Conference

MINFAA 2012 July 25-29, Korea

Refined Jensen's operator inequality and its converses

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Introduction

Let T be a locally compact Hausdorff space and let $\mathcal{A}$ be a C^*-algebra of operators on some Hilbert space H. We say that a field $(x_t)_{t\in T}$ of operators in $\mathcal{A}$ is continuous if the function $t \mapsto x_t$ is norm continuous on T. If in addition μ is a Radon measure on T and the function $t \mapsto \|x_t\|$ is integrable, then we can form *the Bochner integral* $\int_T x_t \, d\mu(t)$, which is the unique element in $\mathcal{A}$ such that $\varphi\left(\int_T x_t \, d\mu(t)\right) = \int_T \varphi(x_t) \, d\mu(t)$ for every linear functional φ in the norm dual $\mathcal{A}^*$.

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Assume furthermore that $(\Phi_t)_{t\in T}$ is a field of positive linear mappings $\Phi_t : \mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ to another C^*-algebra $\mathcal{B}$ of operators on a Hilbert space K. We say that such a field is continuous if the function $t \mapsto \Phi_t(x)$ is continuous for every $x \in \mathcal{A}$. Let the C^*-algebras include the identity operators and the field $t \mapsto \phi_t(1)$ be integrable with $\int_T \phi_t(1) \, d\mu(t) = k1$ for some positive scalar k. *Specially, if $k = 1$, we say that a field $(\phi_t)_{t\in T}$ is unital.*

Let $B(H)$ be the C^*-algebra of all bounded linear operators on a Hilbert space H. We define bounds of an operator $x \in B(H)$ by

$$m_x = \inf_{\|\xi\|=1} \langle x\xi, \xi \rangle \quad \textbf{and} \quad M_x = \sup_{\|\xi\|=1} \langle x\xi, \xi \rangle$$

for $\xi \in H$. If $\mathrm{Sp}(x)$ denotes the spectrum of x, then $\mathrm{Sp}(x) \subseteq [m_x, M_x]$.

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For an operator $x \in B(H)$ we define operators $|x|$, x^+, x^- by

$$|x| = (x^* x)^{1/2}, \qquad x^+ = (|x| + x)/2, \qquad x^- = (|x| - x)/2.$$

Obviously, if x is self-adjoint, then $|x| = (x^2)^{1/2}$ and $x^+, x^- \geq 0$ (called positive and negative parts of $x = x^+ - x^-$).

Preliminary results

If the function f is operator convex on an interval I, then the so-called Jensen operator inequality $\mathbf{f(\phi(x)) \leq \phi(f(x))}$ holds for any unital positive linear mapping $\phi\colon \mathcal{A} \to B(K)$ from a C^*-algebra $\mathcal{A}$ to linear operators on a Hilbert space K, and any self-adjoint element x in $\mathcal{A}$ with spectrum in I. Many other versions of the Jensen operator inequality can be found in:

T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond–Pečarić Method in Operator Inequalities*, Zagreb, Element, 2005.

So, in 1995. B. Mond and J. Pečarić proved the inequality

$$\mathbf{f\left(\sum_{i=1}^{n} w_i \phi_i(x_i)\right) \leq \sum_{i=1}^{n} w_i \phi_i(f(x_i))}$$

for operator convex functions f on I, where $\phi_i : B(H) \to B(K)$ are unital positive linear mappings, $x_1, \ldots, x_n$ are self-adjoint operators with spectra in I and $w_1, \ldots, w_n$ are non-negative real numbers with sum one.

Also, in

F. Hansen, J. Pečarić, I. Perić, *Jensen's operator inequality and it's converses* Math. Scand. **100** (2007) 61–73.

a general formulation of Jensen's operator inequality is given for a bounded continuous field of self-adjoint operators and an unital field of positive linear mappings as follows.

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Let $f : I \to \mathbb{R}$ be an operator convex function defined on an interval I and let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras acting on a Hilbert space H and K respectively. If $(\phi_t)_{t\in T}$ is an unital field of positive linear mappings $\phi_t : \mathcal{A} \to \mathcal{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ, then the inequality

$$\boxed{\mathbf{f}\left(\int_{\mathbf{T}} \phi_{\mathbf{t}}(\mathbf{x}_{\mathbf{t}})\,\mathbf{d}\mu(\mathbf{t})\right) \leq \int_{\mathbf{T}} \phi_{\mathbf{t}}(\mathbf{f}(\mathbf{x}_{\mathbf{t}}))\,\mathbf{d}\mu(\mathbf{t})}$$

holds for every bounded continuous field $(x_t)_{t\in T}$ of self-adjoint elements in $\mathcal{A}$ with spectra contained in I.

Recently, the discrete version of Jensen's operator inequality without operator convexity is proven in

J.Mićić, Z.Pavić and J.Pečarić, *Jensen's inequality for operators without operator convexity*, Linear Algebra Appl. 434 (2011), 1228–1237.

We give a continuous version:

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J.Mićić, Z.Pavić and J.Pečarić, *Jensen's inequality for operators without operator convexity*, Linear Algebra Appl. 434 (2011), 1228–1237.

We give a continuous version:

Let $(x_t)_{t\in T}$ and $(\phi_t)_{t\in T}$ be as above. Let m_t and M_t, $m_t \le M_t$, be the bounds of x_t, $t \in T$. If

$$(\mathbf{m_x}, \mathbf{M_x}) \cap [\mathbf{m_t}, \mathbf{M_t}] = \emptyset, \qquad \mathbf{t} \in \mathbf{T}$$

where m_x and M_x, $m_x \le M_x$, are the bounds of the self-adjoint operator $x = \int_T \phi_t(x_t)\, d\mu(t)$, then

$$\boxed{\mathbf{f}\left(\int_{\mathbf{T}} \phi_{\mathbf{t}}(\mathbf{x_t})\, \mathbf{d}\mu(\mathbf{t})\right) \le \int_{\mathbf{T}} \phi_{\mathbf{t}}(\mathbf{f}(\mathbf{x_t}))\, \mathbf{d}\mu(\mathbf{t})} \tag{1}$$

holds for every continuous convex function $f : I \to \mathbb{R}$ provided that the interval I contains all m_t, M_t.

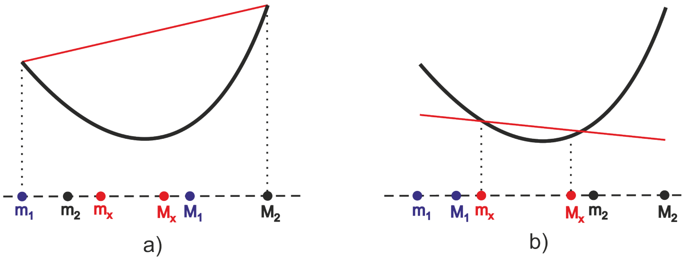


Figure 1 Spectral conditions for a convex function f

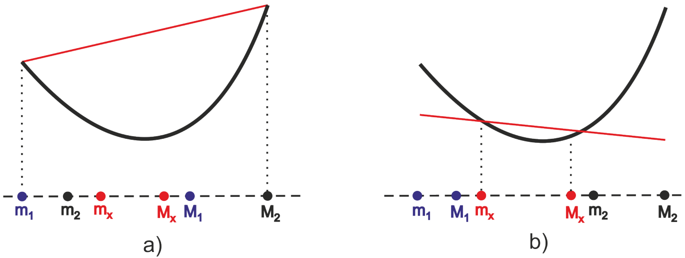


Figure 1 Spectral conditions for a convex function f

Counter-example

Generally, the inequality (1) will be false if we replace the operator convex function by a general convex function. Let $\mathbf{f(t) = t^4}$, $T = \{1, 2\}$, $\Phi_1, \Phi_2 : M_3(\mathbb{C}) \to M_2(\mathbb{C})$ such that $\Phi_k((a_{ij})_{1 \le i,j \le 3}) = \frac{1}{2}(a_{ij})_{1 \le i,j \le 2}$, $k = 1, 2$. If $X_1 = 2\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $X_2 = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ then $(\Phi_1(X_1) + \Phi_2(X_2))^4 \not\leq \Phi_1\left(X_1^4\right) + \Phi_2\left(X_2^4\right)$ and $(\mathbf{m_x}, \mathbf{M_x}) \subset [\mathbf{m_1}, \mathbf{M_1}] \cup [\mathbf{m_2}, \mathbf{M_2}]$ (see Figure 1 a)).

