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GENERALIZATIONS AND IMPROVEMENTS OF CONVERSE JENSEN'S INEQUALITY FOR CONVEX HULLS IN $\mathbb{R}^k$

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1. Introduction

Let U be a convex subset of $\mathbb{R}^k$ and $n \in \mathbb{N}$. If $f : U \to \mathbb{R}$ is a convex function, $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in U$ and $p_1, \ldots, p_n$ nonnegative real numbers with $P_n = \sum_{i=1}^{n} p_i > 0$, then Jensen's inequality

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i \boldsymbol{x}_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(\boldsymbol{x}_i)$$

holds.

The convex hull of vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathbb{R}^k$ is the set

$$\left\{\sum_{i=1}^{n} \alpha_i \boldsymbol{x}_i \middle| \alpha_i \in \mathbb{R}, \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1\right\}$$

and it is denoted by $K = co(\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\})$.

Barycentric coordinates over K are continuous real functions $\lambda_1, \ldots, \lambda_n$ on K with the following properties:

$$\lambda_i(\boldsymbol{x}) \geq 0, i = 1, ..., n$$
$$\sum_{i=1}^{n} \lambda_i(\boldsymbol{x}) = 1$$
$$\boldsymbol{x} = \sum_{i=1}^{n} \lambda_i(\boldsymbol{x})\boldsymbol{x}_i \tag{1}$$

If $\boldsymbol{x}_2 - \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n - \boldsymbol{x}_1$ are linearly independent vectors, then each $\boldsymbol{x} \in K$ can be written in the unique way as a convex combination of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ in the form (1).
We also consider k-simplex $S = co(\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}\})$ in $\mathbb{R}^k$ which is a convex hull of its vertices $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1} \in \mathbb{R}^k$, where vertices $\boldsymbol{v}_2 - \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1} - \boldsymbol{v}_1 \in \mathbb{R}^k$ are lineary independent. In this case we'll denote k-simplex by $S = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1}]$. Barycentric coordinates $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ over S are nonnegative linear polynomials on S and have special form.

Let E be a non-empty set and L be a linear class of real-valued functions $f\colon E \to \mathbb{R}$ having the properties:

$(L1)$ $(\forall f, g \in L)\,(\forall a, b \in \mathbb{R})\;\; af + bg \in L$

$(L2)$ $\mathbf{1} \in L$, that is if $f(t) = 1$ for all $t \in E$, then $f \in L$

We consider positive linear functionals $A\colon L \to \mathbb{R}$, or in other words we assume:

$(A1)$ $(\forall f, g \in L)(\forall a, b \in \mathbb{R}) \quad A(af + bg) = aA(f) + bA(g)$ (linearity)

$(A2)$ $(\forall f \in L)(f \geq 0 \Longrightarrow A(f) \geq 0)$ (positivity)

If additionally the condition $A(\mathbf{1}) = 1$ is satisfied, we say that A is positive normalized linear functional.

With L^k we denote the linear class of functions $\boldsymbol{g}\colon E \to \mathbb{R}^k$ defined by

$$\boldsymbol{g}(t) = (g_1(t), \ldots, g_k(t)), \quad g_i \in L \quad (i = 1, \ldots, k)$$

For given linear functional A, we also consider linear operator $\widetilde{A} = (A, \ldots, A)\colon L^k \to \mathbb{R}^k$ defined by

$$\widetilde{A}(\boldsymbol{g}) = (A(g_1), \ldots, A(g_k)) \tag{2}$$

If $A(\mathbf{1}) = 1$ is satisfied, then using (A1) we also have

(A3) $A(f(\boldsymbol{g})) = f(\widetilde{A}(\boldsymbol{g}))$ for every linear function f on $\mathbb{R}^k$.

The following result is Jessen's generalization of the Jensen's inequality for convex functions which involves positive normalized linear functionals.

Theorem 1.

Let L satisfy L1, L2 on a nonempty set E and let A be a positive normalized linear functional on L. If f is a continuous convex function on an interval $I \subset \mathbb{R}$, then for all $g \in L$ such that $f(g) \in L$ we have $A(g) \in I$ and

$$f(A(g)) \leq A(f(g)).$$

The next theorem, proved by J. Pečarić and P. R. Beesack in 1985, presents generalization of Lah-Ribarič inequality.

Theorem 2 (Lah-Ribarič inequality).

Let L satisfy properties L1, L2 and A be a positive normalized linear functional on L. Let f be a convex function on an interval $I = [m, M] \subset \mathbb{R}$ $(-\infty < m < M < \infty)$. Then for all $g \in L$ such that $g(E) \subset I$ and $f(g) \in L$

$$A(f(g)) \leq \frac{M - A(g)}{M - m} f(m) + \frac{A(g) - m}{M - m} f(M).$$

Using previous theorem, Beesack and Pečarić in 1987. also proved the next result.

Theorem 3.

