Some Identities in the Twisted Group Algebra of Symmetric Groups

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i.e $S_n$ is the set of all permutations of a set $M = \{1, 2, \ldots, n\}$ equipped with a composition as the binary operation on $S_n$ (clearly, the permutations are regarded as bijections from $M$ to itself);
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- $R_n := \mathbb{C}[X_{a\ b} \mid 1 \leq a, b \leq n]$ denote the polynomial ring
  i.e the commutative ring of all polynomials in $n^2$ variables $X_{a\ b}$ over the set $\mathbb{C}$
  with $1 \in R_n$ as a unit element of $R_n$.
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  $\mathbb{C}$ = the set of complex numbers.

- $S_n$ acts on the set $X$ as follows: $g.X_{ab} = X_{g(a)\ g(b)}$.

- This action of $S_n$ on $X$ induces the action of $S_n$ on $R_n$ given by
  $g.p(\ldots, X_{ab}, \ldots) = p(\ldots, X_{g(a)\ g(b)}, \ldots)$
  for every $g \in S_n$ and any $p \in R_n$. 
Recall, the usual group algebra

\[ \mathbb{C}[S_n] = \left\{ \sum_{\sigma \in S_n} c_{\sigma} \sigma \mid c_{\sigma} \in \mathbb{C} \right\} \]

of the symmetric group \( S_n \) is a free vector space (generated with the set \( S_n \)), where the multiplication is given by

\[ \left( \sum_{\sigma \in S_n} c_{\sigma} \sigma \right) \cdot \left( \sum_{\tau \in S_n} d_{\tau} \tau \right) = \sum_{\sigma,\tau \in S_n} (c_{\sigma} d_{\tau}) \sigma \tau. \]

Here we have used the simplified notation: \( \sigma \tau = \sigma \circ \tau \)

for the composition \( \sigma \circ \tau \) i.e. the product of \( \sigma \) and \( \tau \) in \( S_n \).
Now we define more general group algebra

\[ A(S_n) := R_n \rtimes \mathbb{C}[S_n] \]

a twisted group algebra of the symmetric group \( S_n \) with coefficients in the polynomial ring \( R_n \).

Here \( \rtimes \) denotes the semidirect product.
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\[ \sum_{g_i \in S_n} p_i g_i \quad \text{with} \quad p_i \in R_n. \]
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\[
\sum_{g_i \in S_n} p_i \cdot g_i \quad \text{with} \quad p_i \in R_n.
\]

- The multiplication in \( \mathcal{A}(S_n) \) is given by

\[
(p_1 g_1) \cdot (p_2 g_2) := (p_1 \cdot (g_1 \cdot p_2)) \cdot g_1 g_2
\]

where \( g_1 \cdot p_2 \) is defined by:

\[
g \cdot p(\ldots, X_{ab}, \ldots) = p(\ldots, X_{g(a)g(b)}, \ldots)
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where \( g_1 \cdot p_2 \) is defined by:

\[ g \cdot p(\ldots, X_{a,b}, \ldots) = p(\ldots, X_{g(a)g(b)}, \ldots) \]

- The algebra \( \mathcal{A}(S_n) \) is associative but not commutative.
Let

\[ I(g) = \{ (a, b) \mid 1 \leq a < b \leq n, \ g(a) > g(b) \} \]

denote the set of inversions of \( g \in S_n \).

Then to every \( g \in S_n \) we associate a monomial in the ring \( R_n \) defined by

\[
X_g := \prod_{(a, b) \in I(g^{-1})} X_{ab} \left( = \prod_{a < b, \ g^{-1}(a) > g^{-1}(b)} X_{ab} \right),
\]

which encodes all inversions of \( g^{-1} \) (and of \( g \) too).

