
Full paper online: http://www.kirp.chtf.stuba.sk/pcl11/data/abstracts/086.html
Abstract: This paper describes a $H_\infty$ controller design procedure for tensor product based model of gantry crane augmented with friction model in order to minimize friction effects. The Tensor Product (TP) model transformation is a recently proposed technique for transforming given Linear Parameter Varying (LPV) state-space models into polytopic model form, namely, to parameter varying convex combination of Linear Time Invariant (LTI) systems. $H_\infty$ controller guarantee stability and $L_2$ norm bound constraint on disturbance attenuation. $H_\infty$ controller is found using relaxed LMIs which have proof of asymptotic convergence to the global optimal controller under quadratic stability. Control algorithm is experimentally tested on single pendulum gantry (SPG).

Keywords: Parallel Distributed Compensation, Linear Matrix Inequalities, Tensor Product (TP) model transformation, gantry crane control, friction compensation.

1. INTRODUCTION

In modern industrial system, gantry cranes are widely used for the heavy loads transfer. Fast load positioning and load swinging minimization are conflicting requirements imposed to the traveling crane control systems.

For the position and anti-sway control of travelling cranes, there are several solutions, i.e., by fuzzy control, optimal control, pole placement, etc. and each of them is reported to be effective (Popadić et al., 2005), (Nalley and Trabia, 2000), (Omar, 2003). All these approaches are based on linear/linearized model of gantry crane.

Recently, nonlinear control approach based on tensor product model representation (TP) of the process is proposed (Baranyi et al., 2003), (Petres, 2006) and successfully applied to control of Single Pendulum Gantry process (Kolonić et al., 2006). The TP model represents the Linear Parameter Varying (LPV) state-space models by the parameter varying combination of Linear Time Invariant (LTI) models.

However, none of these approaches takes into account friction effect, which is unavoidable in real applications (Olsson et al., 1998). This effect may seriously degrade the performance of the control system, specially when high precision positioning is required. As a consequence steady state error due to static friction is common for all these approaches. Introducing integral action in control loop, to eliminate steady state error due to friction effects, may result in limit cycling called hunting phenomenon, see Hensen et al. (2003).

One way to cope with friction phenomena is to introduce friction compensator based on friction model. The friction model may be a-priory known or learned in an on-line manner (Lee and Tomizuka, 1996), (Huang et al., 2000), (Matuško et al., 2010), (Boras et al., 2010).

Another approach consider the friction effect as an unknown disturbance and design the robust controller. In Cheang and Chen (2000), Burul et al. (2010), $H_\infty$ controller for linearised model is synthesized using loop-shaping methodology, and experimentally proved effective.

In this paper a combination of these two approaches is used. It is assumed that friction model is partially known, and nominal model is identified, while its associated uncertainty is considered as a disturbance. In order to cope with nonlinear nature of the gantry crane model, as well as friction model, non-linear model of gantry crane augmented with nominal friction model is rewritten in LPV form, suitable for TP transformation. As a result of TP model transformation, polytopic model of process is obtained. For such model representation, LMI control approach is used to design a controller (Tanaka and Wang, 2001), (Boyd et al., 1994), (Gahinet et al., 2002). In this paper such approach is used to synthesize $H_\infty$ controller.

The paper is organized as follows: Section II discusses the theoretical background of TP model transformation-based control design. Section III introduces the LPV model augmented with friction and its TP transformation. Section IV discusses the LMI relaxations used in controller design. Section V presents the experimental results obtained on the single pendulum gantry (SPG) experimental model, and Section VI concludes this paper.
2. TENSOR PRODUCT MODEL
TRANSFORMATION-BASED CONTROL DESIGN
METHODOLOGY

Consider the linear parameter-varying state-space model

\[
\begin{pmatrix}
\dot{x}(t) \\
y(t)
\end{pmatrix} = S(p(t)) \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}
\]

(1)

with input \( u(t) \in \mathbb{R}^k \), output \( y(t) \in \mathbb{R}^l \) and state vector \( x(t) \in \mathbb{R}^m \). The system matrix

\[
S(p(t)) = \begin{pmatrix} A(p(t)) & B(p(t)) \\ C(p(t)) & D(p(t)) \end{pmatrix} \in \mathbb{R}^{(m+k) \times (m+l)}
\]

(2)

is a parameter-varying object, where \( p(t) \in \Omega \) is time varying parameter vector, where \( \Omega \) is a closed hypercube in \( \mathbb{R}^N \), \( \Omega = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_N, b_N] \). Parameter \( p(t) \) can also include the elements of the state vector \( x(t) \), therefore LPV system given in Eq. (1) is considered in the class of non-linear dynamic state space models.

