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A NOTE ON THE TRACE THEOREM FOR DOMAINS WHICH ARE LOCALLY SUBGRAPH OF A HÖLDER CONTINUOUS FUNCTION

Boris Muha

Department of Mathematics, Faculty of Science
University of Zagreb
Bijenička 30, 10 000 Zagreb, Croatia

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Abstract. The purpose of this note is to prove a version of the Trace Theorem for domains which are locally subgraph of a Hölder continuous function. More precisely, let $\eta \in C^{0,\alpha}(\omega)$, $0 < \alpha < 1$ and let Ω_η be a domain which is locally subgraph of a function η. We prove that mapping $\gamma_\eta : u \mapsto u(\mathbf{x}, \eta(\mathbf{x}))$ can be extended by continuity to a linear, continuous mapping from $H^1(\Omega_\eta)$ to $H^s(\omega)$, $s < \alpha/2$. This study is motivated by analysis of fluid-structure interaction problems.

1. **Introduction.** The Trace Theorem for Sobolev spaces is well-known and widely used in analysis of boundary and initial-boundary value problems in partial differential equations. Usually, for the Trace Theorem to hold, the minimal assumption is that the domain has a Lipshitz boundary (see e. g. [1, 5, 7]). However, when studying weak solutions to a moving boundary fluid-structure interaction (FSI) problem, domains are not necessary Lipshitz (see [2, 6, 9, 4, 13]). FSI problems have many important applications (for example in biomechanics and aero-elasticity) and therefore have been extensively studied from the analytical, as well as numerical point of view, since the late 1990s (see e.g. [2, 3, 6, 8, 9, 10, 12] and the references within). In FSI problems the fluid domain is unknown, given by an elastic deformation η, and therefore one cannot assume a priori any smoothness of the domain. In [2, 6, 9] an energy inequality implies $\eta \in H^2(\omega)$, $\omega \subset \mathbb{R}^2$. From the Sobolev embeddings one can see that in this case $\eta \in C^{0,\alpha}(\omega)$, $\alpha < 1$, but η is not necessarily Lipschitz. Nevertheless, in Section 1.3 in [2], and Section 1.3. in [6], a version of the Trace Theorem for such domains was proved, which enables the analysis of the considered FSI problems (see also [9], Section 2).

The proof of a version of the Trace Theorem in [6] (Lemma 2) relies on Sobolev embeddings theorems and the fact that $\eta \in H^2(\omega)$ and $\omega \subset \mathbb{R}^2$. Even though the techniques from [6] can be generalized to a broader class of Sobolev class boundaries, the result and techniques from [6] cannot be applied to some other cases of interest

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in FSI problems, for example to the coupling of $2D$ fluid flow with the $1D$ wave equation, where we only have $\eta \in H^1(\omega)$ (see [4, 13]) The purpose of this note is to fill that gap and generalize that result for $\omega \subset \mathbb{R}^{n-1}$, $n > 1$, and arbitrary Hölder continuous functions η. Hence, we prove a version of the Trace Theorem for a domain which is locally a subgraph of a Hölder continuous function. We use real interpolation theory (see [11]) and intrinsic norms for H^s spaces, where s in not an integer.

2. **Notation and preliminaries.**

Let $n \in \mathbb{N}$, $n \geq 2$. Let $\omega \subset \mathbb{R}^{n-1}$ be a Lipschitz domain and let $0 < \alpha < 1$. Furthermore, let η satisfy the following conditions:

$$\eta \in C^{0,\alpha}(\omega),\ \eta(\mathbf{x}) \geq \eta_{min} > 0,\ \mathbf{x} \in \overline{\omega},\ \eta_{|\partial\omega} = 1. \tag{1}$$

We consider the following domain

$$\Omega_\eta = \{(\mathbf{x}, x_n) : \mathbf{x} \in \omega,\ 0 < x_n < \eta(x)\},$$

with its upper boundary

$$\Gamma_\eta = \{(\mathbf{x}, x_n) : \mathbf{x} \in \omega,\ x_n = \eta(x)\}.$$

We define the trace operator $\gamma_\eta : C(\overline{\Omega_\eta}) \to C(\omega)$

$$(\gamma_\eta u)(\mathbf{x}) = u(\mathbf{x}, \eta(\mathbf{x})), \quad \mathbf{x} \in \omega,\ u \in C^0(\overline{\Omega_\eta}). \tag{2}$$

In [2] (Lemma 1) it has been proven that γ_η can be extended by continuity to an operator $\gamma_\eta : H^1(\Omega_\eta) \to L^2(\omega)$. This result holds with an assumption that η is only continuous. Our goal is to extend this result in a way to show that $\mathrm{Im}(\gamma_\eta)$ is a subspace of $H^s(\omega)$, for some $s > 0$, when η is a Hölder continuous function.