More generally, for any subset \( A \subseteq \{1, 2, \ldots, n\} \) we will use the notation

\[
X_A := \prod_{(a, b) \in A \times A, \ a < b} X_{ab} \cdot X_{ba} = \prod_{(a, b) \in A \times A, \ a < b} X_{\{a, b\}},
\]

because

\[
X_{\{a, b\}} := X_{ab} \cdot X_{ba}.
\]
**Definition**

To each $g \in S_n$ we assign a unique element $g^* \in A(S_n)$ defined by

$$g^* := X_g g.$$ 

**Theorem**

For every $g_1^*, g_2^* \in A(S_n)$ we have

$$g_1^* \cdot g_2^* = X(g_1, g_2) (g_1g_2)^*,$$

where the multiplication factor is given by

$$X(g_1, g_2) = \prod_{(a,b) \in I(g_1^{-1}) \setminus I((g_1g_2)^{-1})} X_{\{a, b\}}.$$
Recall,

\[ X(g_1, g_2) = \prod_{(a,b) \in I(g_1^{-1}) \setminus I((g_1 g_2)^{-1})} X_{\{a, b\}} \]

Note that

\[ X(g_1, g_2) = 1 \]

if \( l(g_1 g_2) = l(g_1) + l(g_2) \).

So we have

\[ g_1^* \cdot g_2^* = (g_1 g_2)^* \]

where \( l(g) := \text{Card} I(g) \) is the length of \( g \in S_n \).
Recall,

$$X(g_1, g_2) = \prod_{(a,b) \in I(g_1^{-1}) \setminus I((g_1 g_2)^{-1})} X\{a, b\}$$

- Note that

  $$X(g_1, g_2) = 1$$

  if

  $$l(g_1 g_2) = l(g_1) + l(g_2).$$

So we have

$$g_1^* \cdot g_2^* = (g_1 g_2)^*$$

where

$$l(g) := Card I(g)$$

is the length of \( g \in S_n \).

- The factor \( X(g_1, g_2) \) takes care of the reduced number of inversions in the group product of \( g_1, g_2 \in S_n \).
Recall, \[ g_1^* \cdot g_2^* = X(g_1, g_2)(g_1g_2)^* \] for every \( g_1^*, g_2^* \in A(S_n) \)

**Example**

Let \( g_1 = 132, \quad g_2 = 312 \in S_3 \).

Then \( g_1g_2 = 213, \quad l(g_1) = 1, \quad l(g_2) = 2, \quad l(g_1g_2) = 1 \).

Note that \( g_1^{-1} = 132, \quad g_2^{-1} = 231 \), so

\[
g_1^* \cdot g_2^* = (X_{23} g_1) \cdot (X_{13}X_{23} g_2) = X_{23}X_{12}X_{32} g_1g_2 = X_{\{2, 3\}} X_{12} g_1g_2.
\]

On the other hand we have:

\[
(g_1g_2)^* = X_{12} g_1g_2,
\]

since \((g_1g_2)^{-1} = 213\).

Thus we get

\[
g_1^* \cdot g_2^* = X_{\{2, 3\}} (g_1g_2)^*
\]

and

\[
X(g_1, g_2) = X_{\{2, 3\}}.
\]
Recall,
\[ g_1^* \cdot g_2^* = X(g_1, g_2) (g_1 g_2)^* \] for every \( g_1^*, g_2^* \in \mathcal{A}(S_n) \)

**Example**

For \( g_1 = 132, \quad g_2 = 231 \)

we have \( g_1 g_2 = 321, \quad l(g_1) = 1, \quad l(g_2) = 2, \quad l(g_1 g_2) = 3. \)

Further \( g_1^{-1} = 132, \quad g_2^{-1} = 312 \) and \( (g_1 g_2)^{-1} = 321, \)

so we get:

\[
 g_1^* \cdot g_2^* = (X_{23} g_1) \cdot (X_{12} X_{13} g_2) = X_{23} X_{13} X_{12} g_1 g_2,
\]

\[
 (g_1 g_2)^* = X_{12} X_{13} X_{23} g_1 g_2.
\]