The main idea of TP model transformation is to discretize the given LPV model given in Eq. (1) over hyper rectangular grid \( M \) in \( \Omega \), then via executing Higher Order Singular Value Decomposition, the tensor product structure of given model is obtained. By ignoring singular values, TP model of reduced complexity and accuracy can be obtained. For more details see Petres (2006) and Tikk et al. (2004).

Tensor product structure can be written as follows

\[
S(p(t)) = S \otimes \otimes_{n=1}^N w_n(p_n)
\]

(3)

where \( S \in \mathbb{R}^{I_1 \times I_2 \times \ldots \times I_N \times (m+k) \times (m+l)} \) denotes obtained tensor, \( I_n \) denotes number of LTI systems in n-th dimension \( \Omega \), \( \otimes \) denotes multiple n-mode product of a tensor by a matrix, \( w_n \) is row vector containing \( w_n(i_n) \in [0, 1] \) for each index which is corresponding one variable weighting function defined on the n-th dimension of \( \Omega \) and

\[
S_{i_1, i_2 \ldots i_N} = \begin{pmatrix} A_{i_1, i_2 \ldots i_N} & B_{i_1, i_2 \ldots i_N} \\ C_{i_1, i_2 \ldots i_N} & D_{i_1, i_2 \ldots i_N} \end{pmatrix},
\]

(4)

is LTI system matrix obtained by TP model transformation.

Controller is determined in same form as TP model. Control signal is given by

\[
u = -\sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \ldots \sum_{i_N=1}^{I_N} \prod_{n=1}^{N} w_{n,i_n}(p_n) K_{i_1,i_2 \ldots i_N} x
\]

(5)

where the \( K_{i_1,i_2 \ldots i_N} \) are corresponding LTI feedback gains.

By using i as linear index, equivalent to the multilinear array index with the size of \( I_1 \times I_2 \times \ldots \times I_N \), TP model (3) and control signal (5) can be rewritten in standard polytopic form

\[
S(p) = \sum_{i=1}^{R} w_i(p) S_i, \quad (6)
\]

\[
u = -\sum_{i=1}^{R} w_i(p) K_i, \quad (7)
\]

where \( R = I_1 + I_2 + \ldots + I_N \) and \( w_i(p) \) is corresponding weighting function.

3. TP MODEL-BASED CONTROLLER DESIGN
APPLIED TO THE SINGLE PENDULUM GANTRY
CRANE EXPERIMENTAL MODEL

3.1 Mathematical model of the gantry crane

Experimental laboratory model of Single Pendulum Gantry (SPG) is used to emulate industrial crane application, see Fig 1.

![Fig. 1. SPG photo in mechatronics laboratory a), and schematics of the model b)](image)

Non-linear model of single pendulum gantry (SPG) can be described by following equations:\footnote{Derivation using Lagrangian formulation is omitted for brevity and can be found in (QUANSER User Manual, 1999)}

\[
(M_c + M_p) \ddot{x}_c + M_p l_p \dot{\alpha} \cos(\alpha) - M_p l_p \dot{\alpha}^2 \sin(\alpha) = F_m - F_f - B_{eq} \dot{x}_c,
\]

\[
(I_p + M_p l_p^2) \dot{\alpha} + M_p l_p \dot{x}_c \cos(\alpha) + M_p l_p g \sin(\alpha) = -B_p \dot{\alpha},
\]

(8)

where \( F_m \) is DC motor force while \( F_f \) is the friction force. Driving force obtained from DC motor is given by:

\[
F_m = \frac{\eta F L K_m K_t}{r_{mp}} \frac{1}{R_m} U_m,
\]

(9)

The meanings and the values of other parameters in equations (8) and (9) are given in Table 1.