Let L, A and f be as in Theorem 2. Let J be an interval in $\mathbb{R}$ such that $f(I) \subset J$. If $F\colon J \times J \to \mathbb{R}$ is an increasing function in the first variable, then for all $g \in L$ such that $g(E) \subset I$ and $f(g) \in L$, we have

$$\begin{aligned} & F(A(f(g)), f(A(g))) \\ & \quad \leq \max_{x \in [m,M]} F\left(\frac{M-x}{M-m} f(m) + \frac{x-m}{M-m} f(M), f(x) \right) \\ & \quad = \max_{\theta \in [0,1]} F\left(\theta f(m) + (1-\theta) f(M), f(\theta m + (1-\theta) M) \right). \end{aligned}$$

Remark

If we choose $F(x, y) = x - y$, as a simple consequence of previous theorem it follows

$$A(f(g)) - f(A(g)) \leq \max_{\theta \in [0,1]} \left[\theta f(m) + (1-\theta) f(M) - f(\theta m + (1-\theta) M)\right]. \tag{3}$$

Choosing $F(x, y) = \frac{x}{y}$, for $f > 0$ it follows

$$\frac{A(f(g))}{f(A(g))} \leq \max_{\theta \in [0,1]} \left[\frac{\theta f(m) + (1-\theta) f(M)}{f(\theta m + (1-\theta) M)}\right]. \tag{4}$$

Additional generalization of Jessen's inequality is proved by E. J. McShane in
E. J. McShane, *Jensen's inequality*, Bull. Amer. Math. Soc. 43 (1937),

Theorem 4 (McShane's inequality).

Let L satisfy properties L1, L2, A be a positive normalized linear functional on L and $\widetilde{A}$ defined as in (2). Let f be a continuous convex function on a closed convex set $U \subset \mathbb{R}^k$. Then for all $\boldsymbol{g} \in L^k$ such that $\boldsymbol{g}(E) \subset U$ and $f(\boldsymbol{g}) \in L$, we have that $\widetilde{A}(\boldsymbol{g}) \in U$ and

$$f(\widetilde{A}(g)) \leq A(f(g)).$$

S. Ivelić, J. Pečarić, *Generalizations of Converse Jensen's inequality and related results*, J. Math. Ineq. Volume 5, Number 1 (2011)

Theorem 5.

Let L *satisfy properties L1, L2 on nonempty set* E *and* A *be a positive normalized linear functional on* L. *Let* $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in \mathbb{R}^k$ *and* $K = co(\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_n\})$. *Let* f *be a convex function on* K *and* $\lambda_1, \dots, \lambda_n$ *barycentric coordinates over* K. *Then for all* $\boldsymbol{g} \in L^k$ *such that* $\boldsymbol{g}(E) \subset K$ *and* $f(\boldsymbol{g}), \lambda_i(\boldsymbol{g}) \in L, i = 1, \dots, n$ *we have*

$$A(f(\boldsymbol{g})) \leq \sum_{i=1}^{n} A(\lambda_i(\boldsymbol{g})) f(\boldsymbol{x}_i)$$

Main Results

Our main results are generalizations and improvements of Theorems 2 and 3 which will be obtained using the following lemma.

Lemma 6.

Let ϕ *be a convex function on* U *where* U *is a convex set in* $\mathbb{R}^k$*,* $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \in U^n$ *and* $p = (p_1, \ldots, p_n)$ *is nonnegative* n*-tuple such that* $\sum_{i=1}^{n} p_i = 1$*. Then*

$$\begin{aligned}
\min\{p_1, \ldots, p_n\} &\left[\sum_{i=1}^{n} \phi(\boldsymbol{x}_i) - n\phi\left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i\right)\right] \\
&\leq \sum_{i=1}^{n} p_i \phi(\boldsymbol{x}_i) - \phi\left(\sum_{i=1}^{n} p_i \boldsymbol{x}_i\right) \\
\leq \max\{p_1, \ldots, p_n\} &\left[\sum_{i=1}^{n} \phi(\boldsymbol{x}_i) - n\phi\left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i\right)\right]
\end{aligned}$$

Proof.

This is a simple consequence of Theorem 1 from
J. E. Pečarić, F. Proschan, Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, New York, 1992.

For $n \in \mathbb{N}$ we denote

$$\Delta_{n-1} = \left\{ (\mu_1, \dots, \mu_n) \colon \mu_i \geq 0, i \in \{1, \dots, n\}, \sum_{i=1}^{n} \mu_i = 1 \right\}$$

We also need to equip our linear class L from Introduction with an additional property denoted by (L3):

(L3) $(\forall f, g \in L)\,(\min\{f, g\} \in L \text{ and } \max\{f, g\} \in L)$ (lattice property).

Obviously, $(\mathbb{R}^E, \leq)$ (with standard ordering) is a lattice.

Also, if f is a function defined on an subset $U \subseteq \mathbb{R}^k$ and $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \in U$, we denote

$$S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \sum_{i=1}^{n} f(\boldsymbol{x}_i) - nf\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{x}_i\right)$$

Obviously, if f is convex, $S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \geq 0$
Next theorem presents an improvement of Theorem 5.

Theorem 7.

Let L satisfy properties $L1, L2, L3$ on nonempty set E and A be a positive normalized linear functional on L. Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathbb{R}^k$ and $K = co(\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\})$. Let f be a convex function on K and $\lambda_1, \ldots, \lambda_n$ barycentric coordinates over K. Then for all $\boldsymbol{g} \in L^k$ such that $\boldsymbol{g}(E) \subset K$ and $f(\boldsymbol{g}), \lambda_i(\boldsymbol{g}) \in L, i = 1, \ldots, n$ we have

$$A(f(\boldsymbol{g})) \leq \sum_{i=1}^{n} A\left(\lambda_i(\boldsymbol{g})\right) f\left(\boldsymbol{x}_i\right) - A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$$

Proof.