Thus we get

\[
 g_1^* \cdot g_2^* = (g_1 g_2)^*
\]

and

\[
 X(g_1, g_2) = 1.
\]
We denote by

\( t_{a,b}, \quad 1 \leq a \leq b \leq n \)

the following cyclic permutation in \( S_n \):

\[
t_{a,b}(k) := \begin{cases} 
  k & 1 \leq k \leq a - 1 \text{ or } b + 1 \leq k \leq n \\
  b & k = a \\
  k - 1 & a + 1 \leq k \leq b
\end{cases}
\]

which maps \( b \) to \( b - 1 \) to \( b - 2 \) \( \cdots \) to \( a \) to \( b \) and fixes all \( 1 \leq k \leq a - 1 \) and \( b + 1 \leq k \leq n \) i.e

\[
t_{a,b} = \begin{pmatrix} 
  1 & \ldots & a - 1 & a & a + 1 & \ldots & b - 1 & b & b + 1 & \ldots & n \\
  1 & \ldots & a - 1 & b & a & \ldots & b - 2 & b - 1 & b + 1 & \ldots & n
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t_{a,b} = \begin{pmatrix}
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1 & \ldots & a - 1 & b & a & \ldots & b - 2 & b - 1 & b + 1 & \ldots & n
\end{pmatrix}
\]

\[ t_{b,a} := t_{a,b}^{-1} \quad \text{i.e.}
\]

\[
t_{b,a}(k) = \begin{cases} 
k & 1 \leq k \leq a - 1 \quad \text{or} \quad b + 1 \leq k \leq n \\
k + 1 & a \leq k \leq b - 1 \\
a & k = b \end{cases}
\]
Then the sets of inversions are given by

\[ I(t_{a,b}) = \{(a, j) \mid a + 1 \leq j \leq b\}, \]

\[ I(t_{b,a}) = \{(i, b) \mid a \leq i \leq b - 1\}. \]

so the corresponding elements in \( \mathcal{A}(S_n) \) have the form:

\[ t_{a,b}^* = \left( \prod_{a \leq i \leq b-1} X_{i,b} \right) t_{a,b} \quad \quad \quad t_{b,a}^* = \left( \prod_{a+1 \leq j \leq b} X_{a,j} \right) t_{b,a}. \]

Observe: \( t_{a,a}^* = id \), where \( I(t_{a,a}) = \emptyset \).

Denote: \( t_a = t_{a,a+1} \) (= \( t_{a+1,a} \)), \( 1 \leq a \leq n - 1 \)

(the transposition of adjacent letters \( a \) and \( a + 1 \)).

Then: \( t_a^* = X_{a,a+1} t_a \), with \( I(t_a) = \{(a, a + 1)\} \).
Recall, 

\[ g_1^* \cdot g_2^* = X(g_1, g_2) (g_1 g_2)^* \quad \text{for every} \quad g_1^*, g_2^* \in A(S_n) \]

**Corollary**

**For each** \(1 \leq a \leq n - 1\) **we have**

\[(t_a^*)^2 = X_{\{a, a+1\}} \text{id}.\]

Here we have used that \(t_a t_a = \text{id}\) and \(X_{\{a, a+1\}} = X_{a \cdot a+1} \cdot X_{a+1 \cdot a} \).

**Corollary**

**For each** \(g \in S_n, 1 \leq a < b \leq n\) **we have**

\[ g^* \cdot t_{b,a}^* = \left( \prod_{a < j \leq n, g(a) > g(j)} X_{\{g(j), g(a)\}} \right) (gt_{b,a})^*. \]

**In the case** \(g \in S_j \times S_{n-j}, 1 \leq j \leq k \leq n\) **we have**

\[ g^* \cdot t_{k,j}^* = (gt_{k,j})^*. \]
Recall, \( g_1^* \cdot g_2^* = X(g_1, g_2) (g_1 g_2)^* \) for every \( g_1^*, g_2^* \in A(S_n) \).

**Corollary (Braid relations)**

We have

(i) \( t_a^* \cdot t_{a+1}^* \cdot t_a^* = t_{a+1}^* \cdot t_a^* \cdot t_{a+1}^* \) for each \( 1 \leq a \leq n - 2 \),

(ii) \( t_a^* \cdot t_b^* = t_b^* \cdot t_a^* \) for each \( 1 \leq a, b \leq n - 1 \) with \( |a - b| \geq 2 \).