In this paper Stribeck friction model is used, see Olsson et al. (1998). It can be described by

\[
F_f(\dot{x}_c) = \text{sgn}(\dot{x}_c) \left[ F_C + (F_S - F_C)e^{-((\dot{x}_c)^4)} \right],
\]

(10)
where \( F_C \) is Coulomb friction, \( F_S \) is static friction force, \( \nu_s \) is Striebeck velocity, and \( \delta \) is empirical exponent.

The parameters of the model (10) are experimentally identified and the following values are obtained:
\[
F_C = 2.70, \quad F_S = 3.10, \quad \nu_s = 0.10, \quad \delta = 2.
\]

(11)

### 3.2 LPV model of single pendulum gantry (SPG)

Letting \( x = [x_1 \ x_2 \ x_3 \ x_4]^T = [x_c \ \dot{x}_c \ \alpha \ \dot{\alpha} ]^T, \alpha = U_m \) and \( w \) be bounded disturbance, the equations of motion in linear parameter varying state space form are:
\[
\dot{x} = A(x(t))x + B(x(t))u + E(x(t))w, \quad y = C(x(t))x + D(x(t))u + F(x(t))w.
\]

(12)

The system matrix in LPV form for the model (12) can be written as:
\[
S = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_1/a_x & a_2/a_x & a_3/a_x & b_1/a_x & c_1/a_x & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & a_4/a_x & a_5/a_x & a_6/a_x & b_2/a_x & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

(13)

where:
\[
a_1 = -(I_p + M_p \omega_p^2) \left( \frac{\eta_h \omega m K_2^2 K_i K_m}{r_{mp} R_m} \right) + B_{eq}
\]
\[
a_2 = \frac{M_p \omega_p^2 g \sin(x_3) \cos(x_3)}{x_3}
\]
\[
a_3 = (M_p \omega_p^2 + l_p M_p \omega_p) \sin(x_3)x_4 + M_p \omega_p B_p \cos(x_3)
\]
\[
a_4 = M_p \omega_p \cos(x_3) \cdot \left( B_{eq} - \frac{\eta_h \omega m K_2^2 K_i K_m}{r_{mp} R_m} \right)
\]
\[
a_5 = -\frac{(M_c + M_p) M_p \omega_p \sin(x_3)}{x_3}
\]
\[
a_6 = -(M_c + M_p) B_p - M_p \omega_p^2 \cos(x_3) \sin(x_3)x_4
\]
\[
a_7 = (M_c + M_p) I_p + M_c M_p \omega_p^2 + M_p \omega_p^2 \sin^2(x_3)
\]
\[
b_{11} = -(I_p M_p \omega_p^2)^2 \eta_h \omega m K_2^2 K_i K_m \left[ F_C + (F_S - F_C) e^{-\left( \frac{\eta_h \omega m K_2^2 K_i K_m}{r_{mp} R_m} \right)x_3} \right]
\]
\[
b_{12} = -M_p \omega_p \cos(x_3) \eta_h \omega m K_2^2 K_i K_m \left[ F_C + (F_S - F_C) e^{-\left( \frac{\eta_h \omega m K_2^2 K_i K_m}{r_{mp} R_m} \right)x_3} \right]
\]
\[
e_1 = -(I_p M_p \omega_p^2)^2 \frac{\eta_h \omega m K_2^2 K_i}{R_{mp} r_{mp}} \left[ F_C + (F_S - F_C) e^{-\left( \frac{\eta_h \omega m K_2^2 K_i}{R_{mp} r_{mp}} \right)x_3} \right]
\]
\[
e_2 = -M_p \omega_p \cos(x_3) \eta_h \omega m K_2^2 K_i K_m \left[ F_C + (F_S - F_C) e^{-\left( \frac{\eta_h \omega m K_2^2 K_i}{R_{mp} r_{mp}} \right)x_3} \right]
\]
\[
c_1 = -0.614 \frac{\sin(x_3)}{x_3},
\]

(14)

### 3.3 Single pendulum gantry TP model representation

Operating area for single pendulum gantry, is selected as
\[
\Omega = [\dot{x}_{c_{min}}, \dot{x}_{c_{max}}] \times [\omega_{min}, \omega_{max}] \times [\alpha_{min}, \alpha_{max}]
\]
\[
= [-0.6, 0.6] \times [-0.0873, 0.0873] \times [-0.8, 0.8].
\]

