On Circumradius Equations of Cyclic Polygons

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1 Introduction

Cyclic polygons are the polygons inscribed in a circle. In terms of their side lengths $a_1, a_2, \ldots, a_n$, their area S and circumradius r are given in case of triangles and quadrilaterals explicitly by the following well known formulas: the Heron's formula (60 B.C.) for the area and the circumradius r of triangles (by letting $A = (4S)^2, \rho = 1/r^2$):

$$\begin{aligned} &A - (a+b+c)(a+b-c)(a-b+c)(-a+b+c) = 0 \\ &a^2b^2c^2\rho - (a+b+c)(a+b-c)(a-b+c)(-a+b+c) = 0 \end{aligned} \tag{1.1}$$

and the Brahmagupta's formula, (7 th c. A.D.) for the area and the circumradius of convex ($\varepsilon = 1$) and nonconvex ($\varepsilon = -1$) quadrilaterals:

$$A_\varepsilon - (a+b+c-\varepsilon d)(a+b-c+\varepsilon d)(a-b+c+\varepsilon d)(-a+b+c+\varepsilon d) = 0 \tag{1.2}$$

$$\begin{aligned} &(ab+\varepsilon cd)(ac+\varepsilon bd)(bc+\varepsilon ad)\rho_\varepsilon - \\ &(a+b+c-\varepsilon d)(a+b-c+\varepsilon d)(a-b+c+\varepsilon d)(-a+b+c+\varepsilon d) = 0 \end{aligned} \tag{1.3}$$

In a masterfully written (in german language) thirty pages long paper (and published in 1828 in Crelle's Journal) A. F. Möbius studied some properties of the polynomial equations for the circumradius of arbitrary cyclic polygons (convex and nonconvex) and produced a polynomial of degree $\delta_n = \frac{n}{2}\binom{n-1}{\lfloor (n-1)/2 \rfloor} - 2^{n-2}$ that relates the square of a circumradius (r^2) of a cyclic polygon to the squared side lengths. He also showed that the squared area rationally depends on $r^2, a_1, a_2, \ldots, a_n$. His approach is based, by a clever use of trigonometry, on the rationalization (in terms of the squared sines) of the sine of a sum of n angles (peripheral angles of a cyclic polygon). In this way one obtains a polynomial relating the circumradius to the side lengths squared. These polynomials, known also as generalized Heron $r-$polynomials, are a kind of generalized (symmetric) multivariable Chebyshev polynomials and are quite difficult to be computed explicitly. Möbius obtained nice form for the leading and constant terms for pentagons and hexagons, but no complete answer even for pentagons. By an argument involving series expansions (cf. [MRR]) he proved that the r^2-degree for cyclic $n-$polygon is equal to δ_n. In the final part of the paper he obtained for the squared area a rational function in $r^2, a_1, a_2, \ldots, a_n$ involving partial derivatives, with respect to side length variables, of all the coefficients of the Heron $r-$polynomial. So, in principle, one could get from this formula the area polynomial by using Viete formulas together with a heavy use of symmetric functions.

About ten years ago David Robbins ([R1], [R2]) obtained, for the first time, concise explicit formulas for the areas of cyclic pentagons and hexagons (he mentioned that he computed also the circumradius polynomials for cyclic pentagons and hexagons but was not able to put either formula into a sensible compact form). In [R1] two general conjectures (Conjecture1 and Conjecture2), naturally extending nice Möbius product formulas for the leading and constant terms for pentagons and hexagons are given. We shall give a proof of these conjectures up to $n = 8$.

One of the Additional Conjectures of Robbins, stating that the degree of the minimal A-polynomial equation for cyclic $n-$polygons $\alpha_n(16S^2, a_1^2, \ldots, a_n^2) = 0$, (i.e. of the generalized Heron $A-$ polynomial), is equal to δ_n was established in [FP] first (by relating it to the Sabitov theory of volume polynomials of polyhedra, see nice survey article by Pak) and later in [MRR] (obtained by reviving the argument of Möbius and reproving the Robbins lower bound on the degrees of minimal polynomials).