For each $t \in E$ we have $\boldsymbol{g}(t) \in K$. Using barycentric coordinates we have $\lambda_i(\boldsymbol{g}(t)) \geq 0, i = 1, \ldots, n$, $\sum_{i=1}^{n} \lambda_i(\boldsymbol{g}(t)) = 1$ and

$$\boldsymbol{g}(t) = \sum_{i=1}^{n} \lambda_i(\boldsymbol{g}(t))\boldsymbol{x}_i$$

Since f is convex, we can apply Lemma 6, and then

$$\begin{aligned} f(\boldsymbol{g}(t)) &= f\left(\sum_{i=1}^{n} \lambda_i(\boldsymbol{g}(t))\boldsymbol{x}_i\right) \\ &\leq \sum_{i=1}^{n} \lambda_i(\boldsymbol{g}(t)) f(\boldsymbol{x}_i) - \min\left\{\lambda_i(\boldsymbol{g}(t))\right\}\left[\sum_{i=1}^{n} f(\boldsymbol{x}_i) - n f\left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i\right)\right] \end{aligned}$$

Now, applying the functional A on the last inequality we get

$$A\left(f(\boldsymbol{g})\right) \leq A\left(\sum_{i=1}^{n} \lambda_i(\boldsymbol{g}) f\left(\boldsymbol{x}_i\right) - \min\left\{\lambda_i(\boldsymbol{g})\right\} S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)\right)$$

$$= \sum_{i=1}^{n} A\left(\lambda_i(\boldsymbol{g})\right) f\left(\boldsymbol{x}_i\right) - A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$$

Remark

Theorem 7 is an improvement of Theorem 5 since under the required assumptions we have

$$A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \geq 0$$

Remark

If all the assumptions of Theorem 7 are satisfied and in addittion f is continuous, then

$$f(\widetilde{A}(\boldsymbol{g})) \leq A(f(\boldsymbol{g})) \leq \sum_{i=1}^{n} A\left(\lambda_i(\boldsymbol{g})\right) f\left(\boldsymbol{x}_i\right) - A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$$

The first inequality is consequence of McShane's inequality and the second of previous theorem.

Remark

We know that under assumptions of Theorem 7 we have

$$A(f(\boldsymbol{g})) \leq \sum_{i=1}^{n} A\left(\lambda_i(\boldsymbol{g})\right) f\left(\boldsymbol{x}_i\right) - A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$$

Dividing this by $f(\boldsymbol{g}(t)) = f\left(\sum_{i=1}^{n} \lambda_i(\boldsymbol{g}(t))\boldsymbol{x}_i\right)$, in the case $f > 0$, we obtain

$$\frac{A\left(f\left(\boldsymbol{g}\right)\right)}{f\left(\widetilde{A}\left(\boldsymbol{g}\right)\right)} \leq \frac{\sum_{i=1}^{n} A(\lambda_i(\boldsymbol{g}))f(\boldsymbol{x}_i)}{f\left(\sum_{i=1}^{n} A(\lambda_i(\boldsymbol{g}))\boldsymbol{x}_i\right)} - \frac{A\left(\min\left\{\lambda_i(\boldsymbol{g})\colon i = 1, \ldots, n\right\}\right)}{f\left(\widetilde{A}(\boldsymbol{g})\right)} S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$$

$$\leq \max_{\Delta_{n-1}} \frac{\sum_{i=1}^{n} \mu_i f(\boldsymbol{x}_i)}{f\left(\sum_{i=1}^{n} \mu_i \boldsymbol{x}_i\right)} - \frac{A\left(\min\left\{\lambda_i(\boldsymbol{g})\colon i = 1, \ldots, n\right\}\right)}{f\left(\widetilde{A}(\boldsymbol{g})\right)} S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$$

which is equivalent to

$$\begin{aligned} & A\left(f\left(\boldsymbol{g}\right)\right) \\ & \quad \leq \max_{\Delta_{n-1}} \frac{\sum_{i=1}^{n} \mu_i f(\boldsymbol{x}_i)}{f\left(\sum_{i=1}^{n} \mu_i \boldsymbol{x}_i\right)} f\left(\widetilde{A}\left(\boldsymbol{g}\right)\right) \\ & \quad -A\left(\min\left\{\lambda_i(\boldsymbol{g})\colon i=1,\ldots,n\right\}\right) S_f^n(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n) \end{aligned}$$

This is an improvement of the inequality (2.6) from
S. Ivelić, J. Pečarić, *Generalizations of Converse Jensen's inequality and related results*, J. Math. Ineq. Volume 5, Number 1 (2011).

Using Teorem 7 we prove generalization and improvement of Theorem 3.

Theorem 8.

Let L satisfy properties L1, L2, L3 on nonempty set E, A be a positive normalized linear functional on L and $\widetilde{A}$ defined as in (2). Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathbb{R}^k$ and $K = co(\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\})$. Let f be a convex function on K and $\lambda_1, \ldots, \lambda_n$ barycentric coordinates over K. If J is an interval in $\mathbb{R}$ such that $f(K) \subset J$ and $F\colon J \times J \to \mathbb{R}$ is an increasing function in the first variable, then for all $\boldsymbol{g} \in L^k$ such that $\boldsymbol{g}(E) \subset K$ and $f(\boldsymbol{g}), \lambda_i(\boldsymbol{g}) \in L, i = 1, \ldots, n$ we have

$$
\begin{aligned}
F\left(A(f(\boldsymbol{g})), f(\widetilde{A}(\boldsymbol{g}))\right) &\leq F\left(\sum_{i=1}^{n} A\left(\lambda_i(\boldsymbol{g})\right) f\left(\boldsymbol{x}_i\right)\right. \\
&\quad \left. - A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n), f(\widetilde{A}(\boldsymbol{g}))\right) \\
&\leq \max_{\Delta_{n-1}} F\left(\sum_{i=1}^{n} \mu_i f(\boldsymbol{x}_i)\right. \\
&\quad \left. - A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n), f\left(\sum_{i=1}^{n} \mu_i \boldsymbol{x}_i\right)\right).
\end{aligned}
$$

Proof.