(15)

Applying the TP transformation to the model (13) yield to the TP model representation consisting of 20 LTI models. The LTI system matrices of the TP model are:

\[
S_{1,1,1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -11.64 & 1.512 & 0.009566 & 1.529 & 1.153 \\
0 & 0 & 0 & 1.0 & 0 & 0 \\
0 & 26.71 & -26.05 & -0.09477 & -3.509 & -2.647 \\
0 & 1.0 & 0.6132 & 0 & 0 & 0 \\
0 & 1.0 & 0 & 0 & 0 & 0 \\
0 & -11.64 & 1.512 & 0.009566 & 1.529 & -1.153 \\
0 & 0 & 0 & 1.0 & 0 & 0 \\
\end{bmatrix}
\]

(16)

\[
S_{1,2,1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -11.64 & 1.512 & 0.009566 & 1.529 & 1.153 \\
0 & 0 & 0 & 1.0 & 0 & 0 \\
0 & 26.71 & -26.05 & -0.09477 & -3.509 & -2.647 \\
0 & 1.0 & 0.6132 & 0 & 0 & 0 \\
0 & 1.0 & 0 & 0 & 0 & 0 \\
0 & -11.64 & 1.512 & 0.009566 & 1.529 & -1.153 \\
0 & 0 & 0 & 1.0 & 0 & 0 \\
\end{bmatrix}
\]

(17)

\[
S_{1,3,1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 26.78 & -26.08 & -0.09299 & -3.517 & -2.653 \\
0 & 0 & 0.6136 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.0 & 0 & 0 \\
\end{bmatrix}
\]

(18)
### Table 1. PARAMETERS OF THE SPG SYSTEM

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_m$</td>
<td>DC motor voltage</td>
<td>V</td>
<td></td>
</tr>
<tr>
<td>$I_m$</td>
<td>DC motor armature current</td>
<td>A</td>
<td></td>
</tr>
<tr>
<td>$R_m$</td>
<td>DC motor armature resistance</td>
<td>2.6</td>
<td>Ω</td>
</tr>
<tr>
<td>$L_m$</td>
<td>DC motor armature inductance</td>
<td>0.18</td>
<td>mH</td>
</tr>
<tr>
<td>$K_i$</td>
<td>Motor torque constant</td>
<td>0.00767</td>
<td>N m A$^{-1}$</td>
</tr>
<tr>
<td>$\eta_m$</td>
<td>Motor efficiency</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$K_{em}$</td>
<td>Back electro-motive force constant</td>
<td>0.00767</td>
<td>V s/rad</td>
</tr>
<tr>
<td>$J_m$</td>
<td>Rotor moment of inertia</td>
<td>3.9001 x 10$^{-7}$</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>$I_p$</td>
<td>Pendulum moment of inertia</td>
<td>0.0078838</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>$K_g$</td>
<td>Planetary gearbox gear ratio</td>
<td>3.71</td>
<td></td>
</tr>
<tr>
<td>$\eta_g$</td>
<td>Planetary gearbox efficiency</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$M_c$</td>
<td>Lumped mass of the cart system</td>
<td>1.0731</td>
<td>kg</td>
</tr>
<tr>
<td>$L_p$</td>
<td>Pendulum length from pivot to COG</td>
<td>0.3302</td>
<td>m</td>
</tr>
<tr>
<td>$I_p$</td>
<td>Pendulum mass</td>
<td>0.23</td>
<td>kg</td>
</tr>
<tr>
<td>$r_{mp}$</td>
<td>Motor pinion radius</td>
<td>0.00635</td>
<td>m</td>
</tr>
<tr>
<td>$B_0$</td>
<td>Equivalent viscous damping coeff.</td>
<td>5.4</td>
<td>N m s rad$^{-1}$</td>
</tr>
<tr>
<td>$B_0$</td>
<td>Viscous damping coefficient</td>
<td>0.0024</td>
<td>N m s rad$^{-1}$</td>
</tr>
<tr>
<td>$F_c$</td>
<td>Driving force</td>
<td>N</td>
<td></td>
</tr>
<tr>
<td>$g$</td>
<td>Gravitational constant of earth</td>
<td>9.81</td>
<td>m/s$^2$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Pendulum angle</td>
<td>rad</td>
<td></td>
</tr>
<tr>
<td>$x_c$</td>
<td>Chart position</td>
<td>mm</td>
<td></td>
</tr>
</tbody>
</table>

Weighting functions of the TP model are given in Fig. 2.