**Corollary (Commutation rules)**

We have

(i) \( t_{m,k}^* \cdot t_{p,k}^* = (t_{k}^*)^2 \cdot t_{p,k+1}^* \cdot t_{m-1,k}^* \) if \( 1 \leq k \leq m \leq p \leq n \).

(ii) Let \( w_n (= n \ n - 1 \ \cdots \ 2 \ 1) \) be the longest permutation in \( S_n \). Then for every \( g \in S_n \) we have

\[
(gw_n)^* \cdot w_n^* = w_n^* \cdot (w_n g)^* = \prod_{a < b, g^{-1}(a) < g^{-1}(b)} X\{a, b\} g^*.
\]
Decompositions (from the left) of certain canonical elements in $A(S_n)$

Observe first:

for $\forall g \in S_n$ \exists $g_1 \in S_1 \times S_{n-1}$ and $1 \leq k_1 \leq n$ such that

$$g = g_1 t_{k_1,1}$$

Then $g(k_1) = g_1(t_{k_1,1}(k_1)) = g_1(1) = 1$ implies $k_1 = g^{-1}(1)$. 
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  Then $g(k_1) = g_1(t_{k_1,1}(k_1)) = g_1(1) = 1$ implies $k_1 = g^{-1}(1)$.

• Subsequently, the permutation $g_1 \in S_1 \times S_{n-1}$ can be represented uniquely as $g_1 = g_2 t_{k_2,2}$

  with $g_2 \in S_1 \times S_1 \times S_{n-2}$ and $2 \leq k_2 \leq n$.

  Then $g_1(k_2) = g_2(t_{k_2,2}(k_2)) = g_2(2) = 2$ implies $k_2 = g_1^{-1}(2)$. 

Decompositions (from the left) of certain canonical elements in $\mathcal{A}(S_n)$

- Observe first:
  
  for $\forall g \in S_n$ \quad $\exists g_1 \in S_1 \times S_{n-1}$ and $1 \leq k_1 \leq n$ such that
  
  $g = g_1 t_{k_1,1}$

  Then $g(k_1) = g_1(t_{k_1,1}(k_1)) = g_1(1) = 1$ implies $k_1 = g^{-1}(1)$.

- Subsequently, the permutation $g_1 \in S_1 \times S_{n-1}$ can be represented uniquely as $g_1 = g_2 t_{k_2,2}$ with $g_2 \in S_1 \times S_1 \times S_{n-2}$ and $2 \leq k_2 \leq n$.

  Then $g_1(k_2) = g_2(t_{k_2,2}(k_2)) = g_2(2) = 2$ implies $k_2 = g_1^{-1}(2)$.

- By repeating the above procedure we get the following decomposition:

  $$g = t_{k_n,n} \cdot t_{k_{n-1},n-1} \cdots t_{k_j,j} \cdots t_{k_2,2} \cdot t_{k_1,1} \left( \prod_{1 \leq j \leq n} t_{k_j,j} \right).$$
Recall, \[ g = t_{k_{n,n}} \cdot t_{k_{n-1,n-1}} \cdots t_{k_{j,j}} \cdots t_{k_{2,2}} \cdot t_{k_{1,1}}. \]

**Example**

Let \[ S_3 = \{123, 132, 312, 321, 231, 213\} \] then in \( A(S_3) \) we have:

\[
123^* = t_{3,3}^* \cdot t_{2,2}^* \cdot t_{1,1}^*, \quad 132^* = t_{3,3}^* \cdot t_{3,2}^* \cdot t_{1,1}^*, \quad 312^* = t_{3,3}^* \cdot t_{3,2}^* \cdot t_{2,1}^*,
\]

\[
321^* = t_{3,3}^* \cdot t_{3,2}^* \cdot t_{3,1}^*, \quad 231^* = t_{3,3}^* \cdot t_{2,2}^* \cdot t_{3,1}^*, \quad 213^* = t_{3,3}^* \cdot t_{2,2}^* \cdot t_{2,1}^*.
\]
Recall,  
\[ g = t_{k_n,n} \cdot t_{k_{n-1},n-1} \cdots t_{k_j,j} \cdots t_{k_2,2} \cdot t_{k_1,1}. \]