For each $t \in E$ we have $\boldsymbol{g}(t) \in K$. Using barycentric coordinates we have $\lambda_i(\boldsymbol{g}(t)) \geq 0, i = 1, \ldots, n$, $\sum_{i=1}^{n} \lambda_i(\boldsymbol{g}(t)) = 1$ and

$$\boldsymbol{g}(t) = \sum_{i=1}^{n} \lambda_i(\boldsymbol{g}(t))\boldsymbol{x}_i.$$

Since A is a positive normalized linear functional on L and $\widetilde{A}$ a linear operator on L^k, we have

$$\widetilde{A}\,(\boldsymbol{g}) = (A(g_1), \ldots, A(g_k)) = \sum_{i=1}^{n} A\left(\lambda_i(\boldsymbol{g})\right) x_i$$

where

$$A\left(\lambda_i(\boldsymbol{g})\right) \geq 0, i = 1, \ldots, n$$

and

$$\sum_{i=1}^{n} A\left(\lambda_i(\boldsymbol{g})\right) = A\left(\sum_{i=1}^{n} \lambda_i(\boldsymbol{g})\right) = A(\mathbf{1}) = 1.$$

Therefore, $\widetilde{A}\,(\boldsymbol{g}) \in K$.

Since $F\colon J \times J \to \mathbb{R}$ is an increasing function in the first variable, we have

$$F\left(A(f(\boldsymbol{g})), f(\widetilde{A}(\boldsymbol{g}))\right)$$
$$\leq F\left(\sum_{i=1}^{n} A\left(\lambda_i(\boldsymbol{g})\right) f(\boldsymbol{x}_i) - A\left(\min\left\{\lambda_i(\boldsymbol{g}(t))\right\}\right) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n), f(\widetilde{A}(\boldsymbol{g}))\right)$$

By substitutions

$$A\left(\lambda_i(\boldsymbol{g})\right) = \mu_i, i = 1, \ldots, n,$$

it follows

$$\widetilde{A}\left(\boldsymbol{g}\right) = \sum_{i=1}^{n} \mu_i \boldsymbol{x}_i.$$

Now we have

$$F\left(\sum_{i=1}^{n} A\left(\lambda_i(\boldsymbol{g})\right) f(\boldsymbol{x}_i) - A\left(\min\left\{\lambda_i(\boldsymbol{g}(t))\right\}\right) S_f^n(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n), f(\widetilde{A}(\boldsymbol{g}))\right)$$

$$= F\left(\sum_{i=1}^{n} \mu_i f(\boldsymbol{x}_i) - A\left(\min\left\{\lambda_i(\boldsymbol{g}(t))\right\}\right) S_f^n(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n), f\left(\sum_{i=1}^{n} \mu_i \boldsymbol{x}_i\right)\right)$$

$$\leq \max_{\Delta_{n-1}} F\left(\sum_{i=1}^{n} \mu_i f(\boldsymbol{x}_i)\right.$$

$$\left. - A\left(\min\left\{\lambda_i(\boldsymbol{g}(t))\right\}\right) S_f^n(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n), f\left(\sum_{i=1}^{n} \mu_i \boldsymbol{x}_i\right)\right)$$

By combining last two inequalities we get desired inequality.

Remark

If we choose $F(x,y) = x - y$, as a simple consequence of previous theorem it follows

$$A(f(\boldsymbol{g})) - f(\widetilde{A}(\boldsymbol{g})) \leq \max_{\Delta_{n-1}} \left(\sum_{i=1}^{n} \mu_i f(\boldsymbol{x}_i) - f\left(\sum_{i=1}^{n} \mu_i \boldsymbol{x}_i \right) - A\left(\min\{\lambda_i(\boldsymbol{g})\}\right) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \right)$$

Choosing $F(x,y)=\frac{x}{y}$, for $f>0$ it follows

$$\frac{A(f(\boldsymbol{g}))}{f(\widetilde{A}(\boldsymbol{g}))}\leq \max_{\Delta_{n-1}}\left(\frac{\sum_{i=1}^{n}\mu_i f(\boldsymbol{x}_i)-A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right)S_f^n(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n)}{f\left(\sum_{i=1}^{n}\mu_i\boldsymbol{x}_i\right)}\right).$$

This two inequalities present generalizations and improvements of (3) and (4).

Replacing F by $-F$ in the previous theorem we get next theorem

Theorem 9.