In Robbins work a method of undetermined coefficients is used for pentagons (70 unknowns) and hexagons (134 unknowns). This method seems to be inadequate for heptagons because one would need to handle a linear system with 143307 undetermined coefficients. By using a clever substitution (Robbins t_i's) he was able to write the pentagon and hexagon area equations in a compact form. He wrote his formulas also as a discriminant of some (still mysterious) qubic. Along these lines in [MRR] it is found that for $(2m+1)$–gon or $(2m+2)$–gon, the generalized Heron A-polynomial is the defining polynomial of a certain variety of binary $(2m-1)$–forms with $m-1$ double roots (in some sense it demystify Robbins cubic but its role is still mysterious). In [MRR] a formula for the area polynomial for heptagons and octagons is found in the form of a quotient of two resultants, one of which could be expanded explicitly so far. This exiting result was finished by two of the Robbins collaborators just few months later after Robbins passed away.

Another approach, which uses elimination of diagonals in cyclic polygons, is treated at length in [V1] where among numerous results one also finds an explicit derivation of the Robbins area polynomial for pentagons by using some general properties, developed in that paper, together with a little use of one undetermined coefficient. Independently in [SVV], where an almost forgotten elegant Gauss quadratic pentagon area equation is revived, the Robbins pentagon area formula was obtained with a simpler system of equations by a direct elimination (and MAPLE of course) with no assistance of undetermined coefficient method. In [SVV] also the circumradius and the area times circumradius formulas for pentagons, in terms of symmetric functions of the side lengths squared, are explicitly computed. The diagonal elimination approach seems to be better suited for circumradius computations than for the area computations. By introducing diagonals into play the original side length variables are separated into groups (symmetry breaking) and, after eliminating diagonals, one needs to use immense computations with symmetric functions to regain the symmetry . In [S] we have designed an algorithm, which generalizes the basic algorithm for writing symmetric functions in terms of the elementary symmetric functions, which does not expresses everything in terms of the original variables. Instead it goes only down to the level of symmetric functions of the partial alphabets and leads to global symmetric function expansion. This enabled us to get r–polynomials for hexagons (and hopefully more in the future).

In this paper we illustrate yet another approach to the Robbins problem, especially well suited for obtaining Heron r–polynomials. We have discovered that Robbins problem is somehow related to a Wiener–Hopf factorization. We first associate a Laurent polynomial L_P to a cyclic polygon P,which is invariant under similarity of cyclic polygons (it is a kind of "conformal invariant"). Then there exists a (Wiener-Hopf) factorization of L_P into a product of two polynomials, $\gamma_+(1/z)$ and $\gamma_-(z)$, (in our case it will be $\gamma_- = \gamma_+ =: \gamma$) providing a complex realization of P is given. The factorization (i.e. $\gamma(z)$) is then given in terms of the elementary symmetric functions e_k of the vertex quotients, if we regard vertices of (a realization of) P as complex numbers of equal moduli ($= r$). For $e_k's$, viewed as the unknowns, we then obtain a system of n quadratic equations, arising from our Wiener-Hopf factorization, with $n-1$ unknowns (note that e_n is necessarily equal to 1 as a product of all the vertex quotients (we call this a "cocycle property" or simply "cocyclicity")). The consistency condition (obtained by eliminating all $e_k, k = 1..n-1$) for our "overdetermined" system will then give a relation between the coefficients of our conformal invariant L_P, which in turn will be nothing but the equation relating the inverse square radius of P with the side lengths squared.

In the course of these investigations we found another type of substitutions by expressing the coefficients of L_P in terms of the inverse radius squared (ρ) and the elementary symmetric functions of side lengths squared. By using this substitutions, our Heron ρ-polynomials get remarkably small coefficients. Further simplifications we have obtained by doing computations in some quadratic algebraic extensions. In such quadratic extensions we can simplify our original system (having all but one equations quadratic) by replacing two quadratic equations by two linear ones). Also the final result can be written in a more compact form $\rho_n = A_n^2 - \Delta_n B_n^2$ (a Pell equation). Thus the number of terms is the final formula is roughly a square root of the number of terms in the fully expanded formula. With such tricks we have obtained so far , down to earth, explicit formulas for Heron $\rho-$polynomials, up to $n = 8$.

