Humans have a remarkable capability to reason and make decisions in an environment of uncertainty, imprecision, incompleteness of information, and partiality of knowledge, truth and class membership.

The principal objective of fuzzy logic is formalization of this capability.

Examples
- The weather is cloudy and mostly cloudy
- The water is warm
- This girl is very tall
- The concert starts around 10pm

The fuzzy set theory and possibility theory represent very convenient framework for dealing with knowledge described in this way [Zadeh]

The term fuzzy logic emerged in the development of the theory of fuzzy sets by Lotfi Zadeh (1965)

A type of logic that recognizes more than simple true and false values
- With fuzzy logic, propositions can be represented with degrees of truthfulness
- For example, the statement, today is sunny, might be 100% true if there are no clouds, 80% true if there are a few clouds, 50% true if it's hazy and 0% true if it rains all day.

Two main directions in fuzzy logic have to be distinguished (Zadeh 1994).
- Fuzzy logic in the broad sense (older, better known, heavily applied but not asking deep logical questions) serves mainly as apparatus for fuzzy control, analysis of vagueness in natural language and several other application domains.
- It is one of the techniques of soft-computing, i.e., computational methods tolerant to suboptimality and impreciseness (vagueness) and giving quick, simple and sufficiently good solutions.

Fuzzy logic in the narrow sense is symbolic logic with a comparative notion of truth developed fully in the spirit of classical logic (syntax, semantics, axiomatization, truth-preserving deduction, completeness, etc.; both propositional and predicate logic).
- It is a branch of many-valued logic based on the paradigm of inference under vagueness.
- This fuzzy logic is a relatively young discipline, both serving as a foundation for the fuzzy logic in a broad sense and of independent logical interest, since it turns out that strictly logical investigation of this kind of logical calculi can go rather far.
Fuzzy sets

- **Fuzzy sets** are sets whose elements have degrees of membership.
- Fuzzy set theory permits the gradual assessment of the membership of elements in a set.
  - This is described with the aid of a membership function valued in the real unit interval \([0, 1]\), instead of set \([0, 1]\) in classical (crisp) sets.
- Example: A fuzzy set of young people

\[
\mu_B(x) \text{- membership function}
\]

Classical set

Fuzzy set

A degree of membership of element \(x\) in fuzzy set \(A\) is expressed by membership function.

Let \(X\) be the universal set (universe of discourse), and its elements are denoted by \(x\).

Classical set \(A\) might be expressed as

\[
A = \{(x, \mu_A(x)) \mid x \in X, \mu_A(x) : X \to [0,1]\}
\]

Fuzzy set \(A\) is expressed as

\[
A = \{(x, \mu_A(x)) \mid x \in X, \mu_A(x) : X \to [0,1]\}
\]

Basic operations on fuzzy sets: complement, intersection and union.

- A complement of a fuzzy set \(A \subseteq X\), \(A^c \subseteq X\) is

\[
\mu_{A^c}(x) = 1 - \mu_A(x), \forall x \in X
\]

An intersection of two fuzzy sets \(A \cap B\), \(A, B \subseteq X\), \(A \cap B \subseteq X\) is

\[
\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x)), \forall x \in X
\]

A union of two fuzzy sets \(A \cup B\), \(A, B \subseteq X\), \(A \cup B \subseteq X\) is

\[
\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x)), \forall x \in X
\]

One of the most commonly used examples of a fuzzy set is the set of tall people.

- In this case, the universe of discourse is all potential heights, say from 120 cm to 240 cm, and the word tall would correspond to a curve that defines the degree to which any person is tall.
- However, such a distinction is clearly absurd.
- It may make sense to consider the set of all real numbers greater than 185 because numbers belong on an abstract plane, but when we want to talk about real people, it is unreasonable to call one person short and another one tall when they differ in height by the width of a hair.
Subjective interpretations and appropriate units are built right into fuzzy sets. If you say "She's tall," the membership function tall should already take into account whether you are referring to a six-year-old or a grown woman.

If – then rules

- Fuzzy sets and fuzzy operators are the subjects and verbs of fuzzy logic.
- If–then rule statements are used to formulate the conditional statements that comprise fuzzy logic.