Let L satisfy properties L1, L2, L3 on nonempty set E, A be a positive normalized linear functional on L and $\widetilde{A}$ defined as in (2). Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathbb{R}^k$ and $K = co(\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\})$. Let f be a convex function on K and $\lambda_1, \ldots, \lambda_n$ barycentric coordinates over K. If J is an interval in $\mathbb{R}$ such that $f(K) \subset J$ and $F\colon J \times J \to \mathbb{R}$ is an decreasing function in the first variable, then for all $\boldsymbol{g} \in L^k$ such that $\boldsymbol{g}(E) \subset K$ and $f(\boldsymbol{g}), \lambda_i(\boldsymbol{g}) \in L, i = 1, \ldots, n$ we have

$$F\left(A(f(\boldsymbol{g})), f(\widetilde{A}(\boldsymbol{g}))\right) \geq F\left(\sum_{i=1}^{n} A\left(\lambda_i(\boldsymbol{g})\right) f\left(\boldsymbol{x}_i\right) - \right.$$

$$\left. -A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n), f(\widetilde{A}(\boldsymbol{g}))\right)$$

$$\geq \min_{\Delta_{n-1}} F\left(\sum_{i=1}^{n} \mu_i f(\boldsymbol{x}_i)\right.$$

$$\left. -A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n), f\left(\sum_{i=1}^{n} \mu_i \boldsymbol{x}_i\right)\right).$$

Convex functions on k-simplices in $\mathbb{R}^k$

Let S be a k-simplex in $\mathbb{R}^k$ with vertices $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1} \in \mathbb{R}^k$. The barycentric coordinates $\lambda_1, \ldots \lambda_{k+1}$ over S are nonnegative linear polynomials that satisfy Lagrange's property

$$\lambda_i(\boldsymbol{v}_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} .$$

It is known
(M. Bessenyei, *The Hermite-Hadamard inequality on Simplices,* Amer. Math. Monthly **115** (4) (2008))
that for each $\boldsymbol{x} \in S$ barycentric coordinates $\lambda_1(\boldsymbol{x}), \ldots, \lambda_{k+1}(\boldsymbol{x})$ have the form

$$\begin{aligned}
\lambda_1(\boldsymbol{x}) &= \frac{\mathrm{Vol}_k\left([\boldsymbol{x}, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}]\right)}{\mathrm{Vol}_k(S)}, \\
\lambda_2(\boldsymbol{x}) &= \frac{\mathrm{Vol}_k\left([\boldsymbol{v}_1, \boldsymbol{x}, \boldsymbol{v}_3, \ldots, \boldsymbol{v}_{k+1}]\right)}{\mathrm{Vol}_k(S)}, \\
&\vdots \\
\lambda_{k+1}(\boldsymbol{x}) &= \frac{\mathrm{Vol}_k\left([\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \boldsymbol{x}]\right)}{\mathrm{Vol}_k(S)},
\end{aligned}$$

where Vol_k denotes k-dimensional Lebesgue measure on S. Here, for example, $[\boldsymbol{v}_1, \boldsymbol{x}, \ldots, \boldsymbol{v}_{k+1}]$ denotes the subsimplex obtained by replacing $\boldsymbol{v}_2$ by $\boldsymbol{x}$, i.e. the subsimplex opposite to $\boldsymbol{v}_2$, when adding $\boldsymbol{x}$ as a new vertex.

The signed volume $\mathrm{Vol}_k(S)$ is given by $(k+1)\times(k+1)$ determinant

$$\mathrm{Vol}_k(S) = \frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ v_{11} & v_{21} & & v_{k+11} \\ v_{12} & v_{22} & & v_{k+12} \\ \vdots & \vdots & & \vdots \\ v_{1k} & v_{2k} & \cdots & v_{k+1k} \end{vmatrix},$$

where $\boldsymbol{v}_1 = (v_{11}, v_{12}, \ldots, v_{1k}), \ldots, \boldsymbol{v}_{k+1} = (v_{k+11}, v_{k+12}, \ldots, v_{k+1k})$ (R. T. Rockafellar, *Convex Analysis*, Princeton Math. Ser. No. 28, Princeton Univ. Press, Princeton, New Jersey, 1970.).

Since vectors $\boldsymbol{v}_2 - \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1} - \boldsymbol{v}_1$ are linearly independent, then each $\boldsymbol{x} \in S$ can be written in unique way as convex combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1}$ in the form

$$\boldsymbol{x} = \frac{\mathrm{Vol}_k\left([\boldsymbol{x}, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}]\right)}{\mathrm{Vol}_k(S)}\boldsymbol{v}_1 + \cdots + \frac{\mathrm{Vol}_k\left([\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \boldsymbol{x}]\right)}{\mathrm{Vol}_k(S)}\boldsymbol{v}_{k+1}.$$

Now we present an analog of Theorem 7 for convex functions defined on k-simplices in $\mathbb{R}^k$.

Theorem 10.

Let L satisfy properties L1, L2, L3 on nonempty set E, A be a positive normalized linear functional on L and $\widetilde{A}$ defined as in (2). Let f be a convex function on k-simplex $S = [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}]$ in $\mathbb{R}^k$ and $\lambda_1, \ldots, \lambda_{k+1}$ barycentric coordinates over S. Then for all $\boldsymbol{g} \in L^k$ such that $\boldsymbol{g}(E) \subset S$ and $f(\boldsymbol{g}) \in L$ we have

$$A(f(\boldsymbol{g})) \leq \sum_{i=1}^{k+1} A(\lambda_i(\boldsymbol{g}))\, f(\boldsymbol{v}_i) - A(\min\{\lambda_i(\boldsymbol{g})\})\, S_f^{k+1}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1})$$

$$= \frac{\mathrm{Vol}_k\left(\left[\widetilde{A}(\boldsymbol{g}), \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}\right]\right)}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_1) + \cdots + \frac{\mathrm{Vol}_k\left(\left[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \widetilde{A}(\boldsymbol{g})\right]\right)}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_{k+1})$$

$$-A(\min\{\lambda_i(\boldsymbol{g})\})\, S_f^{k+1}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1}).$$

Proof.