Refined Jensen's inequality

We present a refinement of Jensen's inequality (1). A discrete version of this result is given in

- J. Mićić, J. Pečarić, J. Perić, *Refined Jensen's operator inequality with condition on spectra*, Oper. Matrices. (2012) accepted for publication

To obtain our result we need the following two lemmas.

Lemma 1.

Let f be a convex function on an interval I, $m, M \in I$ and $p_1, p_2 \in [0,1]$ such that $p_1 + p_2 = 1$. Then

$$\min\{p_1, p_2\}\left[f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)\right] \leq p_1 f(m) + p_2 f(M) - f(p_1 m + p_2 M)$$

[follows from Theorem 1, p. 717 in the book: Mitrinović, Pečarić, Fink, *Classical and New Inequalities in Analysis*, 1993.]

Lemma 2.

Let x be a bounded self-adjoint element in an unital C^*-algebra $\mathcal{A}$ of operators on some Hilbert space H. If the spectrum of x is in $[m, M]$, for some scalars $m < M$, then

$$f(x) \leq \frac{M1_H - x}{M - m} f(m) + \frac{x - m1_H}{M - m} f(M) - \delta_f \widetilde{x} \qquad (2)$$

holds for every continuous convex function $f : [m, M] \to \mathbb{R}$, where
$\delta_f := f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)$, $\quad \widetilde{x} := \frac{1}{2}1_H - \frac{1}{M-m}\left|x - \frac{m+M}{2}1_H\right|$.
If f is concave, then the reverse inequality is valid in (2).

Proof

Putting $p_1 = \frac{M-z}{M-m}$ and $p_2 = \frac{z-m}{M-m}$ in Lemma 1 we obtain
$f(z) \leq \frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M) - \left(\frac{1}{2} - \frac{1}{M-m}\left|z - \frac{m+M}{2}\right|\right)\left(f(m) + f(M) - 2f\left(\frac{m+M}{2}\right)\right)$, since
$\min\left\{\frac{M-z}{M-m}, \frac{z-m}{M-m}\right\} = \frac{1}{2} - \frac{1}{M-m}\left|z - \frac{m+M}{2}\right|$. *Finally we use the continuous functional calculus for a self-adjoint operator x.* ∎

Theorem 1. (Refined Jensen's inequality)

Let $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$ be as in the introduction and m_t, M_t be the bounds of x_t. Let

$$(\mathbf{m_x},\mathbf{M_x})\cap[\mathbf{m_t},\mathbf{M_t}]=\emptyset,\ \mathbf{t}\in\mathbf{T}, \qquad \textit{and} \qquad \mathbf{m}<\mathbf{M}$$

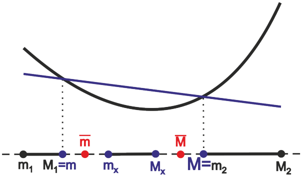
where m_x, M_x be the bounds of the operator $x=\int_T \phi_t(x_t)\,d\mu(t)$ and $m=\sup\{M_t\colon M_t\le m_x, t\in T\}$, $M=\inf\{m_t\colon m_t\ge M_x, t\in T\}$. If $f:I\to\mathbb{R}$ is a continuous convex function provided that the interval I contains all m_t, M_t, then

$$f\left(\int_T \phi_t(x_t)\,d\mu(t)\right)\le\int_T \phi_t(f(x_t))\,d\mu(t)-\delta_f\widetilde{x}\le\int_T \phi_t(f(x_t))\,d\mu(t), \tag{3}$$

where $\delta_f\equiv\delta_f(\bar{m},\bar{M})=f(\bar{m})+f(\bar{M})-2f\left(\frac{\bar{m}+\bar{M}}{2}\right)$,
$\widetilde{x}\equiv\widetilde{x}_x(\bar{m},\bar{M})=\frac{1}{2}1_K-\frac{1}{\bar{M}-\bar{m}}\left|x-\frac{\bar{m}+\bar{M}}{2}1_K\right|$ and $\bar{m}\in[m,m_A]$, $\bar{M}\in[M_A,M]$, $\bar{m}<\bar{M}$, are arbitrary numbers.
If f is concave, then the reverse inequality is valid in (3).

Example

We give an example for the matrix cases and $T = \{1, 2\}$. *Then we have refined inequalities given in Figure 2.*



$$f(\phi_1(x_1)) + f(\phi_2(x_2)) \le \phi_1(f(x_1)) + \phi_2(f(x_2)) - \delta_f \tilde{x},$$

where

$$\delta_f = f(\bar{m}) + f(\bar{M}) - 2f\left((\bar{M}+\bar{m})/2\right),$$

$$\tilde{x} = \frac{1}{2}1_K - \frac{1}{\bar{M}-\bar{m}}\left|\phi_1(x_1) + \phi_2(x_2) - \frac{\bar{M}+\bar{m}}{2}1_K\right|$$

Figure 2 *Refinement for two operators and a convex function f*

Let $f(t) = t^4$, $\Phi_k((a_{ij})_{1\le i,j\le 3}) = \frac{1}{2}(a_{ij})_{1\le i,j\le 2}$, $k = 1, 2$ *and*

$X_1 = \begin{pmatrix} -4 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & -1 \end{pmatrix}$, $X_2 = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 3 \end{pmatrix}$, *then*

$$(\Phi_1(X_1) + \Phi_2(X_2))^4 = \frac{1}{16}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} < \begin{pmatrix} 639.9213 & -255 \\ -255 & 117.8559 \end{pmatrix} =$$

$$\Phi_1\left(X_1^4\right) + \Phi_2\left(X_2^4\right) - \begin{pmatrix} 1.5787 & 0 \\ 0 & 0.6441 \end{pmatrix} < \begin{pmatrix} \frac{1283}{2} & -255 \\ -255 & \frac{237}{2} \end{pmatrix} = \Phi_1\left(X_1^4\right) + \Phi_2\left(X_2^4\right),$$

since $\delta_f = 3.1574$ *and* $\widetilde{X} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.2040 \end{pmatrix}$.

Extension of the refined Jensen's inequality

We present an extension of the discrete version of the inequality given in Theorem 1. This result is given in

J. Mićić, J. Pečarić, J. Perić, *Extension of the refined Jensen's operator inequality with condition on spectra*, Ann. Funct. Anal. **3** (2012), no. 1, 67–85.