2 Equations for cyclic polygons via Wiener–Hopf factorization

Assume that a cyclic polygon P has its vertices on a circle centered at the origin in the complex plane. Suppose that these vertices are in order $v_1, \ldots, v_n$ and that the radius of the circle is r. Also let $v_{n+1} = v_1$ and define the vertex quotients by

$$q_j = \frac{v_{j+1}}{v_j}. \tag{2.4}$$

The geometric meaning of these vertex quotients are $q_j = \cos\varphi_j + I\sin\varphi_j = e^{I\varphi_j}$, where φ_j denotes the central angle $\sphericalangle(v_j O v_{j+1})$ of P. Then we have the following Cocycle identity:

$$\prod_{j=1}^{n} q_j = 1. \tag{2.5}$$

The side lengths a_j (= the distance from v_j to v_{j+1}) of P are given by

$$\begin{aligned} a_j^2 &= |v_j - v_{j+1}|^2 = (v_j - v_{j+1})\overline{(v_j - v_{j+1})} = r^2\left(2 - \tfrac{v_{j+1}}{v_j} - \tfrac{v_j}{v_{j+1}}\right) \\ &= r^2(2 - (q_j + q_j^{-1})). \end{aligned} \tag{2.6}$$

Now we associate to a cyclic polygon P, with side lengths $a_1, \ldots, a_n$, a Laurent polynomial $L_P(z)$ defined by the following formula:

$$L_P(z) := \prod_{j=1}^{n}(z + z^{-1} + 2 - a_j^2\rho) \ \in \mathbb{C}[z, z^{-1}] \tag{2.7}$$

where $\rho = 1/r^2$ denotes the squared curvature of the circle circumscribed to P.

Note that this polynomial is a conformal invariant in the sense that if cyclic polygons P_1 and P_2 are similar, then $L_{P_1}(z) = L_{P_2}(z)$.

Basic notations:

Denote by e_k the elementary symmetric functions of $v_1, \ldots, v_n$ (vertex variables):

$$1 + e_1 t + e_2 t^2 + \cdots + e_n t^n = \prod_{j=1}^{n}(1 + q_j t) \tag{2.8}$$

and by ε_k the elementary symmetric functions of $a_1^2, \ldots, a_n^2$ (side lengths squared):

$$1 + \varepsilon_1 t + \varepsilon_2 t^2 + \cdots + \varepsilon_n t^n = \prod_{j=1}^{n}(1 + a_j^2 t) \tag{2.9}$$

Lemma 2.1 *(Additive form of L_P). We have*

$$L_P(z) = \sum_{-n \le k \le n} \lambda_k z^k = \lambda_0 + \sum_{k=0}^{n} \lambda_k (z^k + z^{-k}) \tag{2.10}$$

where

$$\lambda_{-k} = \lambda_k = \sum_{i=k}^{n} \binom{2i}{i-k} (-1)^{n-i} \varepsilon_{n-i} \rho^{n-i} \quad (0 \le k \le n). \tag{2.11}$$

(Note that $\lambda_N = \lambda_{-n} = 1$.)

Proof .
We compute

$$\begin{aligned}
L_P(z) &= \prod_{j=1}^{n} (z + z^{-1} + 2 - a_j^2 \rho) = \prod_{j=1}^{n} ((1+z)^2 z^{-1} - a_j^2 \rho) \\
&= \sum_{0 \le i \le n} (1+z)^{2i} z^{-i} e_{n-i}(a_1^2, \ldots, a_n^2)(-\rho)^{n-i} \\
&= \sum_{0 \le i \le n} \left(\sum_{0 \le j \le 2i} \binom{2i}{j} z^{i-j} \varepsilon_{n-i} (-\rho)^{n-i} \right) \\
&= \sum_{0 \le i \le n} \binom{2i}{i} \varepsilon_{n-i} (-\rho)^{n-i} + \sum_{1 \le k \le n} \left(\sum_{k \le i \le n} \binom{2i}{i-k} \varepsilon_{n-i} (-\rho)^{n-i} \right) (z^k + z^{-k})
\end{aligned}$$

By equating the coefficients the result follows. ■

If we know the vertex coordinates $v_1, \ldots, v_n$ of P then in terms of the vertex quotients $q_j = v_{j+1}/v_j$ we can factor its Laurent polynomial L_P into a product of two polynomials, one in z and the other in z^{-1}.

