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REMARKS ON THE PAPER "JENSEN'S INEQUALITY AND NEW ENTROPY BOUNDS" OF S. SIMIĆ

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Abstract. The purpose of this paper is twofold. The first is to give a brief account of the results preceding the main results from [14] and [15]. The second is to give generalizations and improvements of these results.

1. Introduction

Let $I \subset \mathbb{R}$ be an interval, $(x_i)_{i=1}^n$ a sequence such that $x_i \in I$, $i = 1,\dots,n$, and $(p_i)_{i=1}^n$ a sequence of positive weights with $\sum_1^n p_i = 1$. For a convex function $f\colon I \to \mathbb{R}$, the Jensen inequality states

$$0 \leqslant \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right).$$

The following theorems are proved in [15].

THEOREM 1. *If f is convex on I, then*

$$\max_{1\leqslant\mu<\nu\leqslant n}\left[p_\mu f(x_\mu) + p_\nu f(x_\nu) - (p_\mu+p_\nu) f\left(\frac{p_\mu x_\mu + p_\nu x_\nu}{p_\mu+p_\nu}\right)\right] \leqslant \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right)$$

and this bound is sharp.

THEOREM 2. *If $(x_i)_{i=1}^n \in [a,b]^n$, then*

$$\sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \leqslant f(a) + f(b) - 2f\left(\frac{a+b}{2}\right) := S_f(a,b)$$

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However, Theorem 1 is contained in the following theorem and corollary, which are in fact Theorem 3.14 and Corollary 3.15 published in [9, p. 87]. The statement on sharpness in Theorem 1 is obvious (equality is attained for $n=2$).

THEOREM 3. *Let $f:U\to\mathbb{R}$ be a convex function, where U is a convex set in a real linear space M. Let I and J be finite subsets in $\mathbb{N}$, such that $I\cap J=\emptyset$. Let $(x_i)_{i\in I\cup J}$ be a sequence such that $x_i\in U$, $i\in I\cup J$ and $(p_i)_{i\in I\cup J}$ a real sequence such that $P_I>0$, $P_J>0$ and $P_{I\cup J}>0$, where $P_K=\sum_{k\in K}p_k$ for $K\subseteq I\cup J$. If $\frac{1}{P_I}\sum_{i\in I}p_ix_i\in U$, $\frac{1}{P_J}\sum_{i\in J}p_ix_i\in U$, $\frac{1}{P_{I\cup J}}\sum_{i\in I\cup J}p_ix_i\in U$. Then*

$$F(I\cup J)\leqslant F(I)+F(J), \tag{1}$$

where $F(K)=P_Kf\left(\frac{1}{P_K}\sum_{k\in K}p_kx_k\right)-\sum_{k\in K}p_kf(x_k)$.

Notice that the function F from Theorem 3 describes the opposite Jensen's difference than the differences in Theorems 1, 2.

COROLLARY 1. *Let f be a convex function on U, where U is a convex set in an arbitrary real linear space M. Let $(x_i)_{i=1}^n$ be a sequence such that $x_i\in U$, $i=1,\dots,n$, and $(p_i)_{i=1}^n$ a real sequence. If $p_i\geqslant 0$, $i=1,\dots,n$ and $I_k=\{1,\dots,k\}$, then*

$$F(I_n)\leqslant F(I_{n-1})\leqslant\cdots\leqslant F(I_2)\leqslant 0$$

and

$$F(I_n)\leqslant\min_{1\leqslant i<j\leqslant n}\left\{(p_i+p_j)f\left(\frac{p_ix_i+p_jx_j}{p_i+p_j}\right)-p_if(x_i)-p_jf(x_j)\right\}.$$

REMARK 4. It should be noted that results related to Theorem 3 and Corollary 1 and implying Theorem 1 were published previously in [12], [13], [6], [2], [8] and [1].

Theorem 2 was proved by the same author in paper published one year before [15]. It was the main result in [14]. This fact wasn't noted in [15]. It was noted in [4] that this main result from [14], and therefore Theorem 2 can be derived from Corollaries 3 and 4 from [5]. Moreover, these Corollaries give the following improvements of Theorem 2.

