Finite 2-groups all of whose proper subgroups have commutator groups of order \( \leq 2 \).

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1. Introduction and known results

Denote by \( G_2 \) the set of finite 2-groups, and for any group \( G \) denote by \( \mathcal{M}(G) \) the set of its maximal subgroups. We consider here the groups which satisfy the condition stated in the title, which is equivalent with the following one:

\[
G \in G_2 \text{ and if } M \in \mathcal{M}(G) \text{ then } |M'| \leq 2.
\]

(1)

We prove the following result, which describes the structure of such groups.

**Theorem.** Let \( G \) be a 2-group whose all proper subgroups have commutator groups of order \( \leq 2 \). Then we have one of the following cases:

1) if \( |G'| = 2 \) then \( G = H_1 \ast \cdots \ast H_n \cdot Z(G), \) \( H_i \) \( (i = 1, 2, \ldots, n) \)

being minimal nonabelian subgroups of \( G \), and \( \ast \) denoting the central product.

2) \( d(G) = 3, \ G' \cong E_8, \ \Phi(G) \leq Z(G), \ G = \langle a_1, a_2, a_3 \rangle, \ [a_2, a_3] = z_1, \ [a_3, a_1] = z_2, \ [a_1, a_2] = z_3, \ G' = \langle z_1, z_2, z_3 \rangle. \) The maximal subgroups of \( G \) are all nonabelian and for maximal \( M, N \leq G, M \neq N \) it is \( M' \neq N'. \)

3) \( d(G) = 3, \ G' \cong E_4, \ \Phi(G) \leq Z(G), G = \langle a_1, a_2, a_3 \rangle, \ [a_2, a_3] = z_1, \ [a_3, a_1] = z_2, \ [a_1, a_2] = 1, \ G' = \langle z_1, z_2 \rangle. \) There is one abelian maximal subgroup \( M_1 = \langle a_1, a_2, \Phi(G) \rangle \) and the remaining 6 are nonabelian and divided in 3 pairs, each pair having the same commutator group.

4) \( d(G) = 2, \ G' \cong E_4 \) or \( G' \cong Z_4, \ G = \langle a_1, a_2 \rangle. \) There are 3 maximal subgroups \( M_1, M_2, M_3 \) such that \( K = M_1^2 M_2^2 M_3^2 = \langle z \rangle \cong Z_2, \ K < G' \)

and \( G/K \) is minimal nonabelian 2-group.
In the proof of this theorem we shall use following known results:

**Proposition 1.** (Janko [2, Proposition 1.7]) Let $G$ be a nonabelian finite 2-group possessing an abelian maximal subgroup. Then $|G| = 2|G'| \cdot |Z(G)|$.

**Proposition 2.** (A. Mann, see Berkovich [1]) Let $G$ be a finite 2-group, $M, N$ any two different maximal subgroups of $G$. Then $|G : M'N'| \leq 2$.

**Proof.** Since $M, N \lhd G$, therefore $M'N' \leq G$. Also $M'N' \leq G' \leq \Phi(G)$. Thus $M/M'N'$ and $N/M'N'$ are abelian and maximal in $G/M/M'N'$. Obviously, $|\overline{G} : Z(\overline{G})| \leq 4$, and so by Proposition 1, $|\overline{G}'| = |G'/M'N'| \leq 2$.

**Proposition 3.** (Janko [3, Proposition 1.6]) Let $G$ be a 2-group with $|G'| = 2$. If $H$ is a minimal nonabelian subgroup of $G$, then $G = HC_G(H)$ and $|G : C_G(H)| = 4$.

**Proof.** Each minimal nonabelian group $H$ is 2-generated: For $x_1, x_2 \in H$, $[x_1, x_2] \neq 1$ is $\langle x_1, x_2 \rangle$ nonabelian and thus $H = \langle x_1, x_2 \rangle$ because of its minimality. Denote $C_i = C_G(x_i)$, $i = 1, 2$. Because of $|G'| = 2$ and $|x^G| = |G : C_G(x)| = |\{x^g | g \in G\}| = |\{x[x, g] | g \in G\}| \leq |G'| = 2$, we have $|C : C_i| = 2$ for $i = 1, 2$. Considering $C = C_1 \cap C_2$, we have $C = C_G(H)$, $|G : C| \leq 4$, $H \cap C = Z(H)$. Since $H$ is nonabelian $|H : Z(H)| \geq 4$ and so $|HC| = (|C| : |H|) : |H \cap C| \geq 4|C|$. Therefore $|G : C| = |H : Z(H)| = 4$ and $G = HC$.

**Proposition 4.** Let $G$ be a finite 2-group, $G' \leq Z(G)$ and $exp G' = 2$. Then $\Phi(G) \leq Z(G)$.

**Proof.** Let $x, g \in G$. Then $[x, g^2] = [x, g][x, g]^g = [x, g]^2 = 1$, as $[x, g] \in G'$. Since $\Phi(G) = \Omega_1(G) = \langle g^2 | g \in G \rangle$ for any 2-group $G$ and $g^2 \in Z(G)$ for all $g \in G$, we have $\Phi(G) \leq Z(G)$.