Lemma 2.2 *(Multiplicative form of L_P) We have*

$$L_P(z) = \gamma(z^{-1}) \gamma(z) \tag{2.12}$$

where $\gamma(z)$ is the following polynomial

$$\gamma(z) = 1 + e_1 z + e_2 z^2 + \cdots + e_n z^n \tag{2.13}$$

with $e_1, \ldots, e_n$ denoting the elementary symmetric functions of vertex quotients $q_1, \ldots, q_n$ of the cyclic polygon P (note that $e_n = q_1 \cdots q_n = 1$!).

Proof .
We apply the identity

$$z + z^{-1} + q + q^{-1} = q^{-1}(1 + q z^{-1})(1 + q z) \tag{2.14}$$

to each factor of the defining formula (2.7) of $L_P(z)$ and then use the cocycle identity (2.5). ■

By combining both Lemma 2.1 and Lemma 2.2 we obtain the following

Theorem 2.3 *The quantities $e_0 = 1, e_1, e_2, \ldots, e_{n-1}, e_n = 1$, associated to a cyclic polygon P, defined by (2.8) satisfy the following quadratic system of equations:*

$$\sum_{j=0}^{k} e_{k-j} e_{n-j} = \lambda_{n-k}, \ k = 1..n \tag{2.15}$$

or more explicitly:

$$\begin{array}{c} e_1 + e_{n-1} = \lambda_{n-1} \\ e_2 + e_1 e_{n-1} + e_{n-2} = \lambda_{n-2} \\ \vdots \\ e_{n-1} + e_{n-2}e_{n-1} + \cdots + e_1 e_2 + e_1 = \lambda_1 \\ 1 + e_1^2 + e_2^2 + \cdots + e_{n-1}^2 + 1 = \lambda_0 \end{array} \tag{2.15'}$$

where $\lambda_0, \lambda_1, \ldots, \lambda_n$ are defined by (2.11).

Proof .
By comparing the coefficients of z^{n-1}, z^{n-2}, ..., z, 1 in the factorization resulting Lemma 2.1 and Lemma 2.2 which explicitly looks as:

$$\begin{array}{l} (1 + \frac{e_1}{z} + \frac{e_2}{z^2} + \cdots + \frac{e_n}{z^n})(1 + e_1 z + e_2 z^2 + \cdots + e_n z^n) = \\ \lambda_0 + \lambda_1(z + z^{-1}) + \lambda_2(z^2 + z^{-2}) + \cdots + \lambda_n(z^n + z^{-n}) \end{array}$$

and using that $e_0 = e_n = 1$. ■

Example 2.4 *For $n = 3$ we get the following system:*

$$\begin{array}{l} e_1 + e_2 = \lambda_2 \\ e_2 + e_1 e_2 + e_1 = \lambda_1 \\ e_1^2 + e_2^2 + 2 = \lambda_0 \end{array} \tag{Eq$_3$}$$

with

$$\begin{array}{lll} \lambda_0 = \sum_{i=0}^{3} \binom{2i}{i} (-1)^{3-i} \varepsilon_{3-i} \rho^{3-i} & = -\varepsilon_3 \rho^3 + 2\varepsilon_2 \rho^2 - 6\varepsilon_1 \rho + 20 \\ \lambda_1 = \sum_{i=1}^{3} \binom{2i}{i-1} (-1)^{3-i} \varepsilon_{3-i} \rho^{3-i} & = \varepsilon_2 \rho^2 - 4\varepsilon_1 \rho + 15 \\ \lambda_2 = \sum_{i=2}^{3} \binom{2i}{i-2} (-1)^{3-i} \varepsilon_{3-i} \rho^{3-i} & = -\varepsilon_1 \rho + 6 \end{array} \tag{Λ_3}$$

