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GENERALIZATIONS AND IMPROVEMENTS OF CONVERSE JENSEN'S INEQUALITY FOR CONVEX HULLS IN $\mathbb{R}^k$

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Abstract. In this paper we prove generalizations and improvements of Lah-Ribarič and related inequalities for convex functions on convex hulls in $\mathbb{R}^k$ and, analogously, for convex functions on k-simplices in $\mathbb{R}^k$. We also verify that one of them is a generalization and an improvement of the Hermite-Hadamard inequality for simplices.

1. Introduction

Let U be a convex subset of $\mathbb{R}^k$ and $n \in \mathbb{N}$. If $f\colon U \to \mathbb{R}$ is a convex function, $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n \in U$ and $p_1,\dots,p_n$ nonnegative real numbers with $P_n = \sum_{i=1}^n p_i > 0$, then Jensen's inequality

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i\boldsymbol{x}_i\right) \leqslant \frac{1}{P_n}\sum_{i=1}^n p_i f(\boldsymbol{x}_i) \tag{1}$$

holds.

Convex hull of vectors $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n \in \mathbb{R}^k$ is the set

$$\left\{\sum_{i=1}^n \alpha_i\boldsymbol{x}_i \middle| \alpha_i \in \mathbb{R}, \alpha_i \geqslant 0, \sum_{i=1}^n \alpha_i = 1\right\}$$

and is denoted by $K = co(\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\})$.

Barycentric coordinates over K are continuous real functions $\lambda_1,\dots,\lambda_n$ on K with the following properties:

$$\begin{aligned}
&\lambda_i(\boldsymbol{x}) \geqslant 0,\ i = 1,\dots,n,\\
&\sum_{i=1}^n \lambda_i(\boldsymbol{x}) = 1,\\
&\boldsymbol{x} = \sum_{i=1}^n \lambda_i(\boldsymbol{x})\boldsymbol{x}_i.
\end{aligned} \tag{2}$$

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If $\boldsymbol{x}_2-\boldsymbol{x}_1,\dots,\boldsymbol{x}_n-\boldsymbol{x}_1$ are linearly independent vectors, then each $\boldsymbol{x}\in K$ can be written in the unique way as a convex combination of $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n$ in the form (2).

We also consider k-simplex $S=co(\{\boldsymbol{v}_1,\boldsymbol{v}_2,\dots,\boldsymbol{v}_{k+1}\})$ in $\mathbb{R}^k$ which is a convex hull of its vertices $\boldsymbol{v}_1,\dots,\boldsymbol{v}_{k+1}\in\mathbb{R}^k$, where $\boldsymbol{v}_2-\boldsymbol{v}_1,\dots,\boldsymbol{v}_{k+1}-\boldsymbol{v}_1\in\mathbb{R}^k$ are linearly independent. In this case we denote the simplex by $S=[\boldsymbol{v}_1,\dots,\boldsymbol{v}_{k+1}]$. Barycentric coordinates $\lambda_1,\lambda_2,\dots,\lambda_{k+1}$ over S are nonnegative linear polynomials on S and have a special form (see [1]).

Let E be a non-empty set and L be a linear class of real-valued functions $f\colon E\to\mathbb{R}$ which contains constant functions, that is, L has the following properties:

$(L1)$ $(\forall f,g\in L)\,(\forall a,b\in\mathbb{R})\ af+bg\in L$

$(L2)$ $1\in L$, that is, if $f(t)=1$ for all $t\in E$, then $f\in L$.

We consider positive linear functionals $A\colon L\to\mathbb{R}$, or in other words we assume:

$(A1)$ $(\forall f,g\in L)(\forall a,b\in\mathbb{R})\quad A(af+bg)=aA(f)+bA(g)$ (linearity)

$(A2)$ $(\forall f\in L)(f\geqslant 0\Longrightarrow A(f)\geqslant 0)$ (positivity).

If additionally the condition $A(1)=1$ is satisfied, we say that A is a positive normalized linear functional.

With L^k we denote the linear class of functions $\boldsymbol{g}\colon E\to\mathbb{R}^k$ defined by

$$\boldsymbol{g}(t)=(g_1(t),\dots,g_k(t)),\quad g_i\in L,\quad i=1,\dots,k.$$

For a given linear functional A, we also consider linear operator $\widetilde{A}=(A,\dots,A)\colon L^k\to\mathbb{R}^k$ defined by

$$\widetilde{A}(\boldsymbol{g})=(A(g_1),\dots,A(g_k)). \tag{3}$$

If $A(1)=1$ is satisfied, then using (A1) we also have:

(A3) $A\left(f(\boldsymbol{g})\right)=f\left(\widetilde{A}(\boldsymbol{g})\right)$ for every linear function f on $\mathbb{R}^k$.

The following result is Jessen's generalization of Jensen's inequality for convex functions (see [8, p. 45]) which involves positive normalized linear functionals.

THEOREM 1. *Let L satisfy $(L1)$ and $(L2)$ on a nonempty set E and let A be a positive normalized linear functional on L. If f is a continuous convex function on an interval $I\subset\mathbb{R}$, then for all $g\in L$ such that $f(g)\in L$ we have $A(g)\in I$ and*

$$f(A(g))\leqslant A(f(g)). \tag{4}$$

The next theorem, proved by J. Pečarić and P. R. Beesack in 1985., presents a generalization of Lah-Ribarič inequality (see [7, p. 98], [8, p. 98]).