- A single fuzzy If–then rule assumes the form
  
  \[ \text{If } x \text{ is } A \text{ then } y \text{ is } B \]

- where \( A \) and \( B \) are linguistic values defined by fuzzy sets on the ranges (universes of discourse) \( X \) and \( Y \), respectively.

- The if-part of the rule "\( x \) is \( A \)" is called the antecedent or premise, while the then-part of the rule "\( y \) is \( B \)" is called the consequent or conclusion.

- An example of such a rule might be:
  
  - If service is good then tip is average.
  - The consequent of a rule can also have multiple parts
  
  e.g. If temperature is cold then hot water valve is open and cold water valve is shut.

Fuzzy inference system

- Fuzzy inference is the process of formulating the mapping from a given input to an output using fuzzy logic.

- The mapping then provides a basis from which decisions can be made, or patterns discerned.

- Fuzzy inference process comprises of five parts:
  
  1. fuzzification of the input variables
  2. application of the fuzzy operator (AND or OR) in the antecedent
  3. implication from the antecedent to the consequent
  4. aggregation of the consequents across the rules
  5. defuzzification

EXAMPLE

- Step 1: Fuzzify Inputs
  
  The first step is to take the inputs and determine the degree to which they belong to each of the appropriate fuzzy sets via membership functions.

  - This example is built on three rules, and each of the rules depends on resolving the inputs into a number of different fuzzy linguistic sets: service is poor, service is good, food is rancid, food is delicious, and so on.

  - Before the rules can be evaluated, the inputs must be fuzzified according to each of these linguistic sets. For example, to what extent is the food really delicious?

- Step 2: Apply Fuzzy Operator
  
  After the inputs are fuzzified, you know the degree to which each part of the antecedent is satisfied for each rule.

  - If the antecedent of a given rule has more than one part, the fuzzy operator is applied to obtain one number that represents the result of the antecedent for that rule.

  - This number is then applied to the output function.

  - The input to the fuzzy operator is two or more membership values from fuzzified input variables.

  - The output is a single truth value.
Step 3. Apply Implication Method
- The consequent is reshaped using a function associated with the antecedent (a single number)
- The input for the implication process is a single number given by the antecedent, and the output is a fuzzy set.
- Implication is implemented for each rule.

Step 4. Aggregate All Outputs
- Decisions are based on the testing of all of the rules in a FIS.
- The rules must be combined in some manner in order to make a decision.
- An aggregation is the process by which the fuzzy sets that represent the outputs of each rule are combined into a single fuzzy set.
- An aggregation only occurs once for each output variable, just prior to the fifth and final step, defuzzification.
- The input of the aggregation process is the list of truncated output functions returned by the implication process for each rule.
- The output of the aggregation process is one fuzzy set for each output variable.

Step 5. Defuzzify
- The input for the defuzzification process is a fuzzy set (the aggregate output fuzzy set) and the output is a single number.
- As much as fuzziness helps the rule evaluation during the intermediate steps, the encompasses a range of output values, and so must be defuzzified in order to resolve a single output value from the set.
- Perhaps the most popular defuzzification method is the centroid calculation, which returns the center of area under the curve.

History
- In 1965, Lotfi Zadeh of the University of California at Berkeley published "Fuzzy Sets," which laid out the mathematics of fuzzy set theory and, by extension, fuzzy logic.
- Zadeh had observed that conventional computer logic couldn't manipulate data that represented subjective or vague ideas, so he created fuzzy logic to allow computers to determine the distinctions among data with shades of gray, similar to the process of human reasoning.
- The first industrial application,
  - A cement kiln built in Denmark, coming on line in 1975.
  - Sendai railway (the first subway system was built which worked with a fuzzy logic)-fuzzy systems were used to control accelerating, braking, and stopping when the line opened in 1987.
- It was a big success and resulted in a fuzzy boom.

Fuzzy boom
- Universities as well as industries got interested in developing the new ideas.
- First, this was mainly the case in Japan.
- Since the religion in Japan accept that things can contain parts of their opposites, it wasn’t such a frightening idea as in most other parts of the world.
- And fuzzy logic promised lots of money to the industries, which was of course welcome too.
- Today, almost every intelligent machine has fuzzy logic technology inside it.
One of the most important applications of Fuzzy Logic was Yamaichi Fuzzy Fund.

