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A NONLINEAR MOVING-BOUNDARY PROBLEM OF PARABOLIC-HYPERBOLIC-HYPERBOLIC TYPE ARISING IN FLUID-MULTI-LAYERED STRUCTURE INTERACTION PROBLEMS

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Abstract. Motivated by modeling blood flow in human arteries, we study a fluid-structure interaction problem in which the structure is composed of multiple layers, each with possibly different mechanical characteristics and thickness. In the problem presented in this manuscript the structure is composed of two layers: a thin layer modeled by the 1D wave equation, and a thick layer modeled by the 2D equations of linear elasticity. The flow of an incompressible, viscous fluid is modeled by the Navier-Stokes equations. The thin structure is in contact with the fluid thereby serving as a fluid-structure interface with mass. The coupling between the fluid and the structure is nonlinear. The resulting problem is a nonlinear, moving-boundary problem of parabolic-hyperbolic-hyperbolic type. We show that the model problem has a well-defined energy, and that the energy is bounded by the work done by the inlet and outlet dynamic pressure data. The spaces of weak solutions reveal that the presence of a thin fluid-structure interface with mass regularizes solutions of the coupled problem. This opens up a new area withing the field of fluid-structure interaction problems, possibly revealing properties of FSI solutions that have not been studied before.

1. Motivation.

Fluid-structure interaction (FSI) problems arise in many applications. They include multi-physics problems in engineering such as aeroelasticity and propeller turbines, as well as biofluidic application such as self-propulsion organisms, fluid-cell interactions, and the interaction between blood flow and cardiovascular tissue. In biofluidic applications, such as the interaction between blood flow and cardiovascular tissue, the density of the structure (arterial walls) is roughly equal to the density of the fluid (blood). In such problems the energy exchange between the fluid and the structure is significant, leading to a highly nonlinear FSI coupling which is responsible for the instabilities in loosely coupled partitioned algorithms [3]. Despite a significant progress within the past decade [1, 2, 7, 8, 10, 13, 14, 6, 5, 4, 11, 15],

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a comprehensive study of these problems remains to be a challenge due to their strong nonlinearity and multi-physics nature. In the blood flow application, the problems are further exacerbated by the fact that arterial walls of major arteries are composed of several layers, each with different mechanical characteristics. The main layers are the tunica intima, the tunica media, and the tunica adventitia. They are separated by the thin elastic laminae, see Figure 1, left. To this date,

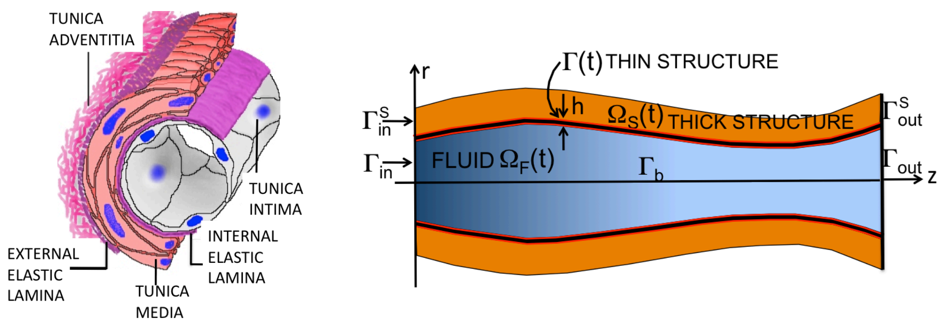

FIGURE 1. Left: Arterial wall structure. Right: Domain sketch.

there are no fluid-structure interaction models or computational solvers in hemodynamics that take into account the multi-layered structure of arterial walls. In this manuscript we take a first step in this direction by proposing a benchmark problem in fluid-multi-layered-structure interaction. The proposed problem is a nonlinear moving-boundary problem of parabolic-hyperbolic-hyperbolic type for which the questions of well-posedness and numerical simulation are wide open. This opens up a new area within the field of FSI problems, in which the structure is composed of multiple layers, each with possibly different mechanical characteristics and thickness.