For each $t \in E$ we have $\boldsymbol{g}(t) \in S$. Using barycentric coordinates we have

$$\lambda_1\left(\boldsymbol{g}(t)\right) = \frac{\mathrm{Vol}_k\left([\boldsymbol{g}(t), \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}]\right)}{\mathrm{Vol}_k(S)} = \frac{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ g_1(t) & v_{21} & & v_{k+11} \\ \vdots & \vdots & & \vdots \\ g_k(t) & v_{2k} & \cdots & v_{k+1k} \end{vmatrix}}{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ v_{11} & v_{21} & & v_{k+11} \\ \vdots & \vdots & & \vdots \\ v_{1k} & v_{2k} & \cdots & v_{k+1k} \end{vmatrix}},$$

$\vdots$

$$\lambda_{k+1}(\boldsymbol{g}(t)) = \frac{\mathrm{Vol}_k\left([\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \boldsymbol{g}(t)]\right)}{\mathrm{Vol}_k(S)} = \frac{\frac{1}{k!}\begin{vmatrix} 1 & \cdots & 1 & 1 \\ v_{11} & & v_{k1} & g_1(t) \\ \vdots & & \vdots & \vdots \\ v_{1k} & \cdots & v_{kk} & g_k(t) \end{vmatrix}}{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ v_{11} & v_{21} & & v_{k+11} \\ \vdots & \vdots & & \vdots \\ v_{1k} & v_{2k} & \cdots & v_{k+1k} \end{vmatrix}},$$

$$\sum_{i=1}^{k+1} \lambda_i(\boldsymbol{g}(t)) = 1 \text{ and } \boldsymbol{g}(t) = \sum_{i=1}^{k+1} \lambda_i(\boldsymbol{g}(t))\boldsymbol{v}_i.$$

Since f is convex on S, then using Lemma 6 we have

$$\begin{aligned} f(\boldsymbol{g}(t)) \le & \sum_{i=1}^{k+1} \lambda_i(\boldsymbol{g}(t)) f(\boldsymbol{v}_i) \\ & - \min\{\lambda_i(\boldsymbol{g}(t))\} \left[\sum_{i=1}^{k+1} f(\boldsymbol{v}_i) - (k+1) f\left(\frac{1}{k+1} \sum_{i=1}^{k+1} \boldsymbol{v}_i \right) \right]. \end{aligned}$$

Using the Laplace expansion of the determinant we can easily check that $\lambda_i(\boldsymbol{g}) \in L$ for all $i = 1, \ldots, k+1$.

Now, applying A on the last inequality we have

$$A(f(\boldsymbol{g})) \leq A\left(\sum_{i=1}^{k+1} \lambda_i(\boldsymbol{g}) f(\boldsymbol{v}_i) \right.$$
$$\left. - \min\left\{\lambda_i(\boldsymbol{g}(t))\right\}\left[\sum_{i=1}^{k+1} f(\boldsymbol{v}_i) - (k+1) f\left(\frac{1}{k+1}\sum_{i=1}^{k+1} \boldsymbol{v}_i\right)\right]\right)$$
$$= \sum_{i=1}^{k+1} A\left(\lambda_i(\boldsymbol{g})\right) f\left(v_i\right) - A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right) S_f^{k+1}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1}).$$

where

$$A\left(\lambda_1\left(\boldsymbol{g}\right)\right) = \frac{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ A(g_1) & v_{21} & & v_{k+11} \\ \vdots & \vdots & & \vdots \\ A(g_k) & v_{2k} & \cdots & v_{k+1k} \end{vmatrix}}{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ v_{11} & v_{21} & & v_{k+11} \\ \vdots & \vdots & & \vdots \\ v_{1k} & v_{2k} & \cdots & v_{k+1k} \end{vmatrix}} = \frac{\mathrm{Vol}_k\left(\left[\widetilde{A}(\boldsymbol{g}), \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}\right]\right)}{\mathrm{Vol}_k(S)},$$

$\vdots$

$$A\left(\lambda_{k+1}\left(\boldsymbol{g}\right)\right)=\frac{\frac{1}{k!}\begin{vmatrix}1 & \cdots & 1 & 1\\ v_{11} & & v_{k1} & A(g_1)\\ \vdots & & \vdots & \vdots\\ v_{1k} & \cdots & v_{kk} & A(g_k)\end{vmatrix}}{\frac{1}{k!}\begin{vmatrix}1 & 1 & \cdots & 1\\ v_{11} & v_{21} & & v_{k+11}\\ \vdots & \vdots & & \vdots\\ v_{1k} & v_{2k} & \cdots & v_{k+1k}\end{vmatrix}}=\frac{\mathrm{Vol}_k\left(\left[\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k,\widetilde{A}(\boldsymbol{g})\right]\right)}{\mathrm{Vol}_k(S)}.$$

Theorem 11.