Remark 1. Notice that γ_η is not a classical trace operator because $\gamma_\eta(u)$ is a function defined on ω, whereas the classical trace would be defined on the upper part of the boundary, Γ_η. However, this version of a trace operator is exactly what one needs in analysis of FSI problems. Namely, in the FSI setting the Trace Theorem is applied to fluid velocity which, at the interface, equals the structure velocity, where the structure velocity is defined on a Lagrangian domain (in our notation ω).

The Sobolev space $H^s(\omega)$, $0 < s < 1$ is defined by the real interpolation method (see [1, 11]). However, $H^s(\omega)$ can be equipped with an equivalent, intrinsic norm (see for example [1, 7]) which is also used in [5]

$$\|u\|^2_{H^s(\omega)} = \|u\|^2_{L^2(\omega)} + \int_{\omega\times\omega} \frac{|u(\mathbf{x}_1) - u(\mathbf{x}_2)|^2}{|\mathbf{x}_1 - \mathbf{x}_2|^{n-1+2s}} d\mathbf{x}_1 d\mathbf{x}_2, \tag{3}$$

where $0 < s < 1$.

3. **Statement and proof of the result.**

Theorem 3.1. *Let $\alpha < 1$ and let η be such that conditions (1) are satisfied. Then operator γ_η, defined by (2), can be extended by continuity to a linear operator from $H^1(\Omega_\eta)$ to $H^s(\omega)$, $0 \leq s < \frac{\alpha}{2}$.*

Proof. We split the main part of the proof into two Lemmas. The main idea of the proof is to transform a function defined on Ω_η to a function defined on $\omega \times (0,1)$ and to apply classical Trace Theorem to a function defined on the domain $\omega \times (0,1)$. Throughout this proof C will denote a generic positive constant that depends only on ω, η and α.

Let $u \in H^1(\Omega_\eta)$. Define

$$\bar{u}(\mathbf{x}, t) = u(\mathbf{x}, \eta(\mathbf{x})t),\ \mathbf{x} \in \omega,\ t \in [0,1]. \tag{4}$$

Let us define function space (see [11], p. 10):

$$W(0,1;s) = \{f : f \in L^2(0,1;H^s(\omega)),\ \partial_t f \in L^2(0,1;L^2(\omega))\},$$

where $0 < s < 1$. Our goal is to prove $\bar{u} \in W(0,1;s)$. However, before that we need to prove the following technical Lemma:

Lemma 3.2. *For every $\mathbf{x}_0$, $\mathbf{x}_1 \in \omega$, there exists a piece-wise smooth curve parameterized by*

$$\Theta_{x_0,x_1} : [0,2] \to \Omega_\eta$$

such that $\Theta_{x_0,x_1}(0) = (\mathbf{x}_0, \eta(\mathbf{x}_0))$, $\Theta_{x_0,x_1}(2) = (\mathbf{x}_1, \eta(\mathbf{x}_1))$ and

$$|\Theta'_{x_0,x_1}(r)| \le C|\mathbf{x}_1 - \mathbf{x}_0|^\alpha, \quad \text{a. e. } r \in [0,2], \tag{5}$$

where C does not depend on $\mathbf{x}_0$, $\mathbf{x}_1$.

Proof. First we define $\mathbf{x}_r$ as a convex combination of $\mathbf{x}_0$ and $\mathbf{x}_1$:

$$\mathbf{x}_r = (1 - r^{1/\alpha})\mathbf{x}_0 + r^{1/\alpha}\mathbf{x}_1 = \mathbf{x}_0 + r^{1/\alpha}(\mathbf{x}_1 - \mathbf{x}_0), \quad r \in [0,1].$$