2. **The benchmark problem.** We study a FSI problem in which the structure consists of two layers: a "thin" structural layer (modeled, e.g., by the linearly elastic Koiter shell equations), and a "thick" layer (modeled, e.g., by the equations of 2D/3D elasticity). To simplify matters, we will be assuming that the elastodynamics of the thin structure is modeled by the 1D linear wave equation. The wave equation model retains the main difficulties associated with the study of solutions to the more general elastodynamics models mentioned above. The thin structural layer is in contact with the flow of an incompressible, viscous fluid, modeled by the Navier-Stokes equations. From an application point of view, it is of interest to study this fluid-multi-structure interaction problem on a cylindrical domain, with the flow driven by the time-dependent dynamic pressure data, see Figure 1, right. The Navier-Stokes equations are defined in a time-dependent fluid domain $\Omega_F(t)$, which is not known *a priori*:

$$\textbf{FLUID}: \qquad \left.\begin{array}{rcl} \rho_F(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) &=& \nabla\cdot\boldsymbol{\sigma}, \\ \nabla\cdot\mathbf{u} &=& 0, \end{array}\right\} \text{ in } \Omega_F(t),\ t\in(0,T), \tag{1}$$

where ρ_F denotes the fluid density; $\mathbf{u}$ the fluid velocity; $\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu_F\mathbf{D}(\mathbf{u})$ is the fluid Cauchy stress tensor; p is the fluid pressure; μ_F is the dynamic viscosity coefficient; and $\mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u})$ is the symmetrized gradient of $\mathbf{u}$.

We assume that the reference fluid domain is a cylinder of radius R and length L, denoted by Ω_F, with the lateral boundary denoted by Γ. To fix ideas, consider the fluid domain to be a subset of $\mathbb{R}^2$ with z and r denoting the axial (horizontal) and radial (vertical) coordinates. The cylinder wall is assumed to be compliant and consisting of two layers: a thin layer, whose location at time t is denoted by $\Gamma(t)$, and a thick structural layer, whose location at time t is denoted by $\Omega_S(t)$, as shown in Figure 1, right. The **thin layer** $\Gamma(t)$ is modeled by the 1D linear wave equation

$$\textbf{THIN STRUCTURE}: \qquad m\partial_{tt}\eta = T\partial_{zz}\eta + f, \quad \text{on } \Gamma \times (0,T). \tag{2}$$

Here $\eta := \eta(t,z)$ denotes the radial (transverse) displacement from the reference position $\Gamma = \{(z,R)|z \in (0,L)\}$, f is the source term (the radial component), m is mass per unit length, and T is tension. The elastodynamics of the **thick structural layer** is governed by the 2D equations of linear elasticity:

$$\textbf{THICK STRUCTURE}: \qquad \rho\partial_{tt}\mathbf{d} = \nabla \cdot \mathbf{S} \quad \text{in } \Omega_S,\ t \in (0,T). \tag{3}$$

Here $\mathbf{d} := (d_r(t,z,r), d_z(t,z,r))$ describes the displacement of a thick elastic structure with respect to a fixed, reference configuration Ω_S, and $\mathbf{S}$ is the first Piola-Kirchoff stress tensor $\mathbf{S} = 2\mu\mathbf{D}(\mathbf{d}) + \lambda(\nabla \cdot \mathbf{d})\mathbf{I}$, with the Lamé constants λ and μ, where $\mathbf{D}(\mathbf{d})$ is the symmetrized gradient of $\mathbf{d}$, and ρ is the mass density.