THEOREM 2. (Lah-Ribarič inequality) *Let L satisfy properties $(L1)$ and $(L2)$ and A be a positive normalized linear functional on L. Let f be a convex function on an interval $I = [m,M] \subset \mathbb{R}$ $(-\infty < m < M < \infty)$. Then for all $g \in L$ such that $g(E) \subset I$ and $f(g) \in L$*

$$A(f(g)) \leqslant \frac{M - A(g)}{M - m} f(m) + \frac{A(g) - m}{M - m} f(M). \tag{5}$$

Using Theorem 2, Beesack and Pečarić in 1987. also proved the next result [8, p. 101].

THEOREM 3. *Let L, A and f be as in Theorem 2. Let J be an interval in $\mathbb{R}$ such that $f(I) \subset J$. If $F : J \times J \to \mathbb{R}$ is an increasing function in the first variable, then for all $g \in L$ such that $g(E) \subset I$ and $f(g) \in L$, we have*

$$\begin{aligned} F(A(f(g)), f(A(g))) &\leqslant \max_{x \in [m,M]} F\left(\frac{M - x}{M - m} f(m) + \frac{x - m}{M - m} f(M), f(x)\right) \\ &= \max_{\theta \in [0,1]} F\left(\theta f(m) + (1 - \theta) f(M), f(\theta m + (1 - \theta) M)\right). \end{aligned} \tag{6}$$

REMARK 1. If we choose $F(x,y) = x - y$, as a simple consequence of Theorem 3 it follows

$$A(f(g)) - f(A(g)) \leqslant \max_{\theta \in [0,1]} [\theta f(m) + (1 - \theta) f(M) - f(\theta m + (1 - \theta) M)]. \tag{7}$$

Taking $F(x,y) = \frac{x}{y}$, for $f > 0$, it follows

$$\frac{A(f(g))}{f(A(g))} \leqslant \max_{\theta \in [0,1]} \left[\frac{\theta f(m) + (1 - \theta) f(M)}{f(\theta m + (1 - \theta) M)}\right]. \tag{8}$$

An additional generalization of Jessen's inequality (4) is proved by E. J. McShane (see [6], [8, p. 48]).

THEOREM 4. (McShane's inequality) *Let L satisfy properties $(L1)$ and $(L2)$, A be a positive normalized linear functional on L and $\widetilde{A}$ defined as in (3). Let f be a continuous convex function on a closed convex set $U \subset \mathbb{R}^k$. Then for all $\boldsymbol{g} \in L^k$ such that $\boldsymbol{g}(E) \subset U$ and $f(\boldsymbol{g}) \in L$, we have that $\widetilde{A}(\boldsymbol{g}) \in U$ and*

$$f(\widetilde{A}(\boldsymbol{g})) \leqslant A(f(\boldsymbol{g})). \tag{9}$$

J. Pečarić and S. Ivelić in [3] proved the following generalization of Theorem 2.

THEOREM 5. *Let L satisfy properties $(L1)$ and $(L2)$ on nonempty set E and A be a positive normalized linear functional on L. Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathbb{R}^k$ and $K = co(\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\})$. Let f be a convex function on K and $\lambda_1, \ldots, \lambda_n$ barycentric coordinates over K. Then for all $\boldsymbol{g} \in L^k$ such that $\boldsymbol{g}(E) \subset K$ and $f(\boldsymbol{g}), \lambda_i(\boldsymbol{g}) \in L, i = 1, \ldots, n$ we have*

$$A(f(\boldsymbol{g})) \leqslant \sum_{i=1}^{n} A(\lambda_i(\boldsymbol{g})) f(\boldsymbol{x}_i).$$

2. Main results

Our main results are generalizations and improvements of Theorems 3 and 5 which will be obtained using the following lemma.

LEMMA 1. *Let ϕ be a convex function on U where U is a convex set in $\mathbb{R}^k$, $(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n)\in U^n$ and $p=(p_1,\dots,p_n)$ be nonnegative n-tuple such that $\sum_{i=1}^{n} p_i = 1$. Then*

$$\begin{aligned}
&\min\{p_1,\dots,p_n\}\left[\sum_{i=1}^{n}\phi(\boldsymbol{x}_i)-n\phi\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{x}_i\right)\right]\\
&\leqslant \sum_{i=1}^{n} p_i\phi(\boldsymbol{x}_i)-\phi\left(\sum_{i=1}^{n}p_i\boldsymbol{x}_i\right)\\
&\leqslant \max\{p_1,\dots,p_n\}\left[\sum_{i=1}^{n}\phi(\boldsymbol{x}_i)-n\phi\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{x}_i\right)\right].
\end{aligned}$$

Proof. This is a simple consequence of [7, p. 717, Theorem 1]. □

For $n\in\mathbb{N}$ we denote

$$\Delta_{n-1}=\left\{(\mu_1,\dots,\mu_n)\colon \mu_i\geqslant 0, i\in\{1,\dots,n\}, \sum_{i=1}^{n}\mu_i=1\right\}.$$

We also need to equip our linear class L from Introduction with an additional property denoted by (L3):

(L3) $(\forall f,g\in L)\,(\min\{f,g\}\in L$ and $\max\{f,g\}\in L)$ (lattice property).

Obviously, $\left(\mathbb{R}^E,\leqslant\right)$ (with standard ordering) is a lattice.