### 4. CONTROLLER DESIGN

#### 4.1 Linear Matrix Inequalities

Recently a class of numerical optimization problems called linear matrix inequality (LMI) problems has received significant attention. These optimization problems can be solved in polynomial time and hence are tractable, at least in a theoretical sense. Interior-point methods, developed for these problems, have been found to be extremely efficient in practice. For systems and control, the importance of LMI optimization stems from the fact that a wide variety of system and control problems can be recast as LMI problems. Except for a few special cases these problems do not have analytical solutions. However, the main point is that through the LMI framework they can be efficiently solved numerically in all cases. Therefore recasting a control problem as an LMI problem is equivalent to finding a solution to the original problem.

Generally, a linear matrix inequality (LMI) has the form

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0,$$

where $x \in R^m$ is the variable and the symmetric matrices $F_i = F_i^T$ are given. The inequality symbol in (20) means that $F(x)$ is positive definite.

#### 4.2 Control objective

In Kolonić et al. (2006) the control objective was to find stabilizing controller under quadratic stability (QS) with prescribed decay rate with minimal overshoot and constrained control signal. In order to obtain stabilizing controller Lyapunov stability approach is used, with Lyapunov function candidate given by:

$$V(x) = x^T P x > 0.$$

The speed of response is related to decay rate $\alpha$, that is, the largest Lyapunov exponent. Therefore, the condition for desired decay rate can be written as

$$\dot{V}(x) \leq -2\alpha V(x).$$

The equilibrium of the continuous system in polytopic form (3) is globally asymptotically stable if there exists a common positive definite matrix $P$ such that

$$A_i^T P + P A_i + 2\alpha P < 0; i \in (1, R).$$

Next, let us consider the stability of the closed-loop control system of TP model of single pendulum gantry. It is globally asymptotically stable if there exists a common positive definite matrix $P$ such that

$$(G_{ij} + G_{ji})^T P + P (G_{ij} + G_{ji}) + 2\alpha P \leq 0, i < j.$$
maximize $\alpha$
subject to

$$X > 0$$
$$-XA_i^T - A_iX + M_i^TB_i^T + B_iM_i + 2\alpha X > 0$$
$$-XA_j^T - A_jX - XA_i^T - A_iX + M_j^TB_j^T + B_jM_j + 4\alpha X \geq 0$$

(26)

where $X = P^{-1}$ and $M_i = F_iX$.

In order to satisfy the constraints on control input and output constraints, the following LMIs are added to the (26).

Constraint on the control value:

Assume that initial condition $x(0)$ is unknown, but its upper bound $\|x(0)\| \leq \phi$ is known, which can be recast as following LMI

$$\phi^2I \leq X,$$

(27)

the constraint $\|u\|_2 \leq \mu$ is enforced $\forall t \geq 0$ if the following LMI holds

$$\begin{pmatrix} X & M_i^T \\ M_i & \mu^2I \end{pmatrix} \geq 0.$$

(28)

Constraint on the output:

Assume that condition (27) is satisfied, the constraint $\|y(t)\|_2 \leq \lambda$ is enforced, $\forall t \geq 0$, if the following LMI holds

$$\begin{pmatrix} X & XC_i^T \\ C_iX & \lambda^2I \end{pmatrix} \geq 0.$$

(29)

LMI conditions (26) - (29) guarantee stability, constrained control signal and constrained output, however since friction effects were neglected it resulted in steady state error. In order to minimize steady state error, $H_\infty$ norm was minimized since it is related to the capacity of the closed-loop system to reject energy bounded disturbance.