By eliminating e_1, e_2 from the (dependent!) system (Eq$_3$) above we obtain

$$\lambda_2^2 + 2\lambda_2 - 2\lambda_1 + 2 - \lambda_0 = 0 \tag{2.16}$$

By substituting for λ_0, λ_1, λ_2 from (Λ_3) into (2.16) we obtain

$$\rho^2(\varepsilon_3 \rho + \varepsilon_1^2 - 4\varepsilon_2) = 0$$

*Since ρ $(= 1/r^2)$ is nonzero we end up with the Heron formula (**??**) for inverse radius squared:*

$$\varepsilon_3 \rho + \varepsilon_1^2 - 4\varepsilon_2 = 0$$

written in terms of elementary symmetric functions $\varepsilon_1 = a_1^2 + a_2^2 + a_3^2$, $\varepsilon_2 = a_1^2 a_2^2 + a_2^2 a_3^2 + a_3^2 a_1^2$, $\varepsilon_2 = a_1^2 a_2^2 a_3^2$.

This example shows the main feature of our Wiener-Hopf type approach to Robbins circumradius of cyclic polygons problem.

We may hope that simply by eliminating $e_1, \dots, e_{n-1}$ from the system (2.15) of Theorem 2.3 we would get an equation for the circumradius of general cyclic polygons. But elimination from such a "simple" quadratic system may be computationally very demanding even for a very powerful computers today. Further notation: The special values for $y = \pm 1$ of the polynomial $\gamma_P(z)$ we denote by

$$Y_n := \gamma_P(1) = 2 + e_1 + e_2 + \cdots + e_{n-1} \tag{2.17}$$

$$\Theta_n := \gamma_P(-1) = 1 + (-1)^n - e_1 + e_2 + \cdots + (-1)^{n-1} e_{n-1} \tag{2.18}$$

$$\Delta_n = \sum_{j=0}^{n} 4^{n-j} (-1)^j \varepsilon_j \rho^j$$

Then, from the factorization $L_P(\pm 1) = \gamma_P(\pm 1)^2$ we immediately get

$$Y_n^2 = \lambda_0 + 2(\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} + 1) = \Delta_n \tag{2.19}$$

$$\Theta_n^2 = (-1)^n \varepsilon_n \rho^n \tag{2.20}$$

If we adjoin to our quadratic system, from Theorem 2.3, two linear equations, resulting from (2.17) and (2.18):
Auxiliary equations:

$$\begin{aligned} e_1 + e_2 + \cdots + e_{n-1} &= Y_n - 2 \\ -e_1 + e_2 + \cdots + (-1)^{n-1} e_{n-1} &= \Theta_n - 1 - (-1)^n \end{aligned} \tag{2.21}$$

For example for $n = 3$ the two auxiliary equations are:

$$\begin{aligned} e_1 + e_2 &= Y_3 - 2 & \text{with } Y_3^2 &= \lambda_0 + 2(\lambda_1 + \lambda_2 + 1) \\ -e_1 + e_2 &= \Theta_3 & \text{with } \Theta_3^2 &= -\varepsilon_3 \rho^3 \end{aligned} \tag{2.22}$$

and we obtain immediately

$$\lambda_2 + 2 - Y_3 = 0 \tag{2.23}$$

This gives us a new form of the classical Heron formula for the circumradius:

$$\rho_3 = \boxed{\rho^{-2}(A_3^2 - \Delta_3 B_3^2) = 0} \tag{2.24}$$

where

$$A_3 := \lambda_2 + 2, \quad B_3 = 1, \; \textit{and} \; \Delta_3 = Y_3^2 = \lambda_0 + 2(\lambda_1 + \lambda_2 + 1). \tag{2.25}$$

This new derivation of the classical Heron formula explains some features of our approach to Robbins problem. We are intending to write a final result in the form

$$\rho_n = \boxed{\rho^{-2^{n-2}}(A_n^2 - \Delta_n B_n^2) = 0} \tag{2.26}$$

which is much shorter than if we would expand A_n^2 and B_n^2. Without auxiliary equations we would get the formula in the expanded form which may not be explicitly computable on a computer at our disposal.