THEOREM 5. *Let $[a,b]\subset\mathbb{R}$, $(x_i)_{i=1}^n$ be a sequence such that $x_i\in[a,b]$, $i=1,\dots,n$ and $(p_i)_{i=1}^n$ a sequence of positive weights with $\sum_{i=1}^n p_i=1$. If $f:[a,b]\to\mathbb{R}$ is a convex function, then*

$$\begin{aligned}
&\sum_{i=1}^n p_if(x_i)-f\left(\sum_{i=1}^n p_ix_i\right)\\
&\leqslant f\left(a+b-\sum_{i=1}^n p_ix_i\right)-2f\left(\frac{a+b}{2}\right)+\sum_{i=1}^n p_if(x_i)\\
&\leqslant f(a)+f(b)-2f\left(\frac{a+b}{2}\right)=S_f(a,b)
\end{aligned}$$

To state further improvements of Theorem 2 and also for the rest of the paper, we need the following notions.

Let E be a non-empty set and L be a linear class of real-valued functions $f\colon E \to \mathbb{R}$ having the properties:

$(L1)$ $(\forall a,b \in \mathbb{R})\,(\forall f,g \in L)\; af+bg \in L$

$(L2)$ $\mathbf{1} \in L$ (that is if $f(t)=1$ for all $t \in E$, then $f \in L$)

We consider positive linear functionals $A\colon L \to \mathbb{R}$, or in other words we assume:

$(A1)$ $(\forall f,g \in L)(\forall a,b \in \mathbb{R})\quad A(af+bg) = aA(f)+bA(g)$ (linearity)

$(A2)$ $(\forall f \in L)(f \geqslant 0 \Longrightarrow A(f) \geqslant 0)$ (positivity)

If additionally the condition $A(\mathbf{1})=1$ is satisfied , we say that A is positive normalized linear functional.

The following generalization and improvement of Theorem 2 was proved in [4].

THEOREM 6. *Let L satisfy $(L1)$ and $(L2)$ and let Φ be a convex function on $I=[a,b]$. Then for any positive normalized linear functional A on L and for any $g \in L$ such that $\Phi(g) \in L$ we have*

$$A(\Phi(g)) - \Phi(A(g)) \leqslant \left\{\frac{1}{2} + \frac{1}{b-a}\left|\frac{a+b}{2} - A(g)\right|\right\} S_{\Phi}(a,b).$$

The main purpose of the paper is to give generalizations and improvements of Theorems 1, 2 and an improvement of Theorem 6.

2. Improvements

In the following theorem we give two generalizations of Theorem 1. For similar results obtained by different methods see [3].

THEOREM 7. *Let f be a convex function on U, where U is a convex set in an arbitrary real linear space M. Let $(x_i)_{i=1}^n$ be a sequence such that $x_i \in U$, $i \in I_n = \{1,\ldots,n\}$, and $(p_i)_{i=1}^n$ a positive sequence.*

(i) *If $\mathscr{S}$ is a family of subsets of I_n, then*

$$F(I_n) \leqslant \min_{S\in\mathscr{S}} \left(F(S) + F(I_n \setminus S)\right) \leqslant \min_{S\in\mathscr{S}} F(S) + \max_{S\in\mathscr{S}} F(I_n \setminus S) \leqslant \min_{S\in\mathscr{S}} F(S). \quad (2)$$

(ii) *If $\mathscr{S}$ is a family of disjoint subsets of I_n, then*

$$F(I_n) \leqslant \sum_{S\in\mathscr{S}} F(S), \quad (3)$$

where F is the function defined in Theorem 3.

Proof. (i) Simple consequence of Theorem 3.
(ii) Obviously $I_n = \cup_{S\in\mathscr{S}} S \cup_{i\notin\cup_{S\in\mathscr{S}}} \{i\}$, so (3) follows using Theorem 3 and $F(\{i\}) = 0$. □

Improvement of Theorem 2 and Theorem 6 will be obtained using the following lemma.

