Robust Structural Design Based on Event-Oriented System Analysis

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Abstract- The aim of this paper is to demonstrate the calculation procedure of the robustness based structural design by analyzing failure events on a self-evident simple example. Therefore, the note at the beginning shortly reviews the probabilistic Event Oriented System Analysis (EOSA) and the definition of the robustness based on the entropy concept in the probability theory. The study jointly employs operational failure modes analysis, effects analysis, and the Advanced First Order Reliability Methods (AFORM). An example of a structural member presented by a number of random failure events illustrates the usage of the conditional entropy as a measure of structural robustness. The conclusion recommends the robustness based design for safety enhancement.

Keywords- Robustness; Entropy; Uncertainty; Reliability; Structural Safety; Structural design; Shipbuilding

I. INTRODUCTION

Robustness in contrast to vulnerability is a favorable property of structural systems. Still, there is not yet conformity among different definitions of robustness and of robust designs [1–7]. Therefore, this study investigates the usefulness of the probabilistic Event Oriented System Analysis (EOSA) [8] in the assessment of the structural robustness [9, 10].

EOSA in general considers systems and subsystems of random operational and failure events in a lifetime of a structure. Therefore, this approach enables a more objective evaluation of uncertainties since it involves the entropy concept for uncertainty assessment in the reliability analysis of the number of observable events. The entropy as a measure of information appears earlier in the information theory [11–14]. The unconditional information entropy has not been recognized earlier as a useful measure of practical importance in engineering. However, the properties of the conditional entropy enable understanding of the system behavior under random circumstances and a practical application of the entropy concept to robustness based as well as in redundancy based design [9, 10, 15, 16, 17].

The example of a basic structural element, a pillar (or a pile, a stanchion, a bar) under compressive load [18] (see Fig. 1) demonstrates in the study how the event-oriented system analysis can provide a more comprehensive assessment of system performance and a more robust structural design.

II. EVENT ORIENTED SYSTEM ANALYSIS EXAMPLE

Event Oriented System Analysis considers structures built of a number of physical components as systems of events [8–10]. Every operational or failure outcome of all operational modes of each of the components is considered as a random event E. The system of events S consists of all observable events Ei with calculated, assessed or at least rationally judged probabilities of occurrence p(Ei), i = 1, 2, … , N, where N is the total number of events in the system. The engineering reliability methods such as FOSM, FORM, AFORM, SORM or Monte Carlo simulation and Bayesian methods are on disposal for probability calculations [e.g. 19–23]. Methods of operational modes and effects analysis such as enumeration, minimal cut-sets, minimal tie-sets, event-tree and fault-tree analysis can identify all or at least the relevant and observable events Ei of a structural system [24–26]. Two types of events are of interest: operational events that represent some functional state or action mode of structure (status o) and failure events that represent some type of structural damage (status f).

System S of N = No+Nf events, where some are operational, denoted \( E^o_i, \ i = 1, 2, \ldots, No \) and some are failure events, denoted \( E^f_j, \ j = No+1, No+2, \ldots, No+Nf \) can be as represented by finite scheme:
Systems are either complete when \( \Sigma p(E_i) = 1, i = 1, 2, \ldots, N \) or incomplete when \( \Sigma p(E_i) < 1 \). In any case, the relation holds:

\[
p(S) = p(O) + p(F) = \sum_{i=1}^{N} p(E_i)
\]

Structural systems consist of a great number of components and the number of possible operational and failure events is growing rapidly sometimes denoted as combinatorial explosion.

EOSA provides the option for analysis of incomplete systems in order to consider only the important and relevant events with respect to the system safety. The event oriented system analysis applies to any relation of sets of events or subsystems, such as exclusive or inclusive sets, as well as dependent and independent events. EOSA requires proper partitioning of the system of events to a set of mutually exclusive events, for example, the well-known exclusion-inclusion expansion of union of events [27].