Also, if f is a function defined on a convex subset $U\subseteq\mathbb{R}^k$ and $\boldsymbol{x}_1,\boldsymbol{x}_2,\dots,\boldsymbol{x}_n\in U$, we denote

$$S_f^n(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n)=\sum_{i=1}^{n}f(\boldsymbol{x}_i)-nf\left(\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{x}_i\right).$$

Obviously, if f is convex, $S_f^n(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n)\geqslant 0$.

Next theorem presents an improvement of Theorem 5.

THEOREM 6. *Let L satisfy properties $(L1)$, $(L2)$ and $(L3)$ on a nonempty set E and A be a positive normalized linear functional on L. Let $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\in\mathbb{R}^k$ and $K=co(\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\})$. Let f be a convex function on K and $\lambda_1,\dots,\lambda_n$ barycentric coordinates over K. Then for all $\boldsymbol{g}\in L^k$ such that $\boldsymbol{g}(E)\subset K$ and $f(\boldsymbol{g})$, $\lambda_i(\boldsymbol{g})\in L$, $i=1,\dots,n$, we have*

$$A(f(\boldsymbol{g}))\leqslant\sum_{i=1}^{n}A(\lambda_i(\boldsymbol{g}))\,f(\boldsymbol{x}_i)-A(\min\{\lambda_i(\boldsymbol{g})\})\,S_f^n(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n). \qquad (10)$$

Proof. For each $t \in E$ we have $\boldsymbol{g}(t) \in K$. Using barycentric coordinates we have $\lambda_i(\boldsymbol{g}(t)) \geqslant 0$, $i = 1, \ldots, n$, $\sum_{i=1}^n \lambda_i(\boldsymbol{g}(t)) = 1$ and

$$\boldsymbol{g}(t) = \sum_{i=1}^n \lambda_i(\boldsymbol{g}(t))\boldsymbol{x}_i.$$

Since f is convex, we can apply Lemma 1, and then

$$\begin{aligned} f(\boldsymbol{g}(t)) &= f\left(\sum_{i=1}^n \lambda_i(\boldsymbol{g}(t))\boldsymbol{x}_i\right) \\ &\leqslant \sum_{i=1}^n \lambda_i(\boldsymbol{g}(t)) f(\boldsymbol{x}_i) - \min\{\lambda_i(\boldsymbol{g}(t))\} \left[\sum_{i=1}^n f(\boldsymbol{x}_i) - nf\left(\frac{1}{n}\sum_{i=1}^n \boldsymbol{x}_i\right)\right]. \end{aligned} \tag{11}$$

Now, applying the functional A on (11), we get

$$\begin{aligned} A(f(\boldsymbol{g})) &\leqslant A\left(\sum_{i=1}^n \lambda_i(\boldsymbol{g}) f(\boldsymbol{x}_i) - \min\{\lambda_i(\boldsymbol{g})\} S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)\right) \\ &= \sum_{i=1}^n A(\lambda_i(\boldsymbol{g})) f(\boldsymbol{x}_i) - A(\min\{\lambda_i(\boldsymbol{g})\}) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n). \quad \square \end{aligned}$$

REMARK 2. Theorem 6 is an improvement of Theorem 5 since under the required assumptions we have

$$A(\min\{\lambda_i(\boldsymbol{g})\}) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \geqslant 0.$$

REMARK 3. If all the assumptions of Theorem 6 are satisfied and additionally f is continuous, then

$$f(\widetilde{A}(\boldsymbol{g})) \leqslant A(f(\boldsymbol{g})) \leqslant \sum_{i=1}^n A(\lambda_i(\boldsymbol{g})) f(\boldsymbol{x}_i) - A(\min\{\lambda_i(\boldsymbol{g})\}) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n).$$

The first inequality is from Theorem 4 and the second from Theorem 6.

REMARK 4. We know that under the assumptions of Theorem 6 we have

$$A(f(\boldsymbol{g})) \leqslant \sum_{i=1}^n A(\lambda_i(\boldsymbol{g})) f(\boldsymbol{x}_i) - A(\min\{\lambda_i(\boldsymbol{g})\}) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n).$$

Dividing this by $f(\widetilde{A}(\boldsymbol{g})) = f\left(\sum_{i=1}^n \lambda_i(\boldsymbol{g}(t))\boldsymbol{x}_i\right)$, when $f > 0$, we obtain

$$\begin{aligned} \frac{A(f(\boldsymbol{g}))}{f\left(\widetilde{A}(\boldsymbol{g})\right)} &\leqslant \frac{\sum_{i=1}^n A(\lambda_i(\boldsymbol{g})) f(\boldsymbol{x}_i)}{f\left(\sum_{i=1}^n A(\lambda_i(\boldsymbol{g}))\boldsymbol{x}_i\right)} - \frac{A(\min\{\lambda_i(\boldsymbol{g})\})}{f\left(\widetilde{A}(\boldsymbol{g})\right)} S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \\ &\leqslant \max_{\Delta_{n-1}} \frac{\sum_{i=1}^n \mu_i f(\boldsymbol{x}_i)}{f\left(\sum_{i=1}^n \mu_i \boldsymbol{x}_i\right)} - \frac{A(\min\{\lambda_i(\boldsymbol{g})\})}{f\left(\widetilde{A}(\boldsymbol{g})\right)} S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n), \end{aligned}$$

which is equivalent to

$$A\left(f\left(\boldsymbol{g}\right)\right) \leqslant \max_{\Delta_{n-1}} \frac{\sum_{i=1}^{n} \mu_i f(\boldsymbol{x}_i)}{f\left(\sum_{i=1}^{n} \mu_i \boldsymbol{x}_i\right)} f\left(\widetilde{A}\left(\boldsymbol{g}\right)\right) - A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right) S_f^n(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n). \tag{12}$$

This is an improvement of the inequality (2.6) from [3].