This is the premier financial application for trading systems, handling 65 industries and a majority of the stocks listed on Nikkei Dow and consists of approximately 800 fuzzy rules. Rules are determined monthly by a group of experts and modified by senior business analysts as necessary. The system was tested for two years, and its performance in terms of the return and growth exceeds the Nikkei Average by over 20%. While in testing, the system recommended "sell" 18 days before the black Monday in 1987 (October 19), when Dow Jones declined 23%.

The system went to commercial operations in 1988.

The greatest success of fuzzy set theory has been in the engineering area of control systems.

One area in which fuzzy controllers quickly established an excellent reputation in the early 1990s is the broad area of consumer products.

Among the earliest consumer products equipped with fuzzy controllers were washing machines.

Next example describes the basic idea of the use of fuzzy controllers in washing machines. In a conventional washing machine, the time of each run is set by the user. If insufficient time is set for a given load of clothes, they are not properly washed. If the washing time is too long, the time and energy are wasted and the machine as well as the clothes are unnecessarily worn out.

Different types of washing machines are commercially available and their control capabilities vary quite substantially. In next example is described the operation of a very simple fuzzy controller, whose purpose is to determine the proper operating time of the washing machine for each load of clothes.

Once the washing machine is loaded with dirty laundry, it begins to calculate how dirty the laundry is and how long it would take to wash it. If a machine takes ten minutes to wash clothes, it calculates how dirty the laundry is. If the load is 100% dirty, then it adds two minutes per piece of dirty laundry to the wash cycle that would have taken ten minutes originally. If the laundry is 50% dirty, then it would add 50% of two minutes. This means a minute to the ten-minute wash cycle. If the laundry is greasy then an additional two minutes are added to the cycle. If the laundry is dirty and greasy, then the machine factors in 4 additional minutes to the entire load.

A fuzzy machine would also have to take into account the amount of soap it takes for each load. A fuzzy washing machine would also have to take into account the amount of soap it takes for each load. A typical rule that could be inserted in a fuzzy logic system is:

"If the FED keeps high rates, and the official discount rate is low, than the short term rates in USA will rise a lot."

The number of these rules is very high (thousands), and needs really powerful computers to evaluate the respective degree of truth, day after day, to give an advice reasonably fast.

The success of fuzzy set theory has been in the engineering area of technical systems, and is now generally recognized.

Western world also accepts fuzzy logic.

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Fuzzy logic has made functions of washing machines more economical, energy-saving and less time-consuming.
The operating time of a washing machine depends on two properties of each given load of clothes:
- It depends on how dirty the clothes are.
- It depends on the type of soil.

The degree of dirtiness is measured by a special sensor via the degree of water transparency:
- The less transparent the water, the dirtier the clothes.

The type of soil is determined by measuring the time needed, after the machine has started, to reach a state in which the water transparency remains constant:
- This time is called saturation time and is different for different types of soil.


Assume that:
- the degree of dirtiness $d$ is expressed by a number in the interval $[0, d_{\text{max}}]$.
- And that we deal only with three levels of dirtiness (expressed in natural language): high, medium, and low.

Assume that:
- the saturation time $s$ is expressed by a number in the interval $[0, s_{\text{max}}]$.
- And that we have only three saturation times (expressed in natural language): short, medium, and long.

The required washing time should be some mathematical function of a degree of dirtiness and a saturation time:
- It is impossible to determine this function exactly.

By using a fuzzy controller we can approximate this function with relative ease on the basis of human intuition and experience.

We need to determine another linguistic variable representing the required washing time $t \in [0, t_{\text{max}}]$.

Now, we need to express knowledge of experienced users by conditional fuzzy propositions of the form:
- If $D = \text{low}$ and $S = \text{short}$ then $T = \text{very long}$.
- If $D = \text{medium}$ and $S = \text{medium}$ then $T = \text{long}$.
- If $D = \text{high}$ and $S = \text{short}$ then $T = \text{very short}$.

We obtain overall conclusion by taking the union of all the individual conclusions (the fuzzy set $C_{d,s}(t)$), whose membership function is defined for each $x \in [0, t_{\text{max}}]$.

The graph of this function is shown in Fig. 10.7.

We can use the center of gravity deffuzification method to determine the desirable operating time of the washing machine as determined by the fuzzy controller for conditions $d^*$ and $s^*$.

The washing timer is set to this value.