To capture a full **two-way coupling** between the fluid and the structure, and between the two structural layers, two sets of boundary conditions need to be prescribed: the kinematic and dynamic coupling conditions. The kinematic condition provides information about the kinematic quantities, such as velocity. We adopt the no-slip condition requiring continuity of velocities at both the fluid-structure interface and at the structure-structure interface. The dynamic coupling condition, on the other hand, describes the second Newton's Law of motion. This condition states that the rate of change of (radial) momentum $\partial_{tt}\eta$ of the interface with mass is a result of the balancing of all the forces exerted onto $\Gamma(t)$, which includes the radial component of the trace of normal stress $\boldsymbol{\sigma}\mathbf{n}$ exerted by the fluid onto $\Gamma(t)$, the trace of the radial component of the normal Piola-Kirchoff stress $\mathbf{S}\mathbf{e}_r$ exerted by the thick structure onto $\Gamma(t)$, and the action of the elastic forces associated with $\Gamma(t)$. Therefore, the coupling conditions are given by:

$$\textbf{COUPLING}: \quad \left.\begin{array}{rcl} \mathbf{u}|_{\Gamma(t)} & = & (\partial_t\eta, 0)^T \\ \mathbf{d}|_\Gamma & = & (\eta, 0)^T \ (or\ \partial_t\mathbf{d}|_\Gamma = (\partial_t\eta, 0)^T), \\ m\partial_{tt}\eta - T\partial_{zz}\eta & = & -J\boldsymbol{\sigma}\mathbf{n}|_{\Gamma(t)} \cdot \mathbf{e}_r + \mathbf{S}|_\Gamma\mathbf{e}_r \cdot \mathbf{e}_r, \end{array}\right\} \begin{array}{l} \text{on } \Gamma \\ \times(0,T), \end{array} \tag{4}$$

where $J = \sqrt{1 + (\partial_z\eta)^2}$ is the Jacobian of the transformation between the Lagrangian coordinates used in the formulation of the structure problem and the Eulerian coordinates used in the formulation of the fluid problem. Vector $\mathbf{e}_r$ is the unit normal to the reference cylinder Γ, while u_r and d_r denote the vertical components of the velocity and displacement of the thick structure, respectively. Notation $\mathbf{u}|_{\Gamma(t)} = (\partial_t\eta, 0)^T$ means $\mathbf{u}(t,z,R+\eta(t,z)) = (\partial_t\eta(t,z), 0)^T$ on $\Gamma \times (0,T)$.

We supplement this problem by the initial and boundary conditions. For example, let the inlet and outlet boundary data for the fluid be given in terms of the dynamic pressure ($p + |\mathbf{u}|^2/2 = P_{in/out}(t)$ on $\Gamma_{in/out}$) and assume that the fluid is entering and leaving the domain parallel to the axis of symmetry ($u_r = 0$ on $\Gamma_{in/out}$). Furthermore, assume that the displacement of both structures is equal to zero at the in/out boundaries ($\eta = d_r = d_z = 0$ on $\Gamma_{in/out}$), and that $\mathbf{S}\mathbf{e}_r = 0$ at the external wall of the thick structure. We can also introduce the symmetry boundary

$\Gamma_b = \{(z,0)|z \in (0,L)\}$ with the symmetry boundary conditions $u_r = \partial_r u_z = 0$, and consider the problem only in the upper half-domain.

The resulting fluid-multi-structure-interaction problem can be summarized as follows (for simplicity we take all the parameters in the problem equal to 1, i.e., $m = T = \rho = \lambda = \mu = \rho_F = \mu_F = 1$): find $\mathbf{u}$, η and $\mathbf{d}$ such that

$$\left.\begin{array}{rcl}(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) & = & \nabla\cdot\boldsymbol{\sigma} \\ \nabla\cdot\mathbf{u} & = & 0\end{array}\right\} \text{ in } \Omega_F(t),\ t\in(0,T), \tag{5}$$

$$\partial_{tt}\mathbf{d} = \nabla\cdot\mathbf{S} \qquad \text{on } \Omega_S\times(0,T), \tag{6}$$

$$\left.\begin{array}{rcl}\mathbf{u}|_{\Gamma(t)} & = & (\partial_t\eta, 0)^T, \\ \mathbf{d}|_{\Gamma} & = & (\eta,0)^T, \\ \partial_{tt}\eta - \partial_{zz}\eta & = & -J\boldsymbol{\sigma}\mathbf{n}|_{\Gamma(t)}\cdot\mathbf{e}_r + \mathbf{S}|_\Gamma\mathbf{e}_r\cdot\mathbf{e}_r\end{array}\right\} \text{ on } \Gamma\times(0,T), \tag{7}$$