REMARK 5. As a special case of Theorem 6 for $k=1$ we get [5, Theorem 12], and if we take p and q nonnegative real numbers such that $A\left(g\right)=\dfrac{pm+qM}{p+q}$ we get right hand side of the inequality (2.3) in [4].

Using Teorem 6 we prove a generalization and an improvement of Theorem 3.

THEOREM 7. *Let L satisfy properties (L1), (L2) and (L3) on nonempty set E, A be a positive normalized linear functional on L and $\widetilde{A}$ defined as in (3). Let $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n \in \mathbb{R}^k$ and $K = co(\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\})$. Let f be a convex function on K and $\lambda_1,\ldots,\lambda_n$ barycentric coordinates over K. If J is an interval in $\mathbb{R}$ such that $f(K) \subset J$ and $F\colon J\times J \to \mathbb{R}$ is an increasing function in the first variable, then for all $\boldsymbol{g}\in L^k$ such that $\boldsymbol{g}\left(E\right)\subset K$ and $f(\boldsymbol{g}),\lambda_i(\boldsymbol{g})\in L, i=1,\ldots,n$ we have*

$$\begin{aligned}
&F\left(A(f(\boldsymbol{g})), f(\widetilde{A}(\boldsymbol{g}))\right)\\
&\leqslant F\left(\sum_{i=1}^{n} A\left(\lambda_i(\boldsymbol{g})\right) f\left(\boldsymbol{x}_i\right) - A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right) S_f^n(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n), f(\widetilde{A}(\boldsymbol{g}))\right)\\
&\leqslant \max_{\Delta_{n-1}} F\left(\sum_{i=1}^{n} \mu_i f(\boldsymbol{x}_i) - A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right) S_f^n(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n), f\left(\sum_{i=1}^{n}\mu_i \boldsymbol{x}_i\right)\right).
\end{aligned} \tag{13}$$

Proof. For each $t\in E$ we have $\boldsymbol{g}(t)\in K$. Using barycentric coordinates we have $\lambda_i(\boldsymbol{g}(t)) \geqslant 0, i=1,\ldots,n$, $\sum_{i=1}^{n}\lambda_i(\boldsymbol{g}(t)) = 1$ and

$$\boldsymbol{g}(t) = \sum_{i=1}^{n} \lambda_i(\boldsymbol{g}(t))\boldsymbol{x}_i.$$

Since A is a positive normalized linear functional on L and $\widetilde{A}$ a linear operator on L^k, we have

$$\widetilde{A}\left(\boldsymbol{g}\right) = (A(g_1),\ldots,A(g_k)) = \sum_{i=1}^{n} A\left(\lambda_i(\boldsymbol{g})\right) x_i,$$

where $A\left(\lambda_i(\boldsymbol{g})\right)\geqslant 0$, $i=1,\ldots,n$ and $\sum_{i=1}^{n} A\left(\lambda_i(\boldsymbol{g})\right) = A\left(\sum_{i=1}^{n}\lambda_i(\boldsymbol{g})\right) = A(1) = 1$. Therefore, $\widetilde{A}\left(\boldsymbol{g}\right)\in K$.

Since $F: J \times J \to \mathbb{R}$ is an increasing function in the first variable, using (10) we have

$$F\left(A(f(\boldsymbol{g})), f(\widetilde{A}(\boldsymbol{g}))\right) \leqslant F\left(\sum_{i=1}^{n} A\left(\lambda_i(\boldsymbol{g})\right) f(\boldsymbol{x}_i) - A\left(\min\left\{\lambda_i(\boldsymbol{g}(t))\right\}\right) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n), f(\widetilde{A}(\boldsymbol{g}))\right).$$

By substitutions

$$A\left(\lambda_i(\boldsymbol{g})\right) = \mu_i, i = 1, \ldots, n,$$

it follows

$$\widetilde{A}(\boldsymbol{g}) = \sum_{i=1}^{n} \mu_i \boldsymbol{x}_i.$$

Now we have

$$\begin{aligned} &F\left(\sum_{i=1}^{n} A\left(\lambda_i(\boldsymbol{g})\right) f(\boldsymbol{x}_i) - A\left(\min\left\{\lambda_i(\boldsymbol{g}(t))\right\}\right) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n), f(\widetilde{A}(\boldsymbol{g}))\right) \\ &= F\left(\sum_{i=1}^{n} \mu_i f(\boldsymbol{x}_i) - A\left(\min\left\{\lambda_i(\boldsymbol{g}(t))\right\}\right) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n), f\left(\sum_{i=1}^{n} \mu_i \boldsymbol{x}_i\right)\right) \\ &\leqslant \max_{\Delta_{n-1}} F\left(\sum_{i=1}^{n} \mu_i f(\boldsymbol{x}_i) - A\left(\min\left\{\lambda_i(\boldsymbol{g}(t))\right\}\right) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n), f\left(\sum_{i=1}^{n} \mu_i \boldsymbol{x}_i\right)\right). \end{aligned}$$