Let L satisfy properties L1, L2, L3 on nonempty set E, A be a positive normalized linear functional on L and $\widetilde{A}$ defined as in (2). Let f be a convex function on k-simplex $S = [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}]$ in $\mathbb{R}^k$ and $\lambda_1, \ldots, \lambda_{k+1}$ barycentric coordinates over S. If J is an interval in $\mathbb{R}$ such that $f(S) \subset J$ and $F \colon J \times J \to \mathbb{R}$ an increasing function in the first variable, then for all $\boldsymbol{g} \in L^k$ such that $\boldsymbol{g}(E) \subset S$ and $f(\boldsymbol{g}) \in L$ we have

$$F\left(A(f(\boldsymbol{g})), f(\widetilde{A}(\boldsymbol{g}))\right) \leq \frac{1}{\mathrm{Vol}_k(S)} \max_{\boldsymbol{x}\in S} F\Bigg(\sum_{i=1}^{k+1} \mathrm{Vol}_k\left([\boldsymbol{v}_1, \ldots, \hat{\boldsymbol{v}}_i, \ldots, \boldsymbol{v}_{k+1}]\right)$$

$$-A\left(\min\{\lambda_i(\boldsymbol{g})\}\right) S_f^{k+1}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1}), f(\boldsymbol{x}) \Bigg)$$

$$= \max_{\Delta_k} F\left(\sum_{i=1}^{k+1} \mu_i f(\boldsymbol{v}_i) - A\left(\min\{\lambda_i(\boldsymbol{g})\}\right) S_f^{k+1}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1}) \right.,$$

$$\left. f\left(\sum_{i=1}^{k+1} \mu_i \boldsymbol{v}_i\right)\right), \quad \textit{where } \hat{\boldsymbol{v}}_i = \boldsymbol{x}$$

Proof.

Since for each $t \in E$ we have $\boldsymbol{g}(t) \in S$, then it follows $\widetilde{A}(\boldsymbol{g}) \in S$ (see the first part of proof of Theorem 8).
Since $F\colon J \times J \to \mathbb{R}$ is an increasing function in the first variable, by Theorem 10 we have

$$
\begin{aligned}
&F\left(A(f(\boldsymbol{g})), f(\widetilde{A}(\boldsymbol{g}))\right)\\
&\quad\leq F\Bigg(\frac{\mathrm{Vol}_k\left(\left[\widetilde{A}(\boldsymbol{g}),\boldsymbol{v}_2,\ldots,\boldsymbol{v}_{k+1}\right]\right)}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_1)+\cdots+\frac{\mathrm{Vol}_k\left(\left[\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k,\widetilde{A}(\boldsymbol{g})\right]\right)}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_{k+1})\\
&\qquad\qquad -A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right) S_f^{k+1}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k+1}), f(\widetilde{A}(\boldsymbol{g}))\Bigg)\\
&\quad\leq \max_{\boldsymbol{x}\in S} F\Bigg(\frac{\mathrm{Vol}_k([\boldsymbol{x},\boldsymbol{v}_2,\ldots,\boldsymbol{v}_{k+1}])}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_1)+\cdots+\frac{\mathrm{Vol}_k([\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k,\boldsymbol{x}])}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_{k+1})\\
&\qquad\qquad -A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right) S_f^{k+1}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k+1}), f\left(\boldsymbol{x}\right)\Bigg).
\end{aligned}
$$

The equality is a simple consequence of substitutions

$$\mu_1 = \frac{\mathrm{Vol}_k\left([\boldsymbol{x}, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}]\right)}{\mathrm{Vol}_k(S)}, \ldots, \mu_{k+1} = \frac{\mathrm{Vol}_k\left([\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \boldsymbol{x}]\right)}{\mathrm{Vol}_k(S)},$$

and

$$\boldsymbol{x} = \sum_{i=1}^{k+1} \mu_i \boldsymbol{v}_i.$$

Remark

If all the assumptions of Theorem 10 are satisfied and in addition f is continuous, then

$$
\begin{aligned}
& f(\widetilde{A}(\boldsymbol{g})) \leq A(f(\boldsymbol{g})) \\
& \quad \leq \sum_{i=1}^{k+1} A\left(\lambda_i(\boldsymbol{g})\right) f\left(\boldsymbol{v}_i\right) - A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right) S_f^{k+1}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1}) \\
& \quad = \frac{\mathrm{Vol}_k\left(\left[\widetilde{A}(\boldsymbol{g}), \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}\right]\right)}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_1) + \cdots + \frac{\mathrm{Vol}_k\left(\left[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \widetilde{A}(\boldsymbol{g})\right]\right)}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_{k+1}) \\
& \qquad - A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right) S_f^{k+1}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1}).
\end{aligned}
$$

Example

Let $S = [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}]$ be a k-simplex in $\mathbb{R}^k$ and f a continuous convex function on S. Let $L = (E, \mathcal{A}, \lambda)$ be a measure space with positive measure λ. We define the functional $A\colon L \to \mathbb{R}$ by

$$A(g) = \frac{1}{\lambda(E)} \int_E g(t) d\lambda(t)$$

It is obvious that A is positive normalized linear functional on L. Then the linear operator $\widetilde{A}$ is defined by

$$\widetilde{A}(\boldsymbol{g}) = \frac{1}{\lambda(E)} \int_E \boldsymbol{g}(t) d\lambda(t).$$

We denote $\overline{\boldsymbol{g}} = \frac{1}{\lambda(E)} \int_E \boldsymbol{g}(t) d\lambda(t)$. If $\boldsymbol{g}(E) \subset S$ and $f(\boldsymbol{g}) \in L$, then from previous remark it follows

$$\begin{aligned} f(\overline{\boldsymbol{g}}) &\le A(f(\boldsymbol{g})) \\ &\le \frac{\mathrm{Vol}_k\left([\overline{\boldsymbol{g}}, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}]\right)}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_1) + \cdots + \frac{\mathrm{Vol}_k\left([\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \overline{\boldsymbol{g}}]\right)}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_{k+1}) \\ &- \left(\frac{1}{\lambda(E)} \int_E \min\left\{\lambda_i(\boldsymbol{g}(t)) \colon i = 1, \ldots, k+1\right\} d\lambda(t) \right) S_f^{k+1}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1}) \end{aligned}$$

Remark

Let $S = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1}]$ be a $k-$simplex in $\mathbb{R}^k$. If we put $E = S, \boldsymbol{g} = \boldsymbol{id_S}$ and λ Lebesgue measure on S in previous example we get

$$\overline{\boldsymbol{id_S}} = \frac{1}{|S|} \int_S t dt = \boldsymbol{v}^* = \frac{1}{k+1} \sum_{i=1}^{k+1} \boldsymbol{v}_i$$

$$A(f(\boldsymbol{id_S})) = \frac{1}{|S|} \int_S f(t) dt$$

where $\boldsymbol{v}^*$ is the barycenter of S.