To give our result we introduce the following assertions:

- $(A_1,\dots,A_n)$ is an $n-$tuple of self-adjoint operators $A_i \in B(H)$ with the bounds m_i and M_i, $m_i \le M_i$, $i=1,\dots,n$.
- $(\Phi_1,\dots,\Phi_n)$ is an $n-$tuple of positive linear mappings $\Phi_i : B(H) \to B(K)$, such that $\sum_{i=1}^{n_1} \Phi_i(1_H) = \alpha 1_K$, $\sum_{i=n_1+1}^{n} \Phi_i(1_H) = \beta 1_K$, where $1 \le n_1 < n$, $\alpha,\beta > 0$ and $\alpha+\beta=1$.
- $m_L = \min\{m_1,\dots,m_{n_1}\}$, $M_R = \max\{M_1,\dots,M_{n_1}\}$ and

$$m = \begin{cases} m_L, & \text{if } \{M_i : M_i \le m_L, i \in \{n_1+1,\dots,n\}\} = \emptyset, \\ \max\{M_i : M_i \le m_L, i \in \{n_1+1,\dots,n\}\}, & \text{otherwise,} \end{cases}$$

$$M = \begin{cases} M_R, & \text{if } \{m_i : m_i \ge M_R, i \in \{n_1+1,\dots,n\}\} = \emptyset, \\ \min\{m_i : m_i \ge M_R, i \in \{n_1+1,\dots,n\}\}, & \text{otherwise.} \end{cases}$$

Theorem 2.

Let the above assertions stand. If $(m_L, M_R) \cap [m_i, M_i] = \emptyset$, $i = n_1 + 1, \ldots, n,$, $m < M$ *and one of two equalities* $\frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(A_i) = \sum_{i=1}^{n} \Phi_i(A_i) = \frac{1}{\beta}\sum_{i=n_1+1}^{n} \Phi_i(A_i)$ *is valid, then*

$$\begin{aligned} \frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) \le \frac{1}{\alpha}\sum_{i=1}^{n_1} \Phi_i(f(A_i)) + \beta\delta_f \widetilde{A} \le \sum_{i=1}^{n} \Phi_i(f(A_i)) \\ \le \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)) - \alpha\delta_f \widetilde{A} \le \frac{1}{\beta} \sum_{i=n_1+1}^{n} \Phi_i(f(A_i)), \end{aligned} \tag{4}$$

holds for every continuous convex function $f : I \to \mathbb{R}$ *provided that the interval* I *contains all* m_i, M_i, $i = 1, \ldots, n$, *where*
$\delta_f \equiv \delta_f(\bar{m}, \bar{M}) = f(\bar{m}) + f(\bar{M}) - 2f\left(\frac{\bar{m}+\bar{M}}{2}\right)$,
$\widetilde{A} \equiv \widetilde{A}_{A,\Phi,n_1,\alpha}(\bar{m}, \bar{M}) = \frac{1}{2}1_K - \frac{1}{\alpha(\bar{M}-\bar{m})}\sum_{i=1}^{n_1} \Phi_i\left(\left|A_i - \frac{\bar{m}+\bar{M}}{2}1_H\right|\right)$ *and*
$\bar{m} \in [m, m_L]$, $\bar{M} \in [M_R, M]$, $\bar{m} < \bar{M}$, *are arbitrary numbers.*
If $f : I \to \mathbb{R}$ *is concave, then the reverse inequality is valid in* (4).

Example

We show an example for $f(t)=t^4$, $n=4$, $n_1=2$ *and* $\Phi_i((a_{jk})_{1\le j,k\le 3})=\frac{1}{4}(a_{jk})_{1\le j,k\le 2}$, $i=1,\dots,4$. *Then* $\alpha=\beta=\frac{1}{2}$.

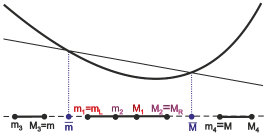


Figure 3

Let

$$A_1=2\begin{pmatrix}2 & 9/8 & 1\\ 9/8 & 2 & 0\\ 1 & 0 & 3\end{pmatrix},\quad A_2=3\begin{pmatrix}2 & 9/8 & 0\\ 9/8 & 1 & 0\\ 0 & 0 & 2\end{pmatrix},$$

$$A_3=-3\begin{pmatrix}4 & 1/2 & 1\\ 1/2 & 4 & 0\\ 1 & 0 & 2\end{pmatrix},\quad A_4=12\begin{pmatrix}5/3 & 1/2 & 0\\ 1/2 & 3/2 & 0\\ 0 & 0 & 3\end{pmatrix}.$$

Then $m_1=1.28607$, $M_1=7.70771$, $m_2=0.53777$, $M_2=5.46221$, $m_3=-14.15050$, $M_3=-4.71071$, $m_4=12.91724$, $M_4=36.$, *so* $m_L=m_2$, $M_R=M_1$, $m=M_3$ *and* $M=m_4$ *(rounded to five decimal places).*
Putting, for example, $\bar{m}=-1$, $\bar{M}=10$, *we obtain*

$$\begin{aligned}\frac{1}{\alpha}\left(\Phi_1(A_1^4)+\Phi_2(A_2^4)\right) &< \frac{1}{2}\left(\Phi_1(A_1^4)+\Phi_2(A_2^4)\right)+2\delta_f\widetilde{A}\\ &< \Phi_1(A_1^4)+\Phi_2(A_2^4)+\Phi_3(A_3^4)+\Phi_4(A_4^4)\\ &< \frac{1}{2}\left(\Phi_3(A_3^4)+\Phi_4(A_4^4)\right)-2\delta_f\widetilde{A} < \frac{1}{2}\left(\Phi_3(A_3^4)+\Phi_4(A_4^4)\right)\end{aligned}$$

since $\delta_f=9180.875$ *and* $\widetilde{A}=\begin{pmatrix}0.28203 & 0.08975\\ 0.08975 & 0.27557\end{pmatrix}$.

Converses of Jensen's operator inequality

B. Mond, J. Pečarić, T. Furuta et al. observed converses of Jensen's inequality. So, using the Mond-Pečarić method, in

J. Mićić, J. Pečarić, Y. Seo, M. Tominaga, *Inequalities of positive linear maps on Hermitian matrices*, Math. Inequal. Appl. **3** (2000) 559–591.

the generalized converse of Jensen's operator inequality is presented:

$$\mathbf{F}\left[\phi\left(\mathbf{f}(\mathbf{A})\right),\mathbf{g}\left(\phi(\mathbf{A})\right)\right] \leq \max_{\mathbf{m}\leq\mathbf{z}\leq\mathbf{M}} \mathbf{F}\left[\mathbf{f}(\mathbf{m}) + \frac{\mathbf{f}(\mathbf{M})-\mathbf{f}(\mathbf{m})}{\mathbf{M}-\mathbf{m}}(\mathbf{z}-\mathbf{m}),\mathbf{g}(\mathbf{z})\right] \mathbf{1}_{\tilde{\mathbf{n}}},$$

for convex functions f defined on an interval $[m, M]$, $m < M$, where g is a real valued continuous function on $[m, M]$, $F(u, v)$ is a real valued function defined on $U \times V$, matrix non-decreasing in u, $U \supset f[m, M]$, $V \supset g[m, M]$, $\phi : H_n \to H_{\tilde{n}}$ is an unital positive linear mapping and A is a Hermitian matrix with spectrum contained in $[m, M]$.