By combining (14) and the last inequality we get (13). □

REMARK 6. If we choose $F(x,y) = x - y$, as a simple consequence of Theorem 7 it follows

$$\begin{aligned} &A(f(\boldsymbol{g})) - f(\widetilde{A}(\boldsymbol{g})) \\ &\leqslant \max_{\Delta_{n-1}} \left(\sum_{i=1}^{n} \mu_i f(\boldsymbol{x}_i) - f\left(\sum_{i=1}^{n} \mu_i \boldsymbol{x}_i\right) - A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)\right). \end{aligned} \tag{14}$$

Taking $F(x,y) = \frac{x}{y}$, for $f > 0$, it follows

$$\frac{A(f(\boldsymbol{g}))}{f(\widetilde{A}(\boldsymbol{g}))} \leqslant \max_{\Delta_{n-1}} \left(\frac{\sum_{i=1}^{n} \mu_i f(\boldsymbol{x}_i) - A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right) S_f^n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)}{f\left(\sum_{i=1}^{n} \mu_i \boldsymbol{x}_i\right)}\right). \tag{15}$$

The inequalities (14) and (15) present generalizations and improvements of (7) and (8).

Replacing F by $-F$ in Theorem 7 we get the next theorem.

THEOREM 8. *Let L satisfy properties $(L1)$, $(L2)$ and $(L3)$ on nonempty set E, A be a positive normalized linear functional on L and $\widetilde{A}$ defined as in (3). Let $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n \in \mathbb{R}^k$ and $K = co(\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\})$. Let f be a convex function on K and $\lambda_1,\dots,\lambda_n$ barycentric coordinates over K. If J is an interval in $\mathbb{R}$ such that $f(K) \subset J$ and $F\colon J\times J \to \mathbb{R}$ is an decreasing function in the first variable, then for all $\boldsymbol{g} \in L^k$ such that $\boldsymbol{g}(E) \subset K$ and $f(\boldsymbol{g}), \lambda_i(\boldsymbol{g}) \in L, i = 1,\dots,n$ we have*

$$\begin{aligned}
&F\left(A(f(\boldsymbol{g})), f(\widetilde{A}(\boldsymbol{g}))\right)\\
&\geqslant F\left(\sum_{i=1}^{n} A(\lambda_i(\boldsymbol{g}))\, f(\boldsymbol{x}_i) - A(\min\{\lambda_i(\boldsymbol{g})\})\, S_f^n(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n), f(\widetilde{A}(\boldsymbol{g}))\right)\\
&\geqslant \min_{\Delta_{n-1}} F\left(\sum_{i=1}^{n} \mu_i f(\boldsymbol{x}_i) - A(\min\{\lambda_i(\boldsymbol{g})\})\, S_f^n(\boldsymbol{x}_1,\dots,\boldsymbol{x}_n), f\left(\sum_{i=1}^{n}\mu_i \boldsymbol{x}_i\right)\right).
\end{aligned} \tag{16}$$

3. Convex functions on k-simplices in $\mathbb{R}^k$

Let S be a k-simplex in $\mathbb{R}^k$ with vertices $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_{k+1} \in \mathbb{R}^k$. The barycentric coordinates $\lambda_1,\dots\lambda_{k+1}$ over S are nonnegative linear polynomials which satisfy Lagrange's property

$$\lambda_i(\boldsymbol{v}_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

It is known (see [1]) that for each $\boldsymbol{x} \in S$ barycentric coordinates $\lambda_1(\boldsymbol{x}),\dots,\lambda_{k+1}(\boldsymbol{x})$ have the form

$$\begin{aligned}
\lambda_1(\boldsymbol{x}) &= \frac{\mathrm{Vol}_k([\boldsymbol{x}, \boldsymbol{v}_2, \dots, \boldsymbol{v}_{k+1}])}{\mathrm{Vol}_k(S)},\\
\lambda_2(\boldsymbol{x}) &= \frac{\mathrm{Vol}_k([\boldsymbol{v}_1, \boldsymbol{x}, \boldsymbol{v}_3, \dots, \boldsymbol{v}_{k+1}])}{\mathrm{Vol}_k(S)},\\
&\vdots\\
\lambda_{k+1}(\boldsymbol{x}) &= \frac{\mathrm{Vol}_k([\boldsymbol{v}_1, \dots, \boldsymbol{v}_k, \boldsymbol{x}])}{\mathrm{Vol}_k(S)},
\end{aligned} \tag{17}$$

where $\mathrm{Vol}_k(F)$ denotes the k-dimensional Lebesgue measure of a measurable set $F \subset \mathbb{R}^k$. Here, for example, $[\boldsymbol{v}_1, \boldsymbol{x}, \dots, \boldsymbol{v}_{k+1}]$ denotes the subsimplex obtained by replacing $\boldsymbol{v}_2$ by $\boldsymbol{x}$, i.e. the subsimplex opposite to $\boldsymbol{v}_2$, when adding $\boldsymbol{x}$ as a new vertex.

The signed volume $\mathrm{Vol}_k(S)$ is given by $(k+1)\times(k+1)$ determinant

$$\mathrm{Vol}_k(S) = \frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ v_{11} & v_{21} & & v_{k+11} \\ v_{12} & v_{22} & & v_{k+12} \\ \vdots & \vdots & & \vdots \\ v_{1k} & v_{2k} & \cdots & v_{k+1k} \end{vmatrix},$$

where $\boldsymbol{v}_1 = (v_{11}, v_{12}, \ldots, v_{1k}), \ldots, \boldsymbol{v}_{k+1} = (v_{k+11}, v_{k+12}, \ldots, v_{k+1k})$ (see [9]).