$$\left.\begin{array}{rcl}p + |\mathbf{u}|^2/2 & = & P_{in/out}(t) \\ u_r & = & 0\end{array}\right\} \text{ on } \Gamma^f_{in/out}\times(0,T), \tag{8}$$

$$\begin{array}{rcl}\eta & = & 0 \text{ on } \partial\Gamma \\ \mathbf{d} & = & \mathbf{0} \text{ on } \Gamma^S_{in/out}\end{array} \tag{9}$$

$$\mathbf{S}\mathbf{e}_r = 0 \text{ on } \Gamma_{ext}, \tag{10}$$

$$\left.\begin{array}{rcl}u_r & = & 0 \\ \partial_r u_z & = & 0\end{array}\right\} \text{ on } \Gamma_b\times(0,T), \tag{11}$$

with $\mathbf{u}(0,\cdot) = \mathbf{u}_0, \eta(0,\cdot) = \eta_0, \partial_t\eta(0,\cdot) = v_0, \mathbf{d}(0,\cdot) = \mathbf{d}_0, \partial\mathbf{d}(0,\cdot) = \mathbf{V_0}$.

Problem (5)-(7) defines a nonlinear, moving boundary problem of mixed, parabolic-hyperbolic-hyperbolic type. The nonlinearity appears both in the equations, as well as in the coupling conditions (7) via the composite function $\mathbf{u}|_{\Gamma(t)} := \mathbf{u}(t,z,R+\eta(t,z))$. The hyperbolic problem in (7) (3rd equation) serves as a lateral boundary condition for both the fluid problem (5) and for the thick-structure problem (6).

Lemma 2.1. *Problem (5)-(10) satisfies the following energy inequality*

$$\frac{d}{dt}\left(E_{kin}(t) + E_{el}(t)\right) + D(t) \le C(P_{in}(t), P_{out}(t)), \tag{12}$$

where

$$\begin{array}{rcl}E_{kin}(t) & := & \|\mathbf{u}\|^2_{L^2(\Omega(t))} + \|\partial_t\eta\|^2_{L^2(\Gamma)} + \|\partial_t\mathbf{d}\|^2_{L^2(\Omega_S)}, \\ E_{el}(t) & := & \|\partial_z\eta\|^2_{L^2(\Gamma)} + 2\|\mathbf{D}(\mathbf{d})\|^2_{L^2(\Omega_S)} + \|\nabla\cdot\mathbf{d}\|^2_{L^2(\Omega_S)},\end{array}$$

denote the kinetic and elastic energy of the coupled problem, respectively, and the term $D(t)$ captures dissipation $D(t) := \|\mathbf{D}(\mathbf{u})\|^2_{L^2(\Omega(t))}$. The bound $C(P_{in}(t), P_{out}(t)))$ depends only on the inlet and outlet pressure data.

Proof. To show that (12) holds,multiply the first equation in (5) by $\mathbf{u}$, integrate over $\Omega_F(t)$, and formally integrate by parts to obtain:

$$\int_{\Omega_F(t)}(\partial_t\mathbf{u}\cdot\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u}) + 2\int_{\Omega_F(t)}|\mathbf{D}\mathbf{u}|^2 - \int_{\partial\Omega_F(t)}(-p\mathbf{I} + 2\mathbf{D}(\mathbf{u}))\mathbf{n}(t)\cdot\mathbf{u} = 0. \tag{13}$$

To deal with the inertia term we first recall that $\Omega_F(t)$ is moving in time and that the velocity of the lateral boundary is given by $\mathbf{u}|_{\Gamma(t)}$. The transport theorem applied to the first term on the left hand-side of the above equation then gives:

$$\int_{\Omega_F(t)}\partial_t\mathbf{u}\cdot\mathbf{u} = \frac{1}{2}\frac{d}{dt}\int_{\Omega_F(t)}|\mathbf{u}|^2 - \frac{1}{2}\int_{\Gamma(t)}|\mathbf{u}|^2\mathbf{u}\cdot\mathbf{n}(t).$$