### III. Uncertainty Measures

The entropy is the only function appropriate for the uncertainty measure according to the uniqueness theorem [12]. Uncertainty of a single random event \( E \) is the simple entropy function \( H \) [28] also representing how an event is unlikely, probable, and the probability of failure is equal to \( p_i = 1/N \), as follows:

\[
H(p) = -\log_2 p(E)
\]

Entropy of a complete probability distribution or of a complete system of events \( H(S) \) expresses the Shannon’s entropy [9] as follows:

\[
H(S) = -\sum_{i=1}^{N} p_i \log p_i = \sum_{i=1}^{N} p_i \log \frac{1}{p_i}
\]

Another measure of uncertainty is the Renyi’s entropy of order one [13] that in its limiting case is relevant to incomplete systems of events:

\[
H_1(S) = \left( \sum_{i=1}^{N} p_i \log p_i \right) / \left( \sum_{i=1}^{N} p_i \right)
\]

The most important properties of entropy regarding EOSA application are listed as follows.

The entropy of a system is equal to zero, when the state of the system can be surely predicted, i.e., uncertainties do not exist at all. This occurs when one of the probabilities of events \( p_i, i = 1, 2, \ldots, N \) is equal to one and all the other probabilities are equal to zero.

The entropy is maximal when all events are equally probable, and the probability of failure is equal to \( p_i = 1/N \), and it amounts to \( H(S)_{\text{max}} = \log N \), the Hartley’s entropy [29].

The entropy increases as the number of events increases.

The entropy of a system of events \( S \) [9] amounts to:
\[ H_n(S) = -\sum_{i=1}^{N} p(E_i) \log p(E_i) \] (13)

If the system is incomplete [13], the entropy is:
\[ H'_n(S) = \frac{H_n(S)}{\rho(S)} \] (14)

Maximum entropy, either for complete of incomplete systems equals to:
\[ H_n(S)_{max} = \log \left[ N / \rho(S) \right] \] (15)

Using conditional probabilities of a subsystem Si consisting of m events of the same status, the conditional entropy is as:
\[ H_m(S / S_i) = -\sum_{i=1}^{N} \frac{p(E_i)}{p(S)} \log \frac{p(E_i)}{p(S)} \] (16)

The maximal conditional entropy of the subsystem Si amounts to
\[ H_m(S / S_i)_{max} = \log m \].

Relative uncertainty \[ \frac{H_n(S)}{H_n(S)_{max}} \] of systems S with same number of events as well as the average number of equally probable events denoted \( F_n(S) = 2^{\mu_e(S)} \) may be useful for practical purposes. Entropy of System S, under the condition that the system is operating (status O), is as shown:
\[ H_n(S / O) = -\sum_{i=1}^{N} \frac{p(E_i')}{p(O)} \log \frac{p(E_i')}{p(O)} \] (17)

Entropy of System S under the condition that the system is failing (status F), is as shown:
\[ H_n(S / F) = -\sum_{i=N+1}^{N'} \frac{p(E_i')}{p(F)} \log \frac{p(E_i')}{p(F)} \] (18)

It is obvious that the entropy of the operational modes in Eq. (17) and of the failure modes in Eq. (18) only depends on the states of the operational and failure modes, not on any other states. The maximum entropy of an operating system is S
\[ H_n(S / O)_{max} = \log N'_o \] and of failing system is
\[ H_n(S / F)_{max} = \log N'_f \].

IV. ROBUST DESIGN

The aim of the structural design in the first step is to assure that the structural strength and all the responses to all operational demands will remain within the elastic capabilities of the applied materials. The failures are expected to occur when the design loads exceed some nominal or working material properties not necessarily leading to structural collapse. In uncertain operational environments, the structural reliability analysis has an important role to assess the probability of failures as defined by the widely adopted rule based design procedures. However, since the designed structural strength is normally below the ultimate strength, additional checking of the ultimate strength is sometimes requested. The reliability analysis of the ultimate strength has the role to assess the probability of structural collapse.