In

J. Mićić, J. Pečarić, Y. Seo, *Converses of Jensen's operator inequality*, Oper. Matrices **4** (2010) 385–403.

a continuous version of the above inequality is presented when $\int_T \phi_t(1_H)\, d\mu(t) = k 1_K$ **for some positive scalar** k**. Recently, in**

J. Mićić, Z. Pavić, J. Pečarić, *Some better bounds in converses of the Jensen operator inequality*, Oper. Matrices. (2011) in print (http://files.ele-math.com/preprints/oam-0488-pre.pdf)

the following bound is obtained, which is better than the above.

Let $(x_t)_{t\in T}$ and $(\phi_t)_{t\in T}$ as in previous sections and the spectra of x_t, $t \in T$, are in $[m, M]$, $m < M$. Let m_x and M_x, $m_x \leq M_x$, be the bounds of $x = \int_T \phi_t(x_t)\, d\mu(t)$ and $f : [m, M] \to \mathbb{R}$, $g : [m_x, M_x] \to \mathbb{R}$, $F : U \times V \to \mathbb{R}$, where $f([m, M]) \subseteq U$, $g([m_x, M_x]) \subseteq V$ and F be bounded. If f is convex and F is operator monotone in the first variable, then

$$\mathbf{F}\left[\int_{\mathbf{T}} \phi_{\mathbf{t}}(\mathbf{f}(\mathbf{x_t}))\, \mathbf{d}\mu(\mathbf{t}),\, \mathbf{g}\left(\int_{\mathbf{T}} \phi_{\mathbf{t}}(\mathbf{x_t})\, \mathbf{d}\mu(\mathbf{t})\right)\right] \leq \mathbf{C_1\, 1_K} \leq \mathbf{C\, 1_K}, \tag{5}$$

where constants

$C_1 \equiv C_1(F, f, g, m, M, m_x, M_x)$ and $C \equiv C(F, f, g, m, M)$ are

$$
\begin{aligned}
C_1 &:= \sup_{m_x \le z \le M_x} F\left[\frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M), g(z)\right] \\
&= \sup_{\frac{M-M_x}{M-m} \le p \le \frac{M-m_x}{M-m}} F[pf(m) + (1-p)f(M), g(pm + (1-p)M)], \\
C &:= \sup_{m \le z \le M} F\left[\frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M), g(z)\right] \\
&= \sup_{0 \le p \le 1} F[pf(m) + (1-p)f(M), g(pm + (1-p)M)].
\end{aligned}
$$

If f is concave, then the reverse inequality is valid in (5) with inf instead of sup in bounds C_1 and C.

General form refined converses

The main result of the improved Mond-Pečarić method:

Lemma 3.

Let $(x_t)_{t\in T}$ and $(\phi_t)_{t\in T}$ as above. If the spectra of x_t, $t \in T$, are in $[m, M]$, for some scalars $m < M$, then

$$\begin{aligned} & \int_T \phi_t(f(x_t))\, d\mu(t) \\ \leq \quad & \frac{M1_K - \int_T \phi_t(x_t)\, d\mu(t)}{M-m} f(m) + \frac{\int_T \phi_t(x_t)\, d\mu(t) - m1_K}{M-m} f(M) - \delta_f \widetilde{x} \\ \leq \quad & \frac{M1_K - \int_T \phi_t(x_t)\, d\mu(t)}{M-m} f(m) + \frac{\int_T \phi_t(x_t)\, d\mu(t) - m1_K}{M-m} f(M) \end{aligned} \tag{6}$$

for every continuous convex function $f : [m, M] \to \mathbb{R}$, where

$$\begin{aligned} \delta_f \equiv \delta_f(m, M) = f(m) + f(M) - 2f\left(\frac{m+M}{2}\right), \\ \widetilde{x} \equiv \widetilde{x}_x(m, M) = \frac{1}{2}1_K - \frac{1}{M-m} \int_T \phi_t\left(\left|x_t - \frac{m+M}{2}1_H\right|\right) d\mu(t). \end{aligned} \tag{7}$$

If f is concave, then the reverse inequality is valid in (6).

Proof of Lemma 3.

We prove only the convex case. By using Lemma 2 for a self-adjoint operator x_t, $Sp(x_t) \subseteq [m, M]$, $t \in T$, *then*

$$f(x_t) \leq \frac{M - x_t}{M - m} f(m) + \frac{x_t - m}{M - m} f(M) - \widetilde{x}_t \left[f(m) + f(M) - 2f\left(\frac{m+M}{2}\right) \right]$$

holds, where $\widetilde{x}_t = \frac{1}{2} 1_H - \frac{1}{M-m} \left| x_t - \frac{m+M}{2} 1_H \right|$. *Applying a positive linear mapping* ϕ_t, *integrating and using* $\int_T \phi_t(1_H)\, d\mu(t) = 1_K$, *we get the first inequality in* (6), *where* δ_f *and* $\widetilde{x}$ *are as in Lemma 3, since*

$$\widetilde{x} = \int_T \phi_t\left(\widetilde{x}_t\right) d\mu(t) = \frac{1}{2} 1_K - \frac{1}{M-m} \int_T \phi_t\left(|x_t - \frac{m+M}{2} 1_H|\right) d\mu(t).$$

Furthermore, $m1_H \leq x_t \leq M1_H$, $t \in T$, *implies* $\int_T \phi_t\left(|x_t - \frac{m+M}{2} 1_H|\right) d\mu(t) \leq \frac{M-m}{2} 1_K$. *It follows* $\widetilde{x} \geq 0$. *Moreover,* $\delta_f \geq 0$, *since* f *is convex. So the second inequality in* (6) *holds.* ∎

We can use Lemma 3 to obtain refinements of some other converse inequalities. First we present a refinement of (5).

Theorem 3. (General form refined converses)

Let $(x_t)_{t\in T}$ and $(\phi_t)_{t\in T}$ as above and the spectra of x_t, $t \in T$, are in $[m,M]$, $m < M$. Let m_x and M_x, $m_x \le M_x$, be the bounds of $x = \int_T \phi_t(x_t)\, d\mu(t)$ and $f : [m,M] \to \mathbb{R}$, $g : [m_x, M_x] \to \mathbb{R}$, $F : U \times V \to \mathbb{R}$, where $f([m,M]) \subseteq U$, $g([m_x,M_x]) \subseteq V$ and F be bounded. If f is convex and F is operator monotone in the first variable, then

$$\begin{aligned} & F\left[\int_T \phi_t(f(x_t))\, d\mu(t),\, g\left(\int_T \phi_t(x_t)\, d\mu(t)\right)\right] \\ \le\ & \sup_{m_x\le z\le M_x} F\left[\tfrac{M-z}{M-m} f(m) + \tfrac{z-m}{M-m} f(M) - \delta_f m_{\widetilde{x}}, g(z)\right] 1_K \\ \le\ & \sup_{m_x\le z\le M_x} F\left[\tfrac{M-z}{M-m} f(m) + \tfrac{z-m}{M-m} f(M), g(z)\right] 1_K, \end{aligned} \qquad (8)$$

where δ_f, $\widetilde{x}$ are defined by (7) and $m_{\widetilde{x}}$ is the lower bound of the operator $\widetilde{x}$. If f is concave, then the reverse inequality is valid in (8) with inf instead of sup.

Difference type converse inequalities

We recall a generalization of Jensen's inequality: If f is operator convex and $\alpha g \leq f$ on $[m, M]$ for some function g and $\alpha \in \mathbb{R}$, then

$$\alpha g\left(\int_T \phi_t(x_t) d\mu(t)\right) \leq \int_T \phi_t\left(f(x_t)\right) d\mu(t).$$

Applying Lemma 3, we obtain its converse as follows.