Furthermore we define $\mathbf{y}_r$ in the following way:

$$y_r = \eta(\mathbf{x}_0) - \|\eta\|_{C^{0,\alpha}(\omega)}|\mathbf{x}_r - \mathbf{x}_0|^\alpha = \eta(\mathbf{x}_0) - \|\eta\|_{C^{0,\alpha}(\omega)} r|\mathbf{x}_1 - \mathbf{x}_0|^\alpha,\ r \in [0,1].$$

By using Hölder continuity of η we get

$$y_r \le \eta(\mathbf{x}_r), \quad r \in [0,1]. \tag{6}$$

Therefore curve $(\mathbf{x}_r, y_r)$ stays bellow the graph of η for $r \in [0,1]$. Now, let us consider whether this curve intersects the hyper-plane $x_n = \eta_{min}$. Since y_r is a strictly decreasing function in r, we distinguish between the two separate cases.

Case 1. $y_r \ge \eta_{min}$, $r \in [0,1]$. We define Θ_{x_0,x_1} in the following way:

$$\Theta_{x_0,x_1}(r) = \begin{cases} (\mathbf{x}_r, y_r) & , \quad 0 \le r \le 1, \\ \\ (\mathbf{x}_1, (2-r)y_1 + (r-1)\eta(\mathbf{x}_1)) & , \quad 1 < r \le 2. \end{cases} \tag{7}$$

From (6), the definition of Θ_{x_0,x_1} (7) and the definition of Ω_η it follows immediately that $\Theta_{x_0,x_1}(0) = (\mathbf{x}_0, \eta(\mathbf{x}_0))$, $\Theta_{x_1,x_2}(2) = (\mathbf{x}_1, \eta(\mathbf{x}_1))$ and $\Theta_{x_0,x_1}(r) \in \Omega_\eta,\ r \in [0,2]$. Therefore it only remains to prove (5). We calculate

$$\Theta'_{x_0,x_1}(r) = \begin{cases} (\frac{1}{\alpha} r^{1/\alpha - 1}(\mathbf{x}_1 - \mathbf{x}_0), -\|\eta\|_{C^{0,\alpha}(\omega)}|\mathbf{x}_1 - \mathbf{x}_0|^\alpha) & , \quad 0 \le r \le 1, \\ \\ (0, \eta(\mathbf{x}_1) - y_1) & , \quad 1 < r \le 2. \end{cases}$$

Since ω is bounded, we can take $C \ge \|\eta\|_{C^{0,\alpha}(\omega)}$ such that

$$|\mathbf{x} - \mathbf{y}| \le C|\mathbf{x} - \mathbf{y}|^\alpha, \quad \mathbf{x},\ \mathbf{y} \in \omega.$$

Using this observation we can get an estimate:

$$|\Theta'_{x_0,x_1}(r)| \le C|\mathbf{x}_0 - \mathbf{x}_1|^\alpha, \quad r \in [0,1).$$

Furthermore, analogously using the definition of y_r and $\eta \in C^{0,\alpha}(\omega)$ we have

$$|\eta(\mathbf{x}_1) - y_1| \le |\eta(\mathbf{x}_1) - \eta(\mathbf{x}_0)| + \|\eta\|_{C^{0,\alpha}(\omega)} r|\mathbf{x}_1 - \mathbf{x}_0|^\alpha \le C|\mathbf{x}_0 - \mathbf{x}_1|^\alpha.$$

Therefore, (5) is proven.

Case 2. There exists $r_0 \in (0,1)$ such that $y_r = \eta_{min}$. In this case we define Θ_{x_0,x_1} in the following way:

$$\Theta_{x_0,x_1}(r) = \begin{cases} (\mathbf{x}_r, y_r) & , \quad 0 \le r \le r_0, \\ (\mathbf{x}_r, \eta_{min}) & , \quad r_0 < r \le 1, \\ (\mathbf{x}_1, (2-r)\eta_{min} + (r-1)\eta(\mathbf{x}_1)) & , \quad 1 < r \le 2. \end{cases} \tag{8}$$

Analogous calculation as in Case 1 shows that estimate (5) is valid in this case as well. This completes the proof of the Lemma. □

Now we are ready to prove the following lemma:

Lemma 3.3. *Let $u \in H^1(\Omega_\eta)$ and let $0 < s < \alpha$. Then $\bar{u} \in W(0,1;s)$, where $\bar{u}$ is defined by formula (4).*