If there exist Lyapunov function (21) such that

$$\dot{V}(x) + y^Ty - \gamma^2w^Tw \leq 0,$$

(30)

closed loop system has guaranteed $H_\infty$ disturbance attenuation less than $\gamma$,

$$\|y(t)\|_2 \leq \gamma\|w(t)\|_2,$$

(31)

besides being quadratically stable. Condition (30) can be rewritten as

$$\sum_{i=1}^R w_i^2(p)T_{ii} + \sum_{i=1}^{R-1} \sum_{j=i+1}^R w_i(p)w_j(p)(T_{ij} + T_{ji}) < 0,$$

(32)

where

$$T_{ij} = \begin{pmatrix} A_iX + XA_i^T + B_iM_i + M_i^TB_j^T + B_jE_i^T \\ C_iX + F_iE_i^T + D_jM_j \\ F_iF_i^T - \gamma^2I \end{pmatrix}^*.$$

(33)
which is in literature known as LMI representation of bounded real lemma (Boyd et al., 1994), (Fridman and Shaked, 2001).

Since $\gamma$ is related to the level of disturbance attenuation it is of great interest to compute controller which ensures minimum value of $\gamma$.

$$\begin{align*}
\text{minimize } & \gamma \\
\text{subject to } & (33)
\end{align*}$$ (34)

Classical LMI conditions test the negative definiteness of (33), by imposing that the coefficients $T_{ii} < 0$ and $T_{ij} + T_{ji} < 0$, which is obviously only sufficient condition. Instead we use condition with proved convergence towards (33) in following theorem

**Theorem 1:** The TP model in (3) is quadratically stabilizable by means of all linear parameter-dependent state feedback control gain $K(p) = \sum_{i=0}^{R} w_i(p)K_i$ with an $H_\infty$ guaranteed cost $> 0$ if and only if there exist a symmetric positive definite matrix $X \in \mathbb{R}^{n \times n}$, matrices $M_i \in \mathbb{R}^{m \times n}$, $i = 1, \ldots, N$, matrices $X_k \in \mathbb{R}^{2n+m+n+q}$, $k \in \mathcal{N}(g)$, a degree $g \geq 1$, $g \in \mathbb{N}$, and a sufficiently large $d \in \mathbb{N}$ such that

$$\begin{align*}
\sum_{k' \in \mathcal{N}(d)} \left( \sum_{i \in \{1, \ldots, N\} | k_i > k_i'} \frac{d!}{\pi(k') \pi(k - k_i + e_i - e_j)} \left( \begin{array}{c} X_{k - k_i'}B_i + B_i^T X_{k - k_i'}^T \end{array} \right) \right) + \\
\sum_{i, j \in \{1, \ldots, N\}} \frac{d!}{\pi(k') \pi(k - k_i + e_i - e_j)} (g - 1)!
\end{align*}$$

$$\begin{bmatrix}
B_i M_j + M_j^T B_i^T & 0 & M_i^T D_i^T \\
0 & I & 0 \\
0 & 0 & -\gamma^2 I
\end{bmatrix} < 0,$$

$\forall k \in \mathcal{N}(g + d + 1),$

(35)

with

$$B_i = \begin{bmatrix}
A_i^T & -I & 0 \\
E_i^T & 0 & -I \\
C_i^T & 0 & E_i^T
\end{bmatrix}.$$ (36)

where $\mathcal{N}(g)$ is set of $N$-tuples obtained as all possible combinations of nonnegative integers $k_i$, $i \in \{1, N\}$, such that $\sum_{i=1}^{N} k_i = g$, for $N$-tuples $k$, $k'$, comparation, summation and substraction are defined componentwise, $N$-tuple $e_i$ denotes $N$-tuple with all components equal 0, except i-th component which equals 1, and where $\pi(k)$ is defined as $\pi(k) = (k_1)! (k_2)! \ldots (k_N)!$.

Minimising $\gamma$ subject to LMI constraints proposed in Theorem 1, have the asymptotic convergence to the minimum $H_\infty$ guaranteed cost under quadratic stabilizability, for more details and proof of Theorem 1 see Montagner et al. (2009), http://www.dt.fee.unicamp.br/~ricfow/robust.htm.

In the affirmative case, local feedback gains are given by $K_i = M_i X^{-1}$, $i = 1, \ldots, N$.

In order to ensure constrained control input (27) and (28) are added to (35).