**Example**

- Let \( S_3 = \{123, 132, 312, 321, 231, 213\} \) then in \( \mathcal{A}(S_3) \) we have:
  
  \[
  123^* = t_{3,3}^* \cdot t_{2,2}^* \cdot t_{1,1}^*, \quad 132^* = t_{3,3}^* \cdot t_{3,2}^* \cdot t_{1,1}^*, \quad 312^* = t_{3,3}^* \cdot t_{3,2}^* \cdot t_{2,1}^*, \]
  
  \[
  321^* = t_{3,3}^* \cdot t_{3,2}^* \cdot t_{3,1}^*, \quad 231^* = t_{3,3}^* \cdot t_{2,2}^* \cdot t_{3,1}^*, \quad 213^* = t_{3,3}^* \cdot t_{2,2}^* \cdot t_{2,1}^*. \]

- Now, assume that
  
  \[ \alpha_3^* = \sum_{g \in S_3} g^*. \]

Then we get the following product form:

\[
\alpha_3^* = \left( t_{3,3}^* \right) \cdot \left( t_{3,2}^* + t_{2,2}^* \right) \cdot \left( t_{3,1}^* + t_{2,1}^* + t_{1,1}^* \right)
\]

\[ \beta_1^* = (id), \quad \beta_2^* \quad \beta_3^* \]

of simpler elements \( \beta_i^*, \ 1 \leq i \leq 3 \) of the algebra \( \mathcal{A}(S_3) \).
The general situation in $\mathcal{A}(S_n)$:

**Definition**

For every $1 \leq k \leq n$ we define

$$\beta^*_{n-k+1} := t^*_{n,k} + t^*_{n-1,k} + \cdots + t^*_{k+1,k} + t^*_{k,k} \left( = \sum_{k \leq m \leq n} t^*_{m,k} \right).$$

**Theorem**

Let

$$\alpha^*_n = \sum_{g \in S_n} g^*.$$

Then

$$\alpha^*_n = \beta^*_1 \cdot \beta^*_2 \cdots \beta^*_n \left( = \prod_{1 \leq k \leq n-1} \beta^*_{n-k+1} \right).$$
In what follows we are going to introduce some new elements in the algebra \( \mathcal{A}(S_n) \) by which we will reduce \( \beta_{n-k+1}^*, 1 \leq k \leq n \).

The motivation is to show that the element \( \alpha_n^* \in \mathcal{A}(S_n) \) can be expressed in turn as products of yet simpler elements of the algebra \( \mathcal{A}(S_n) \).

**Definition**

For every \( 1 \leq k \leq n - 1 \) we define

\[
\gamma_{n-k+1}^* := (id - t_{n,k}^*) \cdot (id - t_{n-1,k}^*) \cdots (id - t_{k+1,k}^*),
\]

\[
\delta_{n-k+1}^* := (id - (t_k^*)^2 t_{n,k+1}^*) \cdot (id - (t_k^*)^2 t_{n-1,k+1}^*) \cdots (id - (t_k^*)^2 t_{k+1,k+1}^*).
\]

Recall, \((t_k^*)^2 = X_{\{k,k+1\}} \cdot id = X_{k+1} \cdot X_{k+1} \cdot id\) and \(t_{k+1,k+1}^* = id\).

**Theorem**

For every \( 1 \leq k \leq n \) we have the following factorization

\[
\beta_{n-k+1}^* = \delta_{n-k+1}^* \cdot (\gamma_{n-k+1}^*)^{-1}.
\]
Recall, 
\[ \alpha_n^* = \beta_1^* \cdot \beta_2^* \cdots \beta_n^* \quad \text{with} \quad \beta_1^* = id \]

\[ \beta_{n-k+1}^* = \delta_{n-k+1}^* \cdot \left( \gamma_{n-k+1}^* \right)^{-1} \]

\[ \gamma_{n-k+1}^* = \left( id - t_{n,k}^* \right) \cdot \left( id - t_{n-1,k}^* \right) \cdots \left( id - t_{k+1,k}^* \right) \]

\[ \delta_{n-k+1}^* = \left( id - (t_k^*)^2 t_{n,k+1}^* \right) \cdot \left( id - (t_k^*)^2 t_{n-1,k+1}^* \right) \cdots \left( id - (t_k^*)^2 t_{k+1,k+1}^* \right) \]