Proof. Let us first take $u \in C_c^\infty(\mathbb{R}^n)$. For $\mathbf{x}_1, \mathbf{x}_2 \in \omega$, $t \in (0,1)$ we have

$$|\bar{u}(\mathbf{x}_1, t) - \bar{u}(\mathbf{x}_2, t)| = |u(\mathbf{x}_1, \eta(\mathbf{x}_1)t) - u(\mathbf{x}_2, \eta(\mathbf{x}_2)t)|$$

Notice that $t\eta \in C^{0,\alpha}(\omega)$ and therefore we can apply Lemma 3.2 to function $t\eta$ (we just need to replace η_{min} with $t\eta_{min}$ in the proof of the Lemma 3.2) to get $\Phi^t_{x_1,x_2} : [0,2] \to \Omega_\eta$ such that:

$$\Theta^t_{x_1,x_2}(0) = (\mathbf{x}_1, \eta(\mathbf{x}_1)t), \quad \Theta^t_{x_1,x_2}(2) = (\mathbf{x}_2, \eta(\mathbf{x}_2)t),$$

$$|\frac{d}{dr}\Theta^t_{x_1,x_2}(r)| \le C|\mathbf{x}_1 - \mathbf{x}_2|^\alpha, \quad \text{a. e. } r \in [0,2],$$

where C does not depend on $\mathbf{x}_1$, $\mathbf{x}_2$ and t. Define

$$f^t_{x_1,x_2}(r) = u(\Theta^t_{x_1,x_2}(r)),\ r \in [0,2].$$

Now we have

$$\begin{aligned} |u(\mathbf{x}_1, \eta(\mathbf{x}_1)t) - u(\mathbf{x}_2, \eta(\mathbf{x}_2)t)|^2 &= |\int_0^2 \frac{d}{dr} f^t_{x_1,x_2}(r)dr|^2 \\ &\le \|\frac{d}{dr}\Theta^t_{x_1,x_2}(r)\|^2_{L^\infty(0,2)} \int_0^2 |\nabla u(\Theta^t_{x_1,x_2}(r))|^2 dr \\ &\le C|\mathbf{x}_1 - \mathbf{x}_2|^{2\alpha} \int_0^2 |\nabla u(\Theta^t_{x_1,x_2}(r))|^2 dr. \end{aligned} \tag{9}$$

Using (9) we get the following estimates:

$$\begin{aligned} \|\bar{u}\|^2_{L^2(0,1:H^s(\omega))} &= \int_0^1 \|\bar{u}(.,t)\|^2_{H^s(\omega)} dt = \int_0^1 dt \int_{\omega\times\omega} \frac{|\bar{u}(\mathbf{x}_1,t) - \bar{u}(\mathbf{x}_2,t)|^2}{|\mathbf{x}_1 - \mathbf{x}_2|^{n-1+2s}} d\mathbf{x}_1 d\mathbf{x}_2 \\ &\le C \int_0^1 dt \int_{\omega\times\omega} \frac{d\mathbf{x}_1 d\mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^{n-1+2(s-\alpha)}} \int_0^2 |\nabla u(\Theta^t_{x_1,x_2}(r))|^2 dr. \\ &\le C\|\nabla u\|^2_{L^2(\Omega_\eta)} \int_{\omega\times\omega} \frac{d\mathbf{x}_1 d\mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^{n-1+2(s-\alpha)}}. \end{aligned} \tag{10}$$

To estimate the last integral in (10), we introduce a new variable $\mathbf{h} = \mathbf{x}_1 - \mathbf{x}_2$ and the change of variables $(\mathbf{x}_1, \mathbf{x}_2) \mapsto (\mathbf{h}, \mathbf{x}_2)$ to get:

$$\int_{\omega\times\omega} \frac{d\mathbf{x}_1 d\mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^{n-1+2(s-\alpha)}} \le C \int_{-R}^{R} \frac{dh}{|h|^{1+2(s-\alpha)}}, \tag{11}$$

where $R = \operatorname{diam}(\omega)$. Recall that $s < \alpha < 1$. Therefore by combining (10) and (11), we get:

$$\|\bar{u}\|_{L^2(0,1:H^s(\omega))} \le C\|u\|_{H^1(\Omega_\eta)}, \quad u \in C_c^\infty(\mathbb{R}^n) \tag{12}$$