EOSA defines robustness as the system’s capability to respond to all failures uniformly. In terms of modern robust design methodologies, the aim is not only to eliminate noise factors, but also to create insensitivity to them [1]. In the EOSA’s way of thinking, the structural robustness is the insensitivity to possible modes of structural failures by structural design equally sensitive to all applied loads. Therefore, the fully robust behavior according to EOSA implies equal failure probabilities of all failure modes under all loading conditions by avoiding weak links and removing vulnerabilities from the operational profile of the system.

When the system responds to all demands uniformly, there is a high uncertainty about which of the failure modes could occur (Eqs. 13 to 18). System robustness in EOSA is simply the conditional entropy of a subsystem of failure events:
\[ ROB(S / S') = ROB(S) = H_n(S / S') \] (19)

A system of events is probabilistically robust in service if there are several failure modes. As the probability distribution of failure events is uniform, there is higher uncertainty about which of the failure event will occur. If there is only one failure event, the uncertainties about which failure will occur vanish, i.e. system’s sensitivity is increasing in terms of robustness. A system with a number of events with smaller failure probabilities is more robust (insensitive to demands) than a system with a single highly probable failure mode. Hence, the higher the entropy of the failure modes is, the higher the robustness is and the lower the system vulnerability is [9, 10].

V. EXAMPLE

The next example demonstrates the redesign of a simply supported pillar of mild shipbuilding steel with rectangular cross section [13] under compressive load F on Fig. 2a, based on nominal elastic material properties using the robustness maximization criterion for uncertain conditions (19) as it is proposed in this technical note.

![Fig. 2a Pillar with a rectangular cross-section under compressive load F](image-url)
Fig. 2b Pillar with a rectangular cross-section under the compressive load

Loads, dimensions and material properties of the pillar on Fig. 2a are uncertain quantities presented in Table 1.

**TABLE I RANDOM VARIABLES FOR ROBUSTNESS CALCULATION OF A PILLAR ON FIG. 2A**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean value</th>
<th>COV</th>
<th>Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>36 mm</td>
<td>0.01</td>
<td>Normal</td>
</tr>
<tr>
<td>b</td>
<td>25 mm</td>
<td>0.01</td>
<td>Normal</td>
</tr>
<tr>
<td>L</td>
<td>500 mm</td>
<td>0.01</td>
<td>Normal</td>
</tr>
<tr>
<td>E</td>
<td>206000 N/mm²</td>
<td>0.01</td>
<td>Normal</td>
</tr>
<tr>
<td>σf</td>
<td>235 N/mm²</td>
<td>0.06</td>
<td>Log-Norm</td>
</tr>
<tr>
<td>F</td>
<td>150 kN</td>
<td>0.30</td>
<td>Normal</td>
</tr>
<tr>
<td>A = a ⋅ b</td>
<td>900 mm²</td>
<td>0.10</td>
<td>Normal</td>
</tr>
<tr>
<td>σCx</td>
<td>202.4 N/mm²</td>
<td>0.07</td>
<td>Log-norm</td>
</tr>
<tr>
<td>σCy</td>
<td>219.3 N/mm²</td>
<td>0.07</td>
<td>Log-norm</td>
</tr>
</tbody>
</table>

Three types of failure of a pillar are considered:
1. Compressive yield (basic event $A_1$);
2. Buckling around x-axis (basic event $A_2$);
3. Buckling around y-axis (basic event $A_3$).

The corresponding limit state functions are of linear character as shown:
1. $g_1 = A_1 \cdot σ_f - F$
2. $g_2 = A_2 \cdot σ_{Cx} - F$
3. $g_3 = A_3 \cdot σ_{Cy} - F$

Here, $σ_{Cx}$ and $σ_{Cy}$ are the Euler’s critical buckling stresses with respect to x and y axes.

Reliability indices $β$ and appropriate probabilities of failure are calculated by AFORM procedure [e.g. 19, 20] by a self-produced computer program as follows:

$β_{A1} = 3.125$, $p_f(A_1) = 0.8888 \times 10^{-3}$
$β_{A2} = 2.141$, $p_f(A_2) = 0.1615 \times 10^{-1}$
$β_{A3} = 3.153$, $p_f(A_3) = 0.8082 \times 10^{-3}$

The number of compound events, $E_i$, is $N = 2^n = 8$ (Table 2). The example represents a typical series system since there is only one operational event, $E_i^a$ - the intact mode, and seven failure events, $E_i^f$, $i = 2, 3, …, 8$.

