition of the set $G \backslash K$. Consider now $L_1 = \langle a, b \rangle$ and $L_2 = \langle c, d \rangle$ - the $Z_4 \times Z_4$ group which contains c. Now $[c, d] = 1$ and we can assume without loss that $d^2 = a^2$, as there is some such element in L_2.
The groups $\langle K, a, d \rangle$ and $\langle K, b, d \rangle$ are nonabelian of order 2^4. Thus $[a, d] = z_1,\ [b, d] = z_2$ for some $z_1, z_2 \in K,\ z_1, z_2 \neq 1$. Obviously $C_G(a) = C_G(b) = \langle a, b \rangle$, and $C_G(c) = C_G(d) = \langle c, d \rangle$. Therefore $[ab, d] = z_1z_2 \neq 1,\ [a, cd] = a^2z_1 \neq 1,\ [ab, cd] = a^2 \cdot z_1 \cdot a^2b^2 \cdot z_2 = b^2z_1z_2 \neq 1$. Also acd should be an element of order 4, so $(acd)^2 = a \cdot (cd)^2 \cdot a^{cd} = a \cdot b^2 \cdot a^3z_1 = b^2z_1 \neq 1$. It follows easily that $z_1 = a^2b^2,\ z_2 = b^2$, and so $[a, d] = a^2b^2,\ [b, d] = b^2$.

The theorem is proved.

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