**Proposition 5.** Let $G$ be a 2-generated finite 2-group and $|G'| = 2$. Then $G$ is minimal nonabelian.

**Proof.** Let $G = \langle a, b \rangle$. By Proposition 4, $\Phi(G) \leq Z(G)$. As $\Phi(G)$ is maximal in all 3 maximal subgroups $M_1 = \langle a, \Phi(G) \rangle$, $M_2 = \langle b, \Phi(G) \rangle$, $M_3 = \langle ab, \Phi(G) \rangle$ of $G$, they are all abelian and so $G$ is minimal abelian.
2. Proof of the Theorem

We prove our Theorem in several steps.

(i) The case $|G'| = 2$.
Let $H_1 = \langle a_1, b_1 \rangle$ be a minimal nonabelian subgroup of $G$.
Then, by Proposition 3, $G = H_1C_G(H_1)$; if $C_G(H_1)$ is abelian, so $C_G(H_1) = Z(G)$ and we have $G = H_1Z(G)$. Otherwise, let $H_2 = \langle a_2, b_2 \rangle$ be a minimal nonabelian subgroup of $C_G(H_1)$. By the same Proposition 3 we have $C_G(H_1) = C_G(H_2) \cdot (C_G(H_1) \cap C_G(H_2)) = H_2 \cdot C_G(⟨H_1, H_2⟩)$, and so $G = H_1 * H_2 \cdot C_G(⟨H_1, H_2⟩)$. Continuing in the same way we get finally $G = H_1 * H_2 * * H_n \cdot C_G(⟨H_1, ..., H_n⟩)$, the last factor being abelian and so equal $Z(G)$. This proves the assertion 1) of the Theorem.

(ii) The order of $G'$ is at most 8.
Proof. Let $M, N$ be two different maximal subgroups of $G$. By assumption $|M'|, |N'| \leq 2$ and $M', N' \leq G$ so $|M'N'| \leq 4$. By Proposition 2 it is $|G' : M'N'| \leq 2$ and so $|G'| \leq 8$.
In the following we denote $K = \langle M' | M \in \mathcal{M}(G) \rangle$, the group generated by commutator groups of all maximal subgroups of $G$. Obviously, $K \leq Z(G)$ and $\exp K = 2$.

(iii) If $G' = K$ and $|K| \geq 4$, then $d(G) = 3$. Moreover $\Phi(G) \leq Z(G)$.
Proof. Let $M, N$ be maximal subgroups of $G$ with $M'N' \cong E_4$, $M' = \langle z_1 \rangle$, $N' = \langle z_2 \rangle$. Then there exist elements $a, b \in M$, $c, d \in N$ such that $[a, b] = z_1$, $[c, d] = z_2$. Now, $H = \langle a, b, c, d \rangle \leq G$ and $H' \geq \langle z_1, z_2 \rangle$, $|H'| \geq 4$.
Consequently, $H = G = \langle a, b, c, d \rangle$. Consider $H_1 = \langle a, b, c \rangle$ and $H_2 = \langle b, c, d \rangle$. Now $[b, c] \leq \langle a, b, c \rangle' \cap \langle b, c, d \rangle'$. If $H_1$ and $H_2$ are different from $G$, then $[b, c] \leq H_1' \cap H_2' = \langle z_1 \rangle \cap \langle z_1 \rangle = 1$, so $[b, c] = 1$.
Similary, $[a, c] = [a, d] = [b, d] = 1$. Consider $H = \langle ac, b, d \rangle$. Here, $[ac, b] = [a, b] = z_1$, $[ac, d] = [c, d] = z_2$, so $|H'| > 2$ and therefore $H_3 = G = \langle ac, b, d \rangle$.
We see that $d(G) \leq 3$. If $d(G) = 2$, then $G = \langle x_1, x_2 \rangle$ and $|x_1, x_2| = z \in G' = K$, implying $G' = \langle [x_1, x_2] \rangle = z \cong Z_3$, a contradiction. So, $d(G) = 3$.
From Proposition 3 we see immediately that $\Phi(G) \leq Z(G)$.