**TABLE II COMPOUND EVENTS DEFINING THE SYSTEM OF OPERATIONAL MODES OF A LOADED PILLAR**

<table>
<thead>
<tr>
<th>Event, $E_i$</th>
<th>Failure Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_i^a$</td>
<td>-</td>
</tr>
<tr>
<td>$E_i^f$</td>
<td>compression</td>
</tr>
<tr>
<td>$E_i^f$</td>
<td>buckling</td>
</tr>
<tr>
<td>$E_i^f$</td>
<td>buckling</td>
</tr>
<tr>
<td>$E_i^f$</td>
<td>Compr&amp;buckling</td>
</tr>
<tr>
<td>$E_i^f$</td>
<td>Compr&amp;buckling</td>
</tr>
<tr>
<td>$E_i^f$</td>
<td>buckling</td>
</tr>
<tr>
<td>$E_i^f$</td>
<td>Compr&amp;buckling</td>
</tr>
</tbody>
</table>

The failure probability matrix is as follows:

$S = \begin{pmatrix}
E_1^a & E_2^f & \cdots & E_8^f \\
p(E_1^a) & p(E_2^f) & \cdots & p(E_8^f)
\end{pmatrix}$

The failure probability matrix is as follows:

$S = \begin{pmatrix}
E_1^a & E_2^f & \cdots & E_8^f \\
0.982972 & 8.74 \times 10^{-4} & 1.53 \times 10^{-2} & 8.08 \times 10^{-4} & 1.44 \times 10^{-7} & 7.18 \times 10^{-7} & 8.08 \times 10^{-7} & 8.08 \times 10^{-4}
\end{pmatrix}$

The upper Ditlevsen’s bound [19] of the failure probability of the system $S$ is calculated:

$$S = \begin{pmatrix}
8.74 \times 10^{-4} & \cdots & \text{sym} \\
1.53 \times 10^{-2} & 8.08 \times 10^{-4} & \cdots & 8.08 \times 10^{-4}
\end{pmatrix}$$

The system reliability is complementary and equals to the probability of operational events:

$$R(S) = p(E_i^a) = 0.983$$

The reliability of the system is calculated as the conditional entropy of the failure modes, Eqs. (18) and (19):

$$H(S) = 0.1342 \quad (\text{Hmax} = 2.8074)$$
Maximum robustness is $ROB_{\text{max}}(S) = \log(N_f) = \log(7) = 2.585$. Hence, the robustness of the prototype is 22% of the attainable robustness.

### A. Robustness Based Structural Design

The aim of the robustness based structural design in this study is to find out the most robust structural configuration of a pillar on Fig. 2a, by employing EOSA.

Therefore, the example firstly investigates the effect of cross-sectional properties $a$ and $b$ on the system robustness. The weight of the component remained constant throughout the analysis, that is:

- cross sectional area is constant, $A = 900 \text{ mm}^2$
- pillar length is constant, $L = 500 \text{ mm}$

The most robust structural configuration obtained expectedly for the symmetrical cross section $a = b = 30 \text{ mm}$ confirms the common engineering reasoning that the pillar with symmetrical cross-section should be most insensitive to the applied load, Fig. 2b. The result also confirms that EOSA provides quantitative measure for the intuited robust structural behavior in engineering.

The robustness of the new configuration (new cross section) amounts to $ROB(S) = 1.78$, which is a significant increase of 69% of the maximum $\log 26$, Fig. 3, compared to the 22% of the prototype. At the same time, the reliability of the system increases slightly changes from 0.983 to 0.996.