Since $C_c^\infty(\mathbb{R}^n)$ is dense in $H^1(\Omega_\eta)$ (see [1], Thm 2, p. 54 with a slight modification near $\partial\omega \times \{1\}$, see also [2], proof of Lemma 1 and [9], Prop A.1.), by a density argument we have

$$\bar{u} \in L^2(0,1;H^s(\omega)), \quad u \in H^1(\Omega_\eta).$$

Now, it only remains to prove $\partial_t \bar{u} \in L^2((0,1)\times\omega)$. However, this can be proven with direct calculation by using the chain rule:

$$\partial_t \bar{u}(\mathbf{x}, t) = \eta(\mathbf{x})\partial_{x_n} u(\mathbf{x}, \eta(\mathbf{x})t).$$

Since η is Hölder continuous on ω, from the above formula we have $\partial_t \bar{u} \in L^2((0,1)\times \omega)$ which completes the proof of the Lemma. □

Now we use continuity properties of $W(0,1;s)$ ([11], p. 19, Thm 3.1.), i.e.

$$W(0,1;s) \hookrightarrow C([0,T];H^{s/2}(\omega)),$$

where this injection is continuous. Therefore, from Lemma 3.3 we have

$$\bar{u} \in C([0,T];H^{s/2}(\omega)), \quad u \in H^1(\Omega_\eta). \tag{13}$$

We finish the proof by noticing that $\gamma_\eta(u) = \bar{u}(.,1)$. □

Remark 2. In [6], Lemma 2, a special case of Theorem 3.1 was proved. Namely, for $n = 3$ and $\eta \in H^2(\omega)$ it was proved that γ_η is a continuous operator from $H^1(\Omega_\eta)$ to $H^s(\omega)$, $0 \le s < \frac{1}{2}$. This result follows from Theorem 3.1 because of the Sobolev embedding $H^2(\omega) \hookrightarrow C^{0,\alpha}(\omega)$, $\alpha < 1$. However, the techniques from [6] rely on Sobolev embeddings and the fact that $\nabla\eta$ is more regular then $L^2(\omega)$ and therefore, cannot be extended for the case of arbitrary Hölder continuous functions.

REFERENCES

[1] R. A. Adams, *Sobolev Spaces*, Pure and Applied Mathematics, 65, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[2] A. Chambolle, B. Desjardins, M. J. Esteban and C. Grandmont, Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate, *J. Math. Fluid Mech.*, **7** (2005), 368–404.

[3] C. H. A. Cheng and S. Shkoller, The interaction of the 3D Navier-Stokes equations with a moving nonlinear Koiter elastic shell, *SIAM J. Math. Anal.*, **42** (2010), 1094–1155.

[4] S. Čanić and B. Muha, A nonlinear moving-boundary problem of parabolic-hyperbolic-hyperbolic type arising in fluid-multi-layered structure interaction problems, to appear in *Proceedings of the Fourteenth International Conference on Hyperbolic Problems: Theory, Numerics and Applications*, American Institute of Mathematical Sciences (AIMS) Publications.

[5] Z. Ding, A proof of the trace theorem of Sobolev spaces on Lipschitz domains, *Proc. Amer. Math. Soc.*, **124** (1996), 591–600.

[6] C. Grandmont, Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate, *SIAM J. Math. Anal.*, **40** (2008), 716–737.

[7] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Monographs and Studies in Mathematics, 24, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[8] I. Kukavica and A. Tuffaha, Solutions to a fluid-structure interaction free boundary problem, *DCDS-A*, **32** (2012), 1355–1389.

[9] D. Lengeler and M. Ružička, Weak solutions for an incompressible newtonian fluid interacting with a linearly elastic koiter shell, *Arch. Ration. Mech. Anal.*, **211** (2014), 205–255.

[10] J. Lequeurre, Existence of strong solutions for a system coupling the Navier-Stokes equations and a damped wave equation, *J. Math. Fluid Mech.*, **15** (2013), 249–271.

[11] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications. Vol. I*, Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181, Springer-Verlag, New York, 1972.

[12] B. Muha and S. Čanić, Existence of a weak solution to a nonlinear fluid-structure interaction problem modeling the flow of an incompressible, viscous fluid in a cylinder with deformable walls, *Arch. Ration. Mech. Anal.*, **207** (2013), 919–968.

[13] B. Muha and S. Čanić, Existence of a solution to a fluid-multi-layered-structure interaction problem, *J. of Diff. Equations*, **256** (2014), 658–706.

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E-mail address: borism@math.hr