LEMMA 1. *Let ϕ be a convex function on an interval I, $x,y \in I$ and $p,q \in [0,1]$ such that $p+q=1$. Then*

$$\min\{p,q\}S_{\phi}(x,y) \leqslant p\phi(x)+q\phi(y)-\phi(px+qy) \leqslant \max\{p,q\}S_{\phi}(x,y). \tag{4}$$

Proof. This is a special case of Theorem 1 from [7, p.717] for $n=2$. □

We will also need to equip our linear class L from above with an additional property denoted by $(L3)$:

$(L3)$ $(\forall f,g \in L)(\min\{f,g\} \in L \wedge \max\{f,g\} \in L)$ (lattice property)

Obviously, $(\mathbb{R}^E, \leqslant)$ (with standard ordering) is a lattice. It can also be easily verified that a subspace $X \subseteq \mathbb{R}^E$ is a lattice if and only if $x \in X$ implies $|x| \in X$. This is a simple consequence of identities

$$\min\{x,y\} = \frac{1}{2}(x+y-|x-y|), \quad \max\{x,y\} = \frac{1}{2}(x+y+|x-y|).$$

Next theorem is our main result.

THEOREM 8. *Let L satisfy $(L1)$, $(L2)$ and $(L3)$ and let Φ be a convex function on $I=[a,b]$. Then for any positive normalized linear functional A on L and for any $g \in L$ such that $\Phi(g) \in L$ we have*

$$\begin{aligned} A(\Phi(g)) - \Phi(A(g)) &\leqslant \\ &\leqslant \frac{1}{b-a}\left\{\left|\frac{a+b}{2} - A(g)\right| + A\left(\left|\frac{a+b}{2} - g\right|\right)\right\} S_{\Phi}(a,b). \end{aligned} \tag{5}$$

Proof. First observe that $\Phi(g) \in L$ also means that the composition $\Phi(g)$ is well defined, hence $g(E) \subseteq [a,b]$. It follows $A(g) \in [a,b]$.
Let the functions $p,q\colon [a,b] \to \mathbb{R}$ be defined by

$$p(x) = \frac{b-x}{b-a}, \quad q(x) = \frac{x-a}{b-a}.$$

For any $x \in [a,b]$ we can write

$$\Phi(x) = \Phi\left(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b\right) = \Phi(p(x)a + q(x)b)$$

Since $A(g) \in [a,b]$ we have $\Phi(A(g)) = \Phi(p(A(g))a + q(A(g))b)$ and by Lemma 1

$$\Phi(A(g)) \geqslant p(A(g))\Phi(a) + q(A(g))\Phi(b) - \max\{p(A(g)), q(A(g))\} S_\Phi(a,b)$$
$$= p(A(g))\Phi(a) + q(A(g))\Phi(b) - \left\{\frac{1}{2} + \frac{\left|\frac{a+b}{2} - A(g)\right|}{b-a}\right\} S_\Phi(a,b) \qquad (6)$$

Again by Lemma 1 we have

$$\Phi(x) \leqslant p(x)\Phi(a) + q(x)\Phi(b) - \min\{p(x), q(x)\} S_\Phi(a,b).$$

We have $p(g), q(g) \in L$ and applying A to the above inequality, we obtain

$$A(\Phi(g)) \leqslant A(p(g))\Phi(a) + A(q(g))\Phi(b) - A\left(\min\{p(g), q(g)\}\right) S_\Phi(a,b)$$
$$= A(p(g))\Phi(a) + A(q(g))\Phi(b) - A\left(\frac{1}{2} - \frac{\left|g - \frac{a+b}{2}\right|}{b-a}\right) S_\Phi(a,b)$$
$$= p(A(g))\Phi(a) + q(A(g))\Phi(b) - A\left(\frac{1}{2} - \frac{\left|g - \frac{a+b}{2}\right|}{b-a}\right) S_\Phi(a,b) \qquad (7)$$

Now, from inequalities (6) and (7) we get desired inequality (5) $\square$

Since $A(g) \in [a,b]$, it is obvious that Theorem 8 is an improvement of Theorem 6 and a generalization and an improvement of Theorem 2.

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