Since vectors $\boldsymbol{v}_2 - \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1} - \boldsymbol{v}_1$ are linearly independent, then each $\boldsymbol{x} \in S$ can be written as a convex combination of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1}$ in the form

$$\boldsymbol{x} = \frac{\mathrm{Vol}_k([\boldsymbol{x}, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}])}{\mathrm{Vol}_k(S)} \boldsymbol{v}_1 + \ldots + \frac{\mathrm{Vol}_k([\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \boldsymbol{x}])}{\mathrm{Vol}_k(S)} \boldsymbol{v}_{k+1}. \tag{18}$$

Now we present an analog of Theorem 6 for convex functions defined on k-simplices in $\mathbb{R}^k$.

THEOREM 9. *Let L satisfy properties $(L1)$, $(L2)$ and $(L3)$ on a nonempty set E, A be a positive normalized linear functional on L and $\widetilde{A}$ defined as in (3). Let f be a convex function on a k-simplex $S = [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}]$ in $\mathbb{R}^k$ and $\lambda_1, \ldots, \lambda_{k+1}$ be barycentric coordinates over S. Then for all $\boldsymbol{g} \in L^k$ such that $\boldsymbol{g}(E) \subset S$ and $f(\boldsymbol{g}) \in L$ we have*

$$\begin{aligned} A(f(\boldsymbol{g})) &\leqslant \sum_{i=1}^{k+1} A(\lambda_i(\boldsymbol{g})) f(\boldsymbol{v}_i) - A(\min\{\lambda_i(\boldsymbol{g})\}) S_f^{k+1}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1}) \qquad (19) \\ &= \frac{\mathrm{Vol}_k\left(\left[\widetilde{A}(\boldsymbol{g}), \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}\right]\right)}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_1) + \ldots + \frac{\mathrm{Vol}_k\left(\left[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \widetilde{A}(\boldsymbol{g})\right]\right)}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_{k+1}) \\ &\quad - A(\min\{\lambda_i(\boldsymbol{g})\}) S_f^{k+1}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k+1}). \end{aligned}$$

Proof. Analogous to the proof of Theorem 6 with

$$\lambda_1(\boldsymbol{g}(t)) = \frac{\mathrm{Vol}_k([\boldsymbol{g}(t), \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k+1}])}{\mathrm{Vol}_k(S)} = \frac{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ g_1(t) & v_{21} & & v_{k+11} \\ \vdots & \vdots & & \vdots \\ g_k(t) & v_{2k} & \cdots & v_{k+1k} \end{vmatrix}}{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ v_{11} & v_{21} & & v_{k+11} \\ \vdots & \vdots & & \vdots \\ v_{1k} & v_{2k} & \cdots & v_{k+1k} \end{vmatrix}},$$

$$\vdots$$

$$\lambda_{k+1}(\boldsymbol{g}(t)) = \frac{\mathrm{Vol}_k([\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \boldsymbol{g}(t)])}{\mathrm{Vol}_k(S)} = \frac{\frac{1}{k!}\begin{vmatrix} 1 & \cdots & 1 & 1 \\ v_{11} & & v_{k1} & g_1(t) \\ \vdots & & \vdots & \vdots \\ v_{1k} & \cdots & v_{kk} & g_k(t) \end{vmatrix}}{\frac{1}{k!}\begin{vmatrix} 1 & 1 & \cdots & 1 \\ v_{11} & v_{21} & & v_{k+11} \\ \vdots & \vdots & & \vdots \\ v_{1k} & v_{2k} & \cdots & v_{k+1k} \end{vmatrix}},$$

and

$$A\left(\lambda_1\left(\boldsymbol{g}\right)\right)=\frac{\frac{1}{k!}\begin{vmatrix}1 & 1 & \cdots & 1\\ A(g_1) & v_{21} & & v_{k+11}\\ \vdots & \vdots & & \vdots\\ A(g_k) & v_{2k} & \cdots & v_{k+1k}\end{vmatrix}}{\frac{1}{k!}\begin{vmatrix}1 & 1 & \cdots & 1\\ v_{11} & v_{21} & & v_{k+11}\\ \vdots & \vdots & & \vdots\\ v_{1k} & v_{2k} & \cdots & v_{k+1k}\end{vmatrix}}=\frac{\mathrm{Vol}_k\left(\left[\widetilde{A}(\boldsymbol{g}),\boldsymbol{v}_2,\ldots,\boldsymbol{v}_{k+1}\right]\right)}{\mathrm{Vol}_k(S)},$$

$$\vdots \tag{20}$$

$$A\left(\lambda_{k+1}\left(\boldsymbol{g}\right)\right)=\frac{\frac{1}{k!}\begin{vmatrix}1 & \cdots & 1 & 1\\ v_{11} & & v_{k1} & A(g_1)\\ \vdots & & \vdots & \vdots\\ v_{1k} & \cdots & v_{kk} & A(g_k)\end{vmatrix}}{\frac{1}{k!}\begin{vmatrix}1 & 1 & \cdots & 1\\ v_{11} & v_{21} & & v_{k+11}\\ \vdots & \vdots & & \vdots\\ v_{1k} & v_{2k} & \cdots & v_{k+1k}\end{vmatrix}}=\frac{\mathrm{Vol}_k\left(\left[\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k,\widetilde{A}(\boldsymbol{g})\right]\right)}{\mathrm{Vol}_k(S)}. \quad \square$$

Using Theorem 9 we prove an analog of Theorem 7 for k-simplices in $\mathbb{R}^k$.