Theorem 4.

Let $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$, m, M, m_x, M_x, be as in Theorem 3 and $f : [m, M] \to \mathbb{R}$, $g : [m_x, M_x] \to \mathbb{R}$ be continuous functions. If f is convex and $\alpha \in \mathbb{R}$, then

$$\begin{aligned} & \int_T \phi_t(f(x_t))\, d\mu(t) - \alpha g\left(\int_T \phi_t(x_t)\, d\mu(t)\right) \\ \leq \ & \max_{m_x \leq z \leq M_x} \left\{ \frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M) - \alpha g(z) \right\} 1_K - \delta_f \widetilde{x} \end{aligned} \tag{9}$$

where $\delta_f \geq 0$, $\widetilde{x} \geq 0$ are defined by (7). The bound in RHS of (9) exists for any m, M, m_x and M_x.
If f is concave, then the reverse inequality with min *instead of* max *is valid in* (9) *with $\delta_f \leq 0$ and $\widetilde{x} \geq 0$.*

We apply Theorem 4 on $f(z) = z^p$ and $g(z) = z^q$:

Corollary 1.

Let $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$, m, M, m_x, M_x, be as in Theorem 3, and additionally let x_t be strictly positive. Let $\widetilde{x}$ is defined by (7).
(i) *Let $p \in (-\infty, 0] \cup [1, \infty)$. Then*

$$\int_T \phi_t(x_t^p)\, d\mu(t) - \alpha \left(\int_T \phi_t(x_t)\, d\mu(t) \right)^q \leq C_\alpha^\star 1_K - \left(m^p + M^p - 2^{1-p}(m+M)^p \right) \widetilde{x},$$

where the bound $C_\alpha^\star$ is determined as follows:

Corollary 1. (continued)

- *if* $\alpha \le 0$ *and* $q \in (-\infty, 0] \cup [1, \infty)$, *then*

$$C_\alpha^\star = \max\left\{ \frac{M-m_x}{M-m} m^p + \frac{m_x - m}{M-m} M^p - \alpha m_x^q, \frac{M-M_x}{M-m} m^p + \frac{M_x - m}{M-m} M^p - \alpha M_x^q \right\}; \tag{10}$$

- $\alpha \le 0$ *and* $q \in (0,1)$, *then*

$$C_\alpha^\star = \begin{cases} \frac{M-m_x}{M-m} m^p + \frac{m_x-m}{M-m} M^p - \alpha m_x^q \ \text{if} \ (\alpha q / k_{t^p})^{1/(1-q)} \le m_x, \\ \frac{Mm^p - mM^p}{M-m} + \alpha(q-1)(\alpha q/k_{t^p})^{q/(1-q)} \ \text{if} \ m_x \le (\alpha q/k_{t^p})^{1/(1-q)} \le M_x, \\ \frac{M-M_x}{M-m} m^p + \frac{M_x-m}{M-m} M^p - \alpha M_x^q \ \text{if} \ (\alpha q/k_{t^p})^{1/(1-q)} \ge M_x, \end{cases} \tag{11}$$

where $k_{t^p} := (M^p - m^p)/(M-m)$.

- $\alpha \ge 0$ *and* $q \in (-\infty, 0] \cup [1, \infty)$, *then* $C_\alpha^\star$ *is defined by* (11),
- $\alpha \ge 0$ *and* $q \in (0,1)$, *then* $C_\alpha^\star$ *is defined by* (10).

Corollary 1. (continued)

(ii) *Let* $p \in (0,1)$. *Then*

$$c_\alpha^\star 1_K + \left(2^{1-p}(m+M)^p - m^p - M^p\right) \widetilde{x} \leq \int_T \phi_t(x_t^p)\, d\mu(t) - \alpha\left(\int_T \phi_t(x_t)\, d\mu(t)\right)^q$$

where the bound $c_\alpha^\star$ *is determined as follows:*

- *if* $\alpha \leq 0$ *and* $q \in (-\infty,0] \cup [1,\infty)$, *then* $c_\alpha^\star$ *is equal to the right side in* (11)*;*
- *if* $\alpha \leq 0$ *and* $q \in (0,1)$, *then* $c_\alpha^\star$ *is equal to the right side in* (10) *with* min *instead of* max*;*
- *if* $\alpha \geq 0$ *and* $q \in (-\infty,0] \cup [1,\infty)$, *then* $c_\alpha^\star$ *is equal to the right side in* (10) *with* min *instead of* max*;*
- *if* $\alpha \geq 0$ *and* $q \in (0,1)$, *then* $c_\alpha^\star$ *is equal to the right side in* (11).

Using Theorem 4 for $g \equiv f$ and $\alpha = 1$ we obtain:

Theorem 5.

Let $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$, m, M, m_x, M_x, be as in Theorem 3 and $f : [m, M] \to \mathbb{R}$ be continuous function. If f is convex, then

$$\begin{aligned} 0 \leq\ & \int_T \phi_t(f(x_t))\, d\mu(t) - f\left(\int_T \phi_t(x_t)\, d\mu(t)\right) \\ \leq\ & \bar{C}1_K - \delta_f \widetilde{x}, \end{aligned} \tag{12}$$

where $\delta_f \geq 0$, $\widetilde{x} \geq 0$ are defined by (7) and

$$\bar{C} = \max_{m_x \leq z \leq M_x} \left\{ \frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M) - f(z) \right\}. \tag{13}$$

Furthermore, if f is strictly convex differentiable, then the bound $\bar{C}1_K - \delta_f \widetilde{x}$ satisfies the following condition $0 \leq \bar{C}1_K - \delta_f \widetilde{x} \leq \left\{ f(M) - f(m) - f'(m)(M-m) - \delta_f m_{\widetilde{x}} \right\} 1_K$, where $m_{\widetilde{x}}$ is the lower bound of the operator $\widetilde{x}$.

Theorem 5. (continued)

We can determine more precisely the value $\bar{C} \equiv \bar{C}(m, M, m_x, M_x, f)$ *in* (13) *as follows*

$$\bar{C} = \frac{M - z_0}{M - m} f(m) + \frac{z_0 - m}{M - m} f(M) - f(z_0), \tag{14}$$

where

$$z_0 = \begin{cases} m_x & \text{if} \quad f'(m_x) \geq \frac{f(M)-f(m)}{M-m}, \\ f'^{-1}\left(\frac{f(M)-f(m)}{M-m}\right) & \text{if} \quad f'(m_x) \leq \frac{f(M)-f(m)}{M-m} \leq f'(M_x), \\ M_x & \text{if} \quad f'(M_x) \leq \frac{f(M)-f(m)}{M-m}. \end{cases} \tag{15}$$

Theorem 5. (continued)

We can determine more precisely the value $\bar{C} \equiv \bar{C}(m, M, m_x, M_x, f)$ *in* (13) *as follows*

$$\bar{C} = \frac{M - z_0}{M - m} f(m) + \frac{z_0 - m}{M - m} f(M) - f(z_0), \tag{14}$$

where

$$z_0 = \begin{cases} m_x & \text{if} \quad f'(m_x) \geq \frac{f(M)-f(m)}{M-m}, \\ f'^{-1}\left(\frac{f(M)-f(m)}{M-m}\right) & \text{if} \quad f'(m_x) \leq \frac{f(M)-f(m)}{M-m} \leq f'(M_x), \\ M_x & \text{if} \quad f'(M_x) \leq \frac{f(M)-f(m)}{M-m}. \end{cases} \tag{15}$$