Now we have

$$
\begin{aligned}
f(\boldsymbol{v}^*) &\leq \frac{1}{|S|}\int_S f(t)dt \\
&\leq \frac{\mathrm{Vol}_k\left([\boldsymbol{v}^*, \boldsymbol{v}_2, \dots, \boldsymbol{v}_{k+1}]\right)}{|S|} f(\boldsymbol{v}_1) + \cdots + \frac{\mathrm{Vol}_k\left([\boldsymbol{v}_1, \dots, \boldsymbol{v}_k, \boldsymbol{v}^*]\right)}{|S|} f(\boldsymbol{v}_{k+1}) \\
&- \left(\frac{1}{|S|}\int_S \min\left\{\lambda_i(t) \colon i = 1, \dots, k+1\right\} dt\right) \left[\sum_{i=1}^{k+1} f(\boldsymbol{v}_i) - (k+1) f(\boldsymbol{v}^*)\right] \\
&= \frac{1}{k+1}\left(\sum_{i=1}^{k+1} f\left(\boldsymbol{v}_i\right)\right) \\
&- \left(\frac{1}{|S|}\int_S \min\left\{\lambda_i(t) \colon i = 1, \dots, k+1\right\} dt\right) \left[\sum_{i=1}^{k+1} f(\boldsymbol{v}_i) - (k+1) f(\boldsymbol{v}^*)\right]
\end{aligned}
$$

For $i = 1, \ldots, k+1$, let S_i be the simplex whose vertices are $\boldsymbol{v}^*$ and all vertices of S except $\boldsymbol{v}_i$. Denote by $\boldsymbol{v}_i^*$ the barycentre of $S_i, i = 1, \ldots, k+1$. Since $\mathrm{Vol}_k\left(S_i\right) = \mathrm{Vol}_k\left(S_j\right), i, j = 1, \ldots, k+1$, it follows from (5) that $t \in S_j$ implies $\min_i \lambda_i(t) = \lambda_j(t)$. It follows

$$\int_S \min_i \lambda_i(t)dt = \sum_{j=1}^{k+1} \int_{S_j} \lambda_j(t)dt. \tag{5}$$

We have

$$\begin{aligned}
\int_{S_j} \lambda_j(t)dt
&= \frac{1}{|S|}\int_{S_j} \mathrm{Vol}_k\left[\boldsymbol{v}_1,\ldots,t,\ldots,\boldsymbol{v}_{k+1}\right]dt\\
&= \frac{1}{|\mathrm{S}|}\mathrm{Vol}_k\left[\boldsymbol{v}_1,\ldots,\int_{S_j} tdt,\ldots,\boldsymbol{v}_{k+1}\right]\\
&= \frac{|S_j|}{|S|}\mathrm{Vol}_k\left[\boldsymbol{v}_1,\ldots,\boldsymbol{v}_j^*,\ldots,\boldsymbol{v}_{k+1}\right] = \frac{1}{k+1}\mathrm{Vol}_k\left[\boldsymbol{v}_1,\ldots,\boldsymbol{v}_j^*,\ldots,\boldsymbol{v}_{k+1}\right]\\
&= \frac{1}{(k+1)^2}\mathrm{Vol}_k\left[\boldsymbol{v}_1,\ldots,\boldsymbol{v}^*,\ldots,\boldsymbol{v}_{k+1}\right] = \frac{1}{(k+1)^3}|S|. \qquad (6)
\end{aligned}$$

Using (5) and (6) we get

$$\int_S \min_i \lambda_i(t)dt = \frac{1}{(k+1)^2}|S|.$$

Now, putting (7) in (5), we have

$$\begin{aligned} f(\boldsymbol{v}^*) &\leq \frac{1}{|S|}\int_S f(t)dt \\ &\leq \frac{k}{(k+1)^2}\sum_{i=1}^{k+1} f(\boldsymbol{v}_i) + \frac{1}{k+1} f(\boldsymbol{v}^*) \end{aligned}$$

which is Theorem 4.1 obtained in
A. Guessab, G. Schmeisser, *Convexity results and sharp error estimates in approximate multivariate integration*, Math. Comp., 2003, Volume 73, Number 247.

It can be easily verified that the right-hand side of this inequality is equivalent to the k-dimensional version of the Hammer-Bullen inequality, namely

$$\frac{1}{|S|}\int_S f(t)dt - f(\boldsymbol{v}^*) \le \frac{k}{k+1}\sum_{i=1}^{k+1} f(\boldsymbol{v}_i) - \frac{k}{|S|}\int_S f(t)dt$$

which is proved, for example in
S. Wąsowicz, A. Witkowski, *On some inequality of Hermite-Hadamard type*, forthcoming paper in Opuscula Math.
In one dimension this is exactly classical Hammer-Bullen inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2} - \frac{1}{4}S_f^2(a,b)$$