(iv) If $|K| \geq 4$ then $G' = K$.
Proof. For $|K| = 8$ it is trivial, as $|G'| \leq 8$. Suppose $|K| = 4$ and $|G'| = 8$. So $G' \cong E_4$. Let $a, b \in G$ such that $[a, b] = c \in G' \cap K$. If $\langle a, b \rangle \leq M$ for some maximal $M \leq G$ then $c = [a, b] \leq M' \leq K$. Thus $\langle a, b \rangle = G$ and $d(G) = 2$. There are 3 maximal subgroups in $G$:

$$M_1 = \langle a, \Phi(G) \rangle, \ M_2 = \langle b, \Phi(G) \rangle, \ \text{and} \ M_3 = \langle ab, \Phi(G) \rangle.$$

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As \( (G/K)' = G'/K \cong Z_2 \), it follows \( G'/K \leq Z(G/K) \) and so \([c, x] \in K \leq Z(G)\) for every \( x \in G\). Therefore \([c, x^2] = [c, x][c, x]^2 = [c, x]^2 = 1\). Now \([a, b^2] = [a, b][a, b] = cc^b \in M_1, \ [a^2, b] = [a, b]^a[a, b] = cc^a \in M_2, \ [a^2, ab] = [a^2, b] = cc^a \in M_3 \cap M'_2\) and \([ab, b^2] = [a, b^2]^b = c^b b^2 = c c^b \in M_2 \cap M'_1\).

If \( cc^a \neq 1 \) or \( cc^b \neq 1 \), then \( M_3 = M'_2 \) or \( M'_3 = M'_1 \), respectively, and so, by Proposition 2, \( |G'| = 4 \). Thus \( cc^a = cc^b = 1 \), and so \( c^a = c^{-1} = c^b \), and \( c^{ab} = c \). It follows that \( G' = \langle c^G \rangle = \langle c \rangle \) is cyclic, a contradiction. Thus \( G' = K \).

\( (v) \) Case \( G' = K \cong E_8 \)

Since \( d(G) = 3 \) and \( M' \neq N' \), \( M'N' \cong E_4 \) for different maximal subgroups \( M, N \), each involution in \( G' \) generates commutator group for exactly one of 7 maximal subgroups in \( G \). Thus we have, without loss of generality:

\[
G = \langle a_1, a_2, a_3 | a_i^{m_i} = z_1^{\delta_i} z_2^{\varepsilon_i} z_3^{\zeta_i}, \ [a_i, a_j] = z_k, \ \langle z_1, z_2, z_3 \rangle = G' \rangle
\]

where \( i = 1, 2, 3 \) and \( \{i, j, k\} = \{1, 2, 3\}, \ \delta_i, \varepsilon_i, \zeta_i \in \{0, 1\} \) and \( m_i \) being the order of \( a_i \) in \( G = G/K \). The established facts prove the part 2) of our Theorem.

\( (vi) \) Case \( G' = K \cong E_4 \)

Again, by (iii) \( d(G) = 3 \) and \( \Phi(G) \leq Z(G) \).

By Proposition 2 there cannot exist more than one abelian maximal subgroup in \( G \). Thus there exist two maximal subgroups \( M_1, M_2 \) with \( M_1' = M_2' = \langle z \rangle \), \( 1 \neq z \in G' \). Denote by \( x_3 \) an element of \( M_1 \cap M_2 \setminus \Phi(G) \) and \( x_1 \in M_1 \setminus M_2, \ x_2 \in M_2 \setminus M_1 \). So \( M_1 = \langle x_1, x_3, \Phi(G) \rangle, \ M_2 = \langle x_2, x_3, \Phi(G) \rangle \). We have \( [x_1, x_3] = [x_2, x_3] = z \) and \( [x_1, x_2, x_3] = [x_1, x_3][x_2, x_3] = z \cdot z = 1 \).

For \( M_3 = \langle x_1, x_2, x_3, \Phi(G) \rangle \), which is also maximal in \( G \), it is \( M_3' = \langle x_1, x_2, x_3 \rangle' = 1 \). We see that there is a unique abelian maximal subgroup in \( G \).

After some renaming of generators we get the following relations for \( G \):

\[
G = \langle a_1, a_2, a_3 | a_i^{m_i} = z_1^{\delta_i} z_2^{\varepsilon_i}, \ [a_1, a_2] = 1, \ [a_1, a_3] = z_1, \ [a_2, a_3] = z_2 \rangle
\]

where \( i = 1, 2, 3, \ \delta_i, \varepsilon_i \in \{0, 1\} \) and \( m_i \) being the order of \( a_i \) in \( G = G/K \).

One can easily check that besides \( M = \langle a_1, a_2, \Phi(G) \rangle \) which is abelian, the other six maximal subgroups are all nonabelian and are divided in 3 pairs with commutator groups \( \langle z_1 \rangle, \langle z_2 \rangle \) and \( \langle z_1, z_2 \rangle \), respectively. This proves the part 3) of the Theorem.

\( (vii) \) Case \( G' > K \)

By (iv) and Proposition 2 we have in this case \( |G'| = 4 \) and \( |K| = 2 \). If \( [a, b] = c \in G' \setminus K \) then \( G = \langle a, b \rangle, d(G) = 2 \), since otherwise \( [a, b] \) would be in \( K = \langle M' \rangle | M \) maximal in \( G \). Now \( (G/K)' = G'/K \cong Z_2 \) and \( G/K \) is 2-generated. Applying Proposition 5 we see that \( G/K \) is minimal nonabelian. This proves the part 4) of the Theorem.
The Theorem is proved.

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REFERENCES


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