In the dual case, when f is concave, then the reverse inequality is valid in (12) *with* min *instead of* max *in* (13). *Furthermore, if f is strictly concave differentiable, then the bound $\bar{C}1_K - \delta_f \widetilde{x}$ satisfies the following condition* $\{f(M) - f(m) - f'(m)(M - m) - \delta_f m_{\widetilde{x}}\} 1_K \leq \bar{C}1_K - \delta_f \widetilde{x} \leq 0$. *We can determine more precisely the value $\bar{C}$ in* (14), *with z_0 which equals the right side in* (15) *with reverse inequality signs.*

Remark

Let the assumptions of Theorem 5 be valid. Let, additionally, the spectra condition $(m_x, M_x) \cap [m_t, M_t] = \emptyset$, $t \in T$, *holds and* $a < b$, *where* m_t *and* M_t, $m_t \leq M_t$, *be the bounds of* x_t, $a := \sup\{M_t \colon M_t \leq m_x, t \in T\}$, $b := \inf\{m_t \colon m_t \geq M_x, t \in T\}$, $m := \inf\{m_t, t \in T\}$, $M := \sup\{M_t, t \in T\}$. *If* $f : I \to \mathbb{R}$ *is a continuous convex, then*

$$\begin{aligned} & f\left(\int_T \phi_t(x_t)\, d\mu(t)\right) \leq \int_T \phi_t(f(x_t))\, d\mu(t) - \delta_f(\bar{m}, \bar{M})\widetilde{\widetilde{x}}(\bar{m}, \bar{M}) \\ \leq\ & \int_T \phi_t(f(x_t))\, d\mu(t) \leq f\left(\int_T \phi_t(x_t)\, d\mu(t)\right) + \bar{C}1_K - \delta_f(m, M)\widetilde{x}(m, M) \\ \leq\ & f\left(\int_T \phi_t(x_t)\, d\mu(t)\right) + \bar{C}1_K, \end{aligned}$$

where $\bar{C}$ *defined by* (13), $\delta_f(p,q) = f(p) + f(q) - 2f\left(\frac{p+q}{2}\right)$,
$\widetilde{\widetilde{x}}(\bar{m}, \bar{M}) = \frac{1}{2}1_K - \frac{1}{\bar{M}-\bar{m}}\left|\int_T \phi_t(x_t)\, d\mu(t) - \frac{\bar{m}+\bar{M}}{2}1_K\right|$,
$\widetilde{x}(m, M) = \frac{1}{2}1_K - \frac{1}{M-m}\int_T \phi_t\left(\left|x_t - \frac{m+M}{2}1_H\right|\right) d\mu(t)$ *and* $\bar{m} \in [a, m_x]$, $\bar{M} \in [M_x, b]$, $\bar{m} < \bar{M}$, *are arbitrary numbers.*

We apply Theorem 5 on $f(z) = z^p$:

Corollary 2.

Let $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$, m, M, m_x, M_x, be as in Theorem 3, and additionally let x_t be strictly positive. If $p \notin (0,1)$, then

$$\begin{aligned} 0 &\le \int_T \phi_t(x_t^p)\,d\mu(t) - \left(\int_T \phi_t(x_t)\,d\mu(t)\right)^p \\ &\le \bar{C}(m_x, M_x, m, M, p)1_K - \left(m^p + M^p - 2^{1-p}(m+M)^p\right)\widetilde{x} \\ &\le \bar{C}(m_x, M_x, m, M, p)1_K \le C(m, M, p)1_K \end{aligned}$$

and if $p \in (0,1)$, then

$$\begin{aligned} C(m, M, p)1_K &\le \bar{c}(m_x, M_x, m, M, p)1_K \\ &\le \bar{c}(m_x, M_x, m, M, p)1_K + \left(2^{1-p}(m+M)^p - m^p - M^p\right)\widetilde{x} \\ &\le \int_T \phi_t(x_t^p)\,d\mu(t) - \left(\int_T \phi_t(x_t)\,d\mu(t)\right)^p \le 0, \end{aligned}$$

Corollary 2. (continued)

where $\widetilde{x}$ *is defined by* (7),

$$\bar{C}(m_x, M_x, m, M, p) = \begin{cases} k_{t^p}\, m_x + \frac{Mm^p - mM^p}{M-m} - m_x^p & \text{if } pm_x^{p-1} \geq k_{t^p}, \\ C(m, M, p) & \text{if } pm_x^{p-1} \leq k_{t^p} \leq pM_x^{p-1}, \\ k_{t^p}\, M_x + \frac{Mm^p - mM^p}{M-m} - M_x^p & \text{if } pM_x^{p-1} \leq k_{t^p}, \end{cases} \tag{16}$$

$k_{t^p} := (M^p - m^p)/(M - m)$ *and* $\bar{c}(m_x, M_x, m, M, p)$ *equals the right side in* (16) *with reverse inequality signs.*

$C(m, M, p)$ *is the known constant Kantorovich type for difference*

$$C(m, M, p) := (p-1)\left(\frac{M^p - m^p}{p(M-m)}\right)^{1/(p-1)} + \frac{Mm^p - mM^p}{M - m}, \qquad \text{for } p \in \mathbb{R}.$$

Ratio type converse inequalities

We recall the ratio type converse of Jensen's inequality:

$$\int_T \phi_t(f(x_t))\,d\mu(t) \le \max_{m_x \le z \le M_x} \left\{ \frac{\frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M)}{g(z)} \right\} g\left(\int_T \phi_t(x_t)\,d\mu(t)\right)$$

if f is convex and $g > 0$. We obtain its two converses:

Theorem 6.

Let $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$, m, M, m_x, M_x, be as in Theorem 3 and $f : [m, M] \to \mathbb{R}$, $g : [m_x, M_x] \to \mathbb{R}$ be continuous functions. If f is convex and $g > 0$, then

$$\begin{aligned} & \int_T \phi_t(f(x_t))\,d\mu(t) \\ \le \; & \max_{m_x \le z \le M_x} \left\{ \frac{\frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M)}{g(z)} \right\} g\left(\int_T \phi_t(x_t)\,d\mu(t)\right) - \delta_f \widetilde{x}, \end{aligned} \tag{17}$$

Theorem 6. (continued)

and

$$\begin{aligned} & \int_T \phi_t(f(x_t))\,d\mu(t) \\ \leq \quad & \max_{m_x \leq z \leq M_x} \left\{ \frac{\frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M) - \delta_f m_{\widetilde{x}}}{g(z)} \right\} g\left(\int_T \phi_t(x_t)\,d\mu(t)\right), \end{aligned} \tag{18}$$

where $m_{\widetilde{x}}$ *be the lower bound of the operator* $\widetilde{x}$ *and* $\delta_f \geq 0$, $\widetilde{x} \geq 0$ *are defined by* (7). *The bounds in the both inequalities* (17) *and* (18) *exist for any* m, M, m_x *and* M_x.

If f *is concave, then the reverse inequalities are valid in* (17) *and* (18) *with* min *instead of* max *and* $\delta_f \leq 0$, $\widetilde{x} \geq 0$.