THEOREM 10. *Let L satisfy properties $(L1)$, $(L2)$ and $(L3)$ on a nonempty set E, A be a positive normalized linear functional on L and $\widetilde{A}$ defined as in (3). Let f be a convex function on a k-simplex $S=[\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_{k+1}]$ in $\mathbb{R}^k$ and $\lambda_1,\ldots,\lambda_{k+1}$ be barycentric coordinates over S. If J is an interval in $\mathbb{R}$ such that $f(S)\subset J$ and $F\colon J\times J\to\mathbb{R}$ an increasing function in the first variable, then for all $\boldsymbol{g}\in L^k$ such that $\boldsymbol{g}(E)\subset S$ and $f(\boldsymbol{g})\in L$ we have*

$$\begin{aligned}&F\left(A(f(\boldsymbol{g})),f(\widetilde{A}(\boldsymbol{g}))\right) \qquad (21)\\ &\leqslant \max_{\boldsymbol{x}\in S}F\Big(\frac{\mathrm{Vol}_k\left([\boldsymbol{x},\boldsymbol{v}_2,\ldots,\boldsymbol{v}_{k+1}]\right)}{\mathrm{Vol}_k(S)}f(\boldsymbol{v}_1)+\ldots+\frac{\mathrm{Vol}_k\left([\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k,\boldsymbol{x}]\right)}{\mathrm{Vol}_k(S)}f(\boldsymbol{v}_{k+1})\\ &\quad -A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right)S_f^{k+1}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k+1}),f\left(\boldsymbol{x}\right)\Big)\\ &=\max_{\Delta_k}F\left(\sum_{i=1}^{k+1}\mu_i f(\boldsymbol{v}_i)-A\left(\min\left\{\lambda_i(\boldsymbol{g})\right\}\right)S_f^{k+1}(\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k+1}),f\left(\sum_{i=1}^{k+1}\mu_i\boldsymbol{v}_i\right)\right).\end{aligned}$$

Proof. Analogous to the proof of Theorem 7 with substitutions

$$\mu_1=\frac{\mathrm{Vol}_k\left([\boldsymbol{x},\boldsymbol{v}_2,\ldots,\boldsymbol{v}_{k+1}]\right)}{\mathrm{Vol}_k(S)},\ldots,\mu_{k+1}=\frac{\mathrm{Vol}_k\left([\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k,\boldsymbol{x}]\right)}{\mathrm{Vol}_k(S)},$$

and

$$\boldsymbol{x} = \sum_{i=1}^{k+1} \mu_i \boldsymbol{v}_i. \quad \square$$

REMARK 7. If all the assumptions of Theorem 9 are satisfied and in addition f is continuous, then

$$\begin{aligned} f(\widetilde{A}(\boldsymbol{g})) &\leqslant A(f(\boldsymbol{g})) \\ &\leqslant \sum_{i=1}^{k+1} A(\lambda_i(\boldsymbol{g})) f(\boldsymbol{v}_i) - A(\min\{\lambda_i(\boldsymbol{g})\}) S_f^{k+1}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{k+1}) \qquad (22) \\ &= \frac{\mathrm{Vol}_k\left(\left[\widetilde{A}(\boldsymbol{g}), \boldsymbol{v}_2, \dots, \boldsymbol{v}_{k+1}\right]\right)}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_1) + \dots + \frac{\mathrm{Vol}_k\left(\left[\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \widetilde{A}(\boldsymbol{g})\right]\right)}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_{k+1}) \\ &\quad - A(\min\{\lambda_i(\boldsymbol{g})\}) S_f^{k+1}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{k+1}). \end{aligned}$$

The first inequality is from Theorem 4 and the second from Theorem 9.

EXAMPLE 1. Let $S = [\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_{k+1}]$ be a k-simplex in $\mathbb{R}^k$ and f a continuous convex function on S. Let $(E, \mathscr{A}, \lambda)$ be a measure space with positive measure λ such that $\lambda(E) < \infty$. Let L be a linear class of measurable real functions on E. We define the functional $A \colon L \to \mathbb{R}$ by

$$A(g) = \frac{1}{\lambda(E)} \int_E g(t) d\lambda(t).$$

It is obvious that A is positive normalized linear functional on L. Then the linear operator $\widetilde{A}$ is defined by

$$\widetilde{A}(\boldsymbol{g}) = \frac{1}{\lambda(E)} \int_E \boldsymbol{g}(t) d\lambda(t).$$

We denote $\overline{\boldsymbol{g}} = \frac{1}{\lambda(E)} \int_E \boldsymbol{g}(t) d\lambda(t)$. If $\boldsymbol{g}(E) \subset S$ and $f(\boldsymbol{g}) \in L$, then from (22) it follows

$$\begin{aligned} f(\overline{\boldsymbol{g}}) &\leqslant A(f(\boldsymbol{g})) \qquad (23) \\ &\leqslant \frac{\mathrm{Vol}_k([\overline{\boldsymbol{g}}, \boldsymbol{v}_2, \dots, \boldsymbol{v}_{k+1}])}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_1) + \dots + \frac{\mathrm{Vol}_k([\boldsymbol{v}_1, \dots, \boldsymbol{v}_k, \overline{\boldsymbol{g}}])}{\mathrm{Vol}_k(S)} f(\boldsymbol{v}_{k+1}) \\ &\quad - \left(\frac{1}{\lambda(E)} \int_E \min\{\lambda_i(\boldsymbol{g}(t))\} d\lambda(t)\right) S_f^{k+1}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{k+1}). \end{aligned}$$