**Example (The factorization of \( \alpha_2^* \in A(S_2) \))**

We have

\[ \alpha_2^* = \beta_2^* \]

i.e

\[ \alpha_2^* = \left( id - (t_1^*)^2 \right) \cdot \left( id - t_{2,1}^* \right)^{-1} \]
Recall, \[ \alpha_n^* = \beta_1^* \cdot \beta_2^* \cdots \beta_n^* \quad \text{with} \quad \beta_1^* = \text{id} \]

\[ \beta_{n-k+1}^* = \delta_{n-k+1}^* \cdot (\gamma_{n-k+1}^*)^{-1} \]

\[ \gamma_{n-k+1}^* = (\text{id} - t_{n,k}^*) \cdot (\text{id} - t_{n-1,k}^*) \cdots (\text{id} - t_{k+1,k}^*) \]

\[ \delta_{n-k+1}^* = (\text{id} - (t_k^*)^2 t_{n,k+1}^*) \cdot (\text{id} - (t_k^*)^2 t_{n-1,k+1}^*) \cdots (\text{id} - (t_k^*)^2 t_{k+1,k+1}^*) \]

**Example (The factorization of \( \alpha_3^* \in A(S_3) \))**

We have \[ \alpha_3^* = \beta_2^* \cdot \beta_3^* \]

where

\[ \beta_2^* = (\text{id} - (t_2^*)^2) \cdot (\text{id} - t_{3,2}^*)^{-1} \]

\[ \beta_3^* = (\text{id} - (t_1^*)^2 t_{3,2}^*) \cdot (\text{id} - (t_1^*)^2) \cdot (\text{id} - t_{2,1}^*)^{-1} \cdot (\text{id} - t_{3,1}^*)^{-1} \]
Recall,
\[ \alpha^*_n = \beta^*_1 \cdot \beta^*_2 \cdots \beta^*_n \quad \text{with} \quad \beta^*_1 = id \]

\[ \beta^*_{n-k+1} = \delta^*_{n-k+1} \cdot (\gamma^*_n)^{-1} \]

\[ \gamma^*_{n-k+1} = (id - t^*_n, k) \cdot (id - t^*_{n-1}, k) \cdots (id - t^*_{k+1}, k) \]

\[ \delta^*_{n-k+1} = (id - (t^*_k)^2 t^*_{n,k+1}) \cdot (id - (t^*_k)^2 t^*_{n-1,k+1}) \cdots (id - (t^*_k)^2 t^*_{k+1,k+1}) \]

**Example (The factorization of \( \alpha^*_4 \in A(S_4) \))**

We have

\[ \alpha^*_4 = \beta^*_2 \cdot \beta^*_3 \cdot \beta^*_4 \]

where

\[ \beta^*_2 = (id - (t^*_3)^2) \cdot (id - t^*_{4,3})^{-1} \]

\[ \beta^*_3 = (id - (t^*_2)^2 \cdot t^*_{4,3}) \cdot (id - (t^*_2)^2) \cdot (id - t^*_{3,2})^{-1} \cdot (id - t^*_{4,2})^{-1} \]

\[ \beta^*_4 = (id - (t^*_1)^2 \cdot t^*_{4,2}) \cdot (id - (t^*_1)^2 \cdot t^*_{3,2}) \cdot (id - (t^*_1)^2) \cdot (id - t^*_{2,1})^{-1} \cdot (id - t^*_{3,1})^{-1} \cdot (id - t^*_{4,1})^{-1} \]
In order to replace the matrix factorizations (from the right) given in \(^1\) by twisted algebra computation, we need to consider similar factorizations (but from the left).
Here we used factorizations from the left, because they are more suitable for computing constants in the algebra of noncommuting polynomials (this will be elaborated in a forthcoming paper).

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