Theorem 6. (continued)

and

$$\int_T \phi_t(f(x_t))\,d\mu(t) \le \max_{m_x \le z \le M_x} \left\{ \frac{\frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M) - \delta_f m_{\widetilde{x}}}{g(z)} \right\} g\left(\int_T \phi_t(x_t)\,d\mu(t)\right), \quad (18)$$

where $m_{\widetilde{x}}$ *be the lower bound of the operator* $\widetilde{x}$ *and* $\delta_f \ge 0$, $\widetilde{x} \ge 0$ *are defined by* (7). *The bounds in the both inequalities* (17) *and* (18) *exist for any* m, M, m_x *and* M_x.

If f *is concave, then the reverse inequalities are valid in* (17) *and* (18) *with* min *instead of* max *and* $\delta_f \le 0$, $\widetilde{x} \ge 0$.

Proof

The inequalities (17) *and* (18) *follow from Theorem 5 and Theorem 3, respectively.* ∎

Remark

Generally, there is no relation between the right sides of the inequalities (17) *and* (18) *under the operator order (see the next example).*

If $g(\int_T \phi_t(x_t)\, d\mu(t)) \leq g(z_0)1_K$, *where* $z_0 \in [m_x, M_x]$ *is the point where it achieves* $\max_{m_x \leq z \leq M_x} \left\{ \frac{\frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M)}{g(z)} \right\}$, *then there is the following order between the right sides of the above inequalities:*

$$\begin{aligned} & \int_T \phi_t(f(x_t))\, d\mu(t) \\ \leq \quad & \max_{m_x \leq z \leq M_x} \left\{ \frac{\frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M)}{g(z)} \right\} g\left(\int_T \phi_t(x_t)\, d\mu(t) \right) - \delta_f \widetilde{x} \\ \leq \quad & \max_{m_x \leq z \leq M_x} \left\{ \frac{\frac{M-z}{M-m} f(m) + \frac{z-m}{M-m} f(M) - \delta_f m_{\widetilde{x}}}{g(z)} \right\} g\left(\int_T \phi_t(x_t)\, d\mu(t) \right). \end{aligned}$$

Example

Let $T=\{1,2\}$, $f(t)=g(t)=t^4$, $\Phi_k((a_{ij})_{1\le i,j\le 3})=\frac{1}{2}(a_{ij})_{1\le i,j\le 2}$, $k=1,2$.

$$\text{If}\quad X_1=\begin{pmatrix}4&1&1\\1&2&0\\1&0&1\end{pmatrix}\quad\text{and}\quad X_2=\begin{pmatrix}5&-1&-1\\-1&2&1\\-1&1&3\end{pmatrix},\quad\text{then}\quad X=\begin{pmatrix}4.5&0\\0&2\end{pmatrix}$$

and $m_1=0.623$, $M_1=4.651$, $m_2=1.345$, $M_2=5.866$, $m=0.623$, $M=5.866$ *(rounded to three decimal places). There is no relation between matrices in the right sides of the equalities* (17) *and* (18), *since*

$$\alpha_1\left(\Phi_1(X_1)+\Phi_2(X_2)\right)^4-\delta_f\widetilde{X} = \begin{pmatrix}7823.449&-53.737\\-53.737&139.768\end{pmatrix}$$

$$\alpha_1\left(\Phi_1(X_1)+\Phi_2(X_2)\right)^4 = \begin{pmatrix}7974.38&0\\0&311.148\end{pmatrix},$$

using that $\alpha_1=19.447$ *(in the RHS of* (18)*)*, $\delta_f=962.73$, $\widetilde{X}=\begin{pmatrix}0.157&0.056\\0.056&0.178\end{pmatrix}$ *and* $\alpha_2=12.794$ *(in the RHS of* (17)*).*

Using Theorem 6 for $g \equiv f$ we obtain:

Theorem 7.

Let $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$, m, M, m_x, M_x, be as in Theorem 3. Let $f:[m,M]\to\mathbb{R}$ be a continuous function and strictly positive on $[m_x, M_x]$ and $k_f=(f(M)-f(m))/(M-m)$ and $l_f=(f(m)M-f(M)m)/(M-m)$. If f is convex, then

$$\int_T \phi_t(f(x_t))\,d\mu(t) \le \max_{m_x\le z\le M_x}\left\{\frac{k_f z+l_f-\delta_f m_{\widetilde{x}}}{f(z)}\right\} f\left(\int_T \phi_t(x_t)\,d\mu(t)\right), \tag{19}$$

$$\int_T \phi_t(f(x_t))\,d\mu(t) \le \max_{m_x\le z\le M_x}\left\{\frac{k_f z+l_f}{f(z)}\right\} f\left(\int_T \phi_t(x_t)\,d\mu(t)\right)-\delta_f\widetilde{x}, \tag{20}$$

where $\delta_f\ge 0$, $\widetilde{x}\ge 0$ are defined by (7) and $m_{\widetilde{x}}$ is the lower bound of the operator $\widetilde{x}$.

In the dual case, if f is concave, then the reverse inequality are valid in (19) and (20) with min *instead of* max, *where $\delta_f\le 0$.*

Theorem 7. (continued)

Furthermore, if f is convex differentiable on $[m_x, M_x]$, we can determine the bound

$$\alpha_1 \equiv \alpha_1(m, M, m_x, M_x, f) = \max_{m_x \le z \le M_x} \left\{ \frac{k_f z + l_f - \delta_f m_{\tilde{x}}}{f(z)} \right\}$$

in (19) *more precisely as follows: If f is convex, then*

$$\alpha_1 = \begin{cases} \frac{k_f m_x + l_f - \delta_f m_{\tilde{x}}}{f(m_x)} & \text{if} \quad f'(z) \ge \frac{k_f f(z)}{k_f z + l_f - \delta_f m_{\tilde{x}}} \text{ for every } z \in (m_x, M_x), \\ \frac{k_f z_0 + l_f - \delta_f m_{\tilde{x}}}{f(z_0)} & \text{if} \quad f'(z_0) = \frac{k_f f(z_0)}{k_f z_0 + l_f - \delta_f m_{\tilde{x}}} \text{ for some } z_0 \in (m_x, M_x), \\ \frac{k_f M_x + l_f - \delta_f m_{\tilde{x}}}{f(M_x)} & \text{if} \quad f'(z) \le \frac{k_f f(z)}{k_f z + l_f - \delta_f m_{\tilde{x}}} \text{ for every } z \in (m_x, M_x). \end{cases} \tag{21}$$

Also, if f is strictly convex twice differentiable on $[m_x, M_x]$, then we can determine the bound

$$\alpha_2 \equiv \alpha_2(m, M, m_x, M_x, f) = \max_{m_x \le z \le M_x} \left\{ \frac{k_f z + l_f}{f(z)} \right\}$$

in (20) *more precisely as follows:*

$$\alpha_2 = (k_f z_0 + l_f)/f(z_0), \tag{22}$$

Theorem 7. (continued)

where $z_0 \in (m_x, M_x)$ is defined as the unique solution of the equation $k_f f(z) = (k_f z + l_f) f'(z)$ provided $(k_f m_x + l_f) f'(m_x)/f(m_x) \leq k_f \leq (k_f M_x + l_f) f'(M_x)/f(M_x)$, otherwise z_0 is defined as m_x or M_x provided $k_f \leq (k_f m_x + l_f) f'(m_x)/f(m_x)$ or $k_f \geq (k_f M_x + l_f) f'(M_x)/f(M_x)$, respectively.