REMARK 8. Let $S = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_{k+1}]$ be a $k-$simplex in $\mathbb{R}^k$. If we put $E = S$, $\boldsymbol{g} = \boldsymbol{id_S}$ and λ is Lebesgue measure on S, from Example 1 we get

$$\begin{aligned} \overline{\boldsymbol{id_S}} &= \frac{1}{|S|} \int_S t\, dt = \boldsymbol{v}^* = \frac{1}{k+1} \sum_{i=1}^{k+1} \boldsymbol{v}_i \\ A(f(\boldsymbol{id_S})) &= \frac{1}{|S|} \int_S f(t) dt, \end{aligned}$$

where $\boldsymbol{v}^*$ is the barycenter of S. Now we have

$$\begin{aligned}
f(\boldsymbol{v}^*) &\leqslant \frac{1}{|S|}\int_S f(t)dt \\
&\leqslant \frac{\mathrm{Vol}_k([\boldsymbol{v}^*,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_{k+1}])}{|S|}f(\boldsymbol{v}_1)+\ldots+\frac{\mathrm{Vol}_k([\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k,\boldsymbol{v}^*])}{|S|}f(\boldsymbol{v}_{k+1}) \\
&\quad -\left(\frac{1}{|S|}\int_S \min\{\lambda_i(t)\}\,dt\right)\left[\sum_{i=1}^{k+1} f(\boldsymbol{v}_i)-(k+1)f(\boldsymbol{v}^*)\right] \qquad (24) \\
&= \frac{1}{k+1}\left(\sum_{i=1}^{k+1} f(\boldsymbol{v}_i)\right)-\left(\frac{1}{|S|}\int_S \min\{\lambda_i(t)\}\,dt\right)\left[\sum_{i=1}^{k+1} f(\boldsymbol{v}_i)-(k+1)f(\boldsymbol{v}^*)\right].
\end{aligned}$$

For $i=1,\ldots,k+1$, let S_i be the simplex whose vertices are $\boldsymbol{v}^*$ and all vertices of S except $\boldsymbol{v}_i$. Denote by $\boldsymbol{v}_i^*$ the barycentre of $S_i, i=1,\ldots,k+1$. Since $\mathrm{Vol}_k(S_i)=\mathrm{Vol}_k(S_j), i,j=1,\ldots,k+1$, it follows from (17) that $t\in S_j$ implies $\min\{\lambda_i(t)\}=\lambda_j(t)$. It follows

$$\int_S \min\{\lambda_i(t)\}dt=\sum_{j=1}^{k+1}\int_{S_j}\lambda_j(t)dt. \qquad (25)$$

We have

$$\begin{aligned}
\int_{S_j}\lambda_j(t)dt &= \frac{1}{|S|}\int_{S_j}\mathrm{Vol}_k[\boldsymbol{v}_1,\ldots,t,\ldots,\boldsymbol{v}_{k+1}]\,dt \\
&= \frac{1}{|S|}\mathrm{Vol}_k\left[\boldsymbol{v}_1,\ldots,\int_{S_j} t\,dt,\ldots,\boldsymbol{v}_{k+1}\right] \\
&= \frac{|S_j|}{|S|}\mathrm{Vol}_k\left[\boldsymbol{v}_1,\ldots,\boldsymbol{v}_j^*,\ldots,\boldsymbol{v}_{k+1}\right]=\frac{1}{k+1}\mathrm{Vol}_k\left[\boldsymbol{v}_1,\ldots,\boldsymbol{v}_j^*,\ldots,\boldsymbol{v}_{k+1}\right] \\
&= \frac{1}{(k+1)^2}\mathrm{Vol}_k[\boldsymbol{v}_1,\ldots,\boldsymbol{v}^*,\ldots,\boldsymbol{v}_{k+1}]=\frac{1}{(k+1)^3}|S|. \qquad (26)
\end{aligned}$$

Using (25) and (26) we get

$$\int_S \min\{\lambda_i(t)\}dt=\frac{1}{(k+1)^2}|S|. \qquad (27)$$

Now, putting (27) in (24), we have

$$\begin{aligned}
f(\boldsymbol{v}^*) &\leqslant \frac{1}{|S|}\int_S f(t)dt \\
&\leqslant \frac{k}{(k+1)^2}\sum_{i=1}^{k+1} f(\boldsymbol{v}_i)+\frac{1}{k+1}f(\boldsymbol{v}^*),
\end{aligned}$$

which is obtained in [2, Theorem 4.1].

It can be easily verified that the right-hand side of this inequality is equivalent to the k-dimensional version of the Hammer-Bullen inequality, namely

$$\frac{1}{|S|}\int_S f(t)dt-f(\boldsymbol{v}^*)\leqslant \frac{k}{k+1}\sum_{i=1}^{k+1} f(\boldsymbol{v}_i)-\frac{k}{|S|}\int_S f(t)dt$$

which is proved, for example in [10].

In one dimension this is an improvement of classical Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)dt \leqslant \frac{f(a)+f(b)}{2} - \frac{1}{4}S_f^2(a,b).$$

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