Secondly, the example investigates the effect of the pillar length $L$ on the system robustness for symmetrical cross section $a = b = 30 \text{ mm}$ under the condition that the system reliability is $R(S) \geq 0.99$. This was investigated due to practical reasons, since the minimum required reliability is often a limiting parameter in structural design. That is even more pronounced for lengths ranged from 50 mm to 250 mm, where the reliability is constantly $R(S) = 0.9991$, Fig. 4. The attained robustness $ROB(S) = 2.01$ is 78% of the maximal value for $L = 423 \text{ mm}$. That is a significant increase with respect to the robustness of the prototype of length $L = 500 \text{ mm}$. The result indicates that the pillar shortening at 0.846 $L$ of the initial length, which can be arranged by adequate bracketing at both ends, Fig. 2b, significantly increases the system robustness, Fig. 3. The system of events that represents pillar with maximum robustness ($L = 423 \text{ mm}$, $a = b = 30 \text{ mm}$) is:

- System of events for the pillar with highest robustness clearly shows a more uniform distribution of failure probabilities of the prototype:

  $$ S = \left\{ E_1^f, E_{11}^f, E_{12}^f, E_1^f, E_{13}^f, E_{13}^f \right\} $$

  $$ S = \left\{ 0.996041, 8.83 \times 10^{-4}, 3.08 \times 10^{-5}, 3.08 \times 10^{-5}, 2.73 \times 10^{-4}, 2.73 \times 10^{-4}, 3.08 \times 10^{-3} \right\} $$

  Reliability indices and probabilities of failure for the system with maximum robustness are calculated as previously:

  $\beta A1 = 3.1251$, $\beta A2 = 2.73961$, $\beta A3 = 2.73961$

  $pf(A1) = 0.8888 \times 10^{-3}$, $pf(A2) = 0.3075 \times 10^{-2}$, $pf(A3) = 0.3075 \times 10^{-2}$.

The increase in structural robustness results in more uniform failure probability distribution, Fig. 5. Typical pillar end support is presented on Fig. 6.
VI. CONCLUSIONS

The paper investigated how the Event Oriented System Analysis that employs methods of operational modes and effects analysis, engineering reliability methods and the entropy concept in probability theory could enhance the robustness of structural systems, that is, how to reduce sensitivity to possible failure modes.

The procedure implies on one hand theoretically and experimentally well approved and widely adopted mechanical criteria for yielding and buckling failures of structural elements. On the other hand the procedure implies less certain statistical data about random loads and uncertain material properties combined with probabilistic theory of well established structural reliability to deal with operational uncertainties. It is taken for granted that statistics and probabilistic theories are well established in the past and their results can be extended to engineering problems of structural safety without additional proofs. However, it is almost impossible and for sure fully impractical to prove the accuracy of probabilistic reliability analysis of complex structural systems by experimenting. The lack of proofs can be compensated by systematic data collection about structural uncertainties of loads, workmanship and material properties as well as by well thought-out usage of the potentials of the probabilistic reliability theory for structural safety assessments.

Nevertheless, the methods of probabilistic reliability analysis are at present the only practical approach that can deal with structural safety assessment under uncertain operational and service conditions.

The computational procedure on digital computers combines two programmable modules, one for calculation of structural failure modes and another for probabilistic system analysis using some of the well-known methods for reliability analysis. The first part is problem dependent and has to be programmed for particular structural requirements. The second one is normally available as a software solution for any of structural reliability methods that suits best. Here, the combinatorial explosion in case of greater number of events under consideration might be the limiting condition in selection of appropriate hardware configurations.

The example of a robust design methodology based on event-oriented system analysis, exhibited in the paper, brings forward following observations:

The entropy measure of robustness is highly sensitive to variations in mechanical and material properties of structural elements.

The system robustness may distinguish different distributions of failure probabilities even among structures of the same reliability.

The robustness of a structural configuration may indicate maximum robustness under various imposed structural and operational constraints.

The increase in structural robustness provides more uniform distribution of failure probabilities.

At the end, the paper recommends the robust design based on event-oriented system analysis as a potentially useful tool in structural safety enhancement.

REFERENCES


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