In the dual case, if f is concave differentiable, then the value α_1 is equal to the right side in (21) *with reverse inequality signs. Also, if f is strictly concave twice differentiable, then we can determine more precisely the value α_2 in* (22)*, with z_0 which equals the right side in* (22) *with reverse inequality signs.*

Finally, we apply Theorem 7 on $f(z) = z^p$. For this result, we will define a refinement of the known Kantorovich constant $K(h,p)$ presented in §2.7 in the book:

T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond–Pečarić Method in Operator Inequalities*, Zagreb, Element, 2005.

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Definition 1.

Let $h > 0$. A refinement of the Kantorovich constant $K(h,p,c)$ is defined by

$$K(h,p,c) := \frac{h^p - h + c(h^p + 1 - 2^{1-p}(h+1)^p)(h-1)}{(p-1)(h-1)} \left(\frac{p-1}{p} \frac{h^p - 1}{h^p - h + c(h^p + 1 - 2^{1-p}(h+1)^p)(h-1)} \right)^p$$

for any real number $p \in \mathbb{R}$ and any $0 \le c \le 0.5$.

The constant $K(h,p,c)$ is sometimes denoted by $K(p,c)$ briefly.

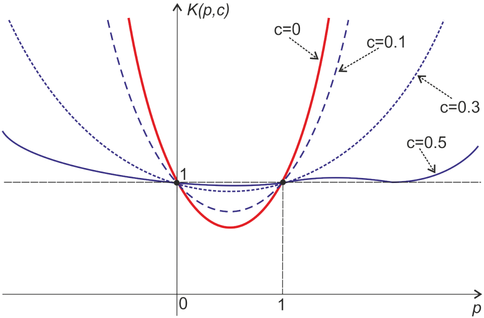

Figure 4 Relation between $K(p,c)$ for $p \in \mathbb{R}$ and $0 \leq c \leq 0.5$

Putting $c = 0$ in $K(h,p,c)$ we obtain the known Kantorovich constant $K(h,p)$.

Moreover, the constant $K(m,M,p,c)$ defined in the next corollary is just coincides with $K(h,p,c)$ by putting $h = M/m > 1$.

Lemma 4.

Let $h > 0$. A refinement of the Kantorovich constant $K(h,p,c)$ has the following properties.

(i) $K(h,p,c) = K(\frac{1}{h},p,c)$ *for all* $p \in \mathbb{R}$

(ii) $K(h,0,c) = K(h,1,c) = 1$ *for all* $0 \le c \le 0.5$ *and* $K(1,p,c) = 1$ *for all* $p \in \mathbb{R}$

(iii) $K(h,p,c)$ *is decreasing of* c *for* $p \notin (0,1)$ *and increasing for* $p \in (0,1)$

(iv) $K(h,p,c) \ge 1$ *for all* $p \notin (0,1)$ *and* $0 < K(h,0.5,0) \le K(h,p,c) \le 1$ *for all* $p \in (0,1)$

(v) $K(h,p,c) \le h^{p-1}$ *for all* $p \ge 1$

Ratio type converse inequalities for the power function:

Corollary 3.

Let $(x_t)_{t\in T}$, $(\phi_t)_{t\in T}$, m, M, m_x, M_x, be as in Theorem 3, and additionally and additionally let $0 < m < M$. Let $\delta_p := m^p + M^p - 2^{1-p}(m+M)^p$, $\widetilde{x} := \frac{1}{2}1_K - \frac{1}{M-m}\int_T \phi_t\left(\left|x_t - \frac{m+M}{2}1_H\right|\right) d\mu(t)$ and $m_{\widetilde{x}}$ be the lower bound of the operator $\widetilde{x}$.

If $p \notin (0,1)$, then

$$\begin{aligned}0 &\le \int_T \phi_t(x_t^p)\, d\mu(t) \le \bar{K}(m_x, M_x, m, M, p, 0)\left(\int_T \phi_t(x_t)\, d\mu(t)\right)^p - \delta_p \widetilde{x} \\ &\le \bar{K}(m_x, M_x, m, M, p, 0)\left(\int_T \phi_t(x_t)\, d\mu(t)\right)^p \\ &\le K(m, M, p)\left(\int_T \phi_t(x_t)\, d\mu(t)\right)^p\end{aligned}$$

and

$$\begin{aligned}0 &\le \int_T \phi_t(x_t^p)\, d\mu(t) \le \bar{K}(m_x, M_x, m, M, p, m_{\widetilde{x}})\left(\int_T \phi_t(x_t)\, d\mu(t)\right)^p \\ &\le \bar{K}(m_x, M_x, m, M, p, 0)\left(\int_T \phi_t(x_t)\, d\mu(t)\right)^p \\ &\le K(m, M, p)\left(\int_T \phi_t(x_t)\, d\mu(t)\right)^p,\end{aligned}$$

Corollary 3. (continued)

where

$$\bar{K}(m_x, M_x, m, M, p, c)$$

$$=: \begin{cases} \frac{k_{t^p}\, m_x + l_{t^p} - c\delta_p}{m_x^p} & \text{if} \quad \frac{p(l_{t^p} - c\delta_p)}{m_x} \geq (1-p)\, k_{t^p}, \\ K(m, M, p, c) & \text{if} \quad \frac{p(l_{t^p} - c\delta_p)}{m_x} < (1-p)\, k_{t^p} < \frac{p(l_{t^p} - c\delta_p)}{M_x}, \\ \frac{k_{t^p}\, M_x + l_{t^p} - c\delta_p}{M_x^p} & \text{if} \quad \frac{p(l_{t^p} - c\delta_p)}{M_x} \leq (1-p)\, k_{t^p}. \end{cases} \tag{23}$$

$K(m, M, p, c)$ is a refinement of the known Kantorovich constant

$$K(m, M, p, c) := \frac{mM^p - Mm^p + c\delta_p(M-m)}{(p-1)(M-m)} \left(\frac{p-1}{p} \frac{M^p - m^p}{mM^p - Mm^p + c\delta_p(M-m)} \right)^p,$$

for $p \in \mathbb{R}$ and $0 \leq c \leq 0.5$.

Corollary 3. (continued)

If $p \in (0,1)$, *then*

$$\begin{aligned} \int_T \phi_t(x_t^p)\, d\mu(t) &\geq \bar{k}(m_x, M_x, m, M, p, 0) \left(\int_T \phi_t(x_t)\, d\mu(t)\right)^p - \delta_p \widetilde{x} \\ &\geq \bar{k}(m_x, M_x, m, M, p, 0) \left(\int_T \phi_t(x_t)\, d\mu(t)\right)^p \\ &\geq K(m, M, p) \left(\int_T \phi_t(x_t)\, d\mu(t)\right)^p \geq 0 \end{aligned}$$

and

$$\begin{aligned} \int_T \phi_t(x_t^p)\, d\mu(t) &\geq \bar{k}(m_x, M_x, m, M, p, m_{\widetilde{x}}) \left(\int_T \phi_t(x_t)\, d\mu(t)\right)^p \\ &\geq \bar{k}(m_x, M_x, m, M, p, 0) \left(\int_T \phi_t(x_t)\, d\mu(t)\right)^p \\ &\geq K(m, M, p) \left(\int_T \phi_t(x_t)\, d\mu(t)\right)^p \geq 0, \end{aligned}$$

where $\bar{k}(m_x, M_x, m, M, p, c)$ *equals the right side in* (23) *with reverse inequality signs.*

Conference

MINFAA 2012 July 25-29, Korea

Thank you very much for your attention