Cyclic quadrilaterals ($n = 4$)
Now by eliminating e_1, e_2, e_3 from the basic system

$$EQ4 = \{e_1+e_3-\lambda_3, e_2+e_1e_3+e_2-\lambda_2, e_3+e_2e_3+e_1e_2+e_1-\lambda_1, e_1^2+e_2^2+e_3^2-\lambda_0\}$$

$$\lambda_3^4 - 2\lambda_3^2\lambda_2 - \lambda_3^2\lambda_0 - \lambda_3^2 + 2\lambda_3\lambda_1 + \lambda_1^2 = 0$$

With auxiliary equations

$$e_1 + e_2 + e_3 = Y_4 - 2, \quad Y_4^2 = \lambda_0 + 2(\lambda_1 + \lambda_2 + \lambda_3 + 1)$$

we get

$$\rho_4 = \rho^{-4}(A_4^2 - \Delta_4 B_4^2) = 0$$

where

$$A_4 := \lambda_3^2 + \lambda_3 + \lambda_1, \quad B_4 = \lambda_3$$

Remark 2.5 *If we substitute* $\lambda_0 = \varepsilon_4\rho^4 - 2\varepsilon_3\rho^3 + 6\varepsilon_2\rho^2 - 20\varepsilon_1\rho + 70$, $\lambda_1 = -\varepsilon_3\rho^3 + 4\varepsilon_2\rho^2 - 15\varepsilon_1\rho + 56$, $\lambda_2 = \varepsilon_2\rho^2 - 6\varepsilon_1\rho + 28$, $\lambda_3 = 8 - \varepsilon_1\rho$, *and* $\varepsilon_4 = \eta_4^2$ *we obtain*

$$\begin{aligned}\rho_4 &= (\varepsilon_3\rho + \varepsilon_1^2 - 4\varepsilon_2 + \eta_4(8 - \varepsilon_1\rho))(\varepsilon_3\rho + \varepsilon_1^2 - 4\varepsilon_2 - \eta_4(8 - \varepsilon_1\rho)) \\ &= \rho_4^+\rho_4^-\end{aligned}$$

where ρ_4^+ *corresponds to convex quadrilaterals and* ρ_4^- *to nonconvex quadrilaterals. Note also the following property:*

$$\rho_4 = \rho_3^2 - \varepsilon_4(8 - \varepsilon_1\rho)^2.$$

Cyclic pentagons ($n = 5$)
By eliminating $e_1, \ldots, e_4$ from the basic system for cyclic pentagon we obtain a polynomial in $\lambda_0, \ldots, \lambda_4$ having 119 terms and coefficients between -20 and 32. By substituting $\lambda_k = \sum_{i=k}^{5} \binom{2i}{i-k}(-1)^{5-i}\varepsilon_{5-i}\rho^{5-i}$ $(0 \le k \le 4)$ we obtain a ρ^8 times a polynomial of degree 7 in ρ having 81 terms and coefficients between -16384 and 8192.

By using auxiliary equations we obtain a much shorter expression (with coefficients ±1, ±2, ±3, ±4)

$$\rho_5 = \rho^{-8}(A_5^2 - B_5^2\Delta_5)$$

where

$$\begin{aligned}A_5 &= \lambda_4^4 + (-3\lambda_3 + 2\lambda_2 + \lambda_1 - 3)\lambda_4^2 + (-2\lambda_3 - 4\lambda_1 + 2)\lambda_4 + \\ &\quad + 2\lambda_3^2 + (-2\lambda_2 - 2\lambda_1 + 4)\lambda_3 + \lambda_2^2 + 2\lambda_2 - 2\lambda_1 + (\lambda_3 + 3)\lambda_0 + 2 \\ B_5 &= -\lambda_4^3 + 2\lambda_4^2 + (2\lambda_3 - \lambda_2)\lambda_4 - 2\lambda_3 + 2\lambda_1 - \lambda_0 - 2 \\ \Delta_5 &= Y_5^2 = \lambda_0 + 2(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 1)\end{aligned}$$

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