By using Yalmip$^2$ and Sedumi$^3$ 1.3, the following feasible solution and feedback gains are obtained:

$$
\begin{align*}
K_{1,1,1} &= \begin{bmatrix}
64.55 & 22.34 & -5.605 & 8.128
\end{bmatrix}^T \\
K_{1,2,1} &= \begin{bmatrix}
64.55 & 22.34 & -5.605 & 8.128
\end{bmatrix}^T \\
K_{2,1,1} &= \begin{bmatrix}
64.52 & 22.32 & -5.616 & 8.118
\end{bmatrix}^T \\
K_{2,2,1} &= \begin{bmatrix}
64.52 & 22.32 & -5.616 & 8.118
\end{bmatrix}^T \\
K_{1,3,1} &= \begin{bmatrix}
65.29 & 22.52 & -5.287 & 8.213
\end{bmatrix}^T \\
K_{2,3,1} &= \begin{bmatrix}
65.29 & 22.52 & -5.287 & 8.213
\end{bmatrix}^T \\
K_{1,4,1} &= \begin{bmatrix}
65.34 & 22.51 & -5.267 & 8.212
\end{bmatrix}^T \\
K_{2,4,1} &= \begin{bmatrix}
65.34 & 22.51 & -5.267 & 8.212
\end{bmatrix}^T \\
K_{1,5,1} &= \begin{bmatrix}
66.71 & 22.86 & -4.684 & 8.374
\end{bmatrix}^T \\
K_{2,5,1} &= \begin{bmatrix}
66.71 & 22.86 & -4.684 & 8.374
\end{bmatrix}^T \\
K_{1,1,2} &= \begin{bmatrix}
64.52 & 22.32 & -5.616 & 8.118
\end{bmatrix}^T \\
K_{1,2,2} &= \begin{bmatrix}
64.52 & 22.32 & -5.616 & 8.118
\end{bmatrix}^T \\
K_{2,1,2} &= \begin{bmatrix}
65.55 & 22.34 & -5.605 & 8.128
\end{bmatrix}^T \\
K_{2,2,2} &= \begin{bmatrix}
65.55 & 22.34 & -5.605 & 8.128
\end{bmatrix}^T \\
K_{1,3,2} &= \begin{bmatrix}
65.27 & 22.5 & -5.296 & 8.296
\end{bmatrix}^T \\
K_{2,3,2} &= \begin{bmatrix}
65.27 & 22.5 & -5.296 & 8.296
\end{bmatrix}^T \\
K_{1,4,2} &= \begin{bmatrix}
65.38 & 22.54 & -5.251 & 8.225
\end{bmatrix}^T \\
K_{2,4,2} &= \begin{bmatrix}
65.38 & 22.54 & -5.251 & 8.225
\end{bmatrix}^T \\
K_{1,5,2} &= \begin{bmatrix}
66.71 & 22.86 & -4.687 & 8.371
\end{bmatrix}^T \\
K_{2,5,2} &= \begin{bmatrix}
66.71 & 22.86 & -4.687 & 8.371
\end{bmatrix}^T \\
\end{align*}

(37)

4.3 Results

Simulation and experimental results are shown on Fig 3. During simulation and experimental tests proposed $H_\infty$ approach is compared to original TP model based approach (QS) described in Kolonić et al. (2006). It can be seen that proposed approach has successfully minimized steady state error of chart position from 14.55 mm to 2.4 mm in simulation, and from 7mm to 0.4 mm in experimental results. However, slower decay rate as well as oscillatory angle response is obtained due to increased control action needed to overcome static friction force.

5. CONCLUSION

TP transform (HOSVD) creates polytopic model suitable for PDC controller synthesis via LMIs. Presented results for gantry crane are obtained by using exact TP transform. Simulation and experimental results verify that $H_\infty$ controller has successfully minimized steady state error due to friction effect, however it resulted in oscillatory angle response and slower decay rate. In order to minimize oscillations $H_\infty$ controller could be extended with robust pole placement inside LMI regions.

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$^2$ YALMIP is a modelling language for advanced modeling and solution of convex and nonconvex optimization problems.

$^3$ SeDuMi: a Matlab toolbox for optimization over symmetric cones.
Fig. 3. Simulation and experimental results after step reference position, where $y$ is pendulum tip position, $x_c$ is cart position, $\alpha$ is pendulum angle and $U_m$ is control input, respectively.
REFERENCES