To deal with the nonlinear advection term in (13) we integrate by parts, and use the divergence-free condition to obtain:

$$\int_{\Omega_F(t)}(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u} = \frac{1}{2}\int_{\partial\Omega_F(t)}|\mathbf{u}|^2\mathbf{u}\cdot\mathbf{n}(t) = \frac{1}{2}\Big(\int_{\Gamma(t)}|\mathbf{u}|^2\mathbf{u}\cdot\mathbf{n}(t) - \int_{\Gamma_{in}}|\mathbf{u}|^2u_z + \int_{\Gamma_{out}}|\mathbf{u}|^2u_z\Big).$$

These two terms added together give

$$\int_{\Omega_\eta(t)}\partial_t\mathbf{u}\cdot\mathbf{u} + \int_{\Omega_\eta(t)}(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{u} = \frac{1}{2}\frac{d}{dt}\int_{\Omega_\eta(t)}|\mathbf{u}|^2 - \frac{1}{2}\int_{\Gamma_{in}}|\mathbf{u}|^2u_z + \frac{1}{2}\int_{\Gamma_{out}}|\mathbf{u}|^2u_z. \tag{14}$$

Notice the importance of nonlinear advection in canceling the cubic term $\int_{\Gamma(t)}|\mathbf{u}|^2\mathbf{u}\cdot\mathbf{n}(t)$!

To deal with the boundary integral over $\partial\Omega_F(t)$ of the normal stress in (13) we first employ the boundary condition $u_r = 0$ form (8) in combination with the divergence-free condition to obtain $\partial_z u_z = -\partial_r u_r = 0$. Now, using the fact that the normal to $\Gamma_{in/out}$ is $\mathbf{n} = (\mp 1, 0)$, we get:

$$\int_{\Gamma_{in/out}}(-p\mathbf{I} + 2\mathbf{D}(\mathbf{u}))\mathbf{n}\cdot\mathbf{u} = \int_{\Gamma_{in}}P_{in}u_z - \int_{\Gamma_{out}}P_{out}u_z. \tag{15}$$

In a similar way, using the symmetry boundary condition (11), we obtain

$$\int_{\Gamma_b}(-p\mathbf{I} + 2\mathbf{D}(\mathbf{u}))\mathbf{n}\cdot\mathbf{u} = 0.$$

What is left is to integrate the normal stress over $\Gamma(t)$. For this purpose we consider the wave equation (2), multiply it by $\partial_t\eta$, and integrate by parts to obtain

$$\int_\Gamma f\partial_t\eta = \frac{1}{2}\frac{d}{dt}\|\partial_t\eta\|^2_{L^2(\Gamma)} + \frac{1}{2}\frac{d}{dt}\|\partial_z\eta\|^2_{L^2(\Gamma)} \tag{16}$$

Furthermore, we consider the elasticity equation (6), multiply it by $\partial_t\mathbf{d}$ and integrate by parts over Ω_S to obtain:

$$\frac{1}{2}\frac{d}{dt}\ \Big(\|\partial_t\mathbf{d}\|^2_{L^2(\Omega_S)} + 2\|\mathbf{D}(\mathbf{d})\|^2_{L^2(\Omega_S)} + \|\nabla\cdot\mathbf{d}\|^2_{L^2(\Omega_S)}\Big) = -\int_\Gamma \mathbf{S}\mathbf{e}_r\cdot\partial_t\mathbf{d}. \tag{17}$$

By enforcing the dynamic and kinematic coupling conditions (7) we obtain

$$-\int_{\Gamma(t)}\sigma\mathbf{n}(t)\cdot\mathbf{u} = -\int_\Gamma J\sigma\mathbf{n}\cdot\mathbf{u} = \int_\Gamma(f - \mathbf{S}\mathbf{e}_r)\partial_t\eta. \tag{18}$$

Finally, by combining (18) with (16), (17), and by adding the remaining contributions to the energy of the FSI problem one obtains the following **energy equality**:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_F(t)}|\mathbf{u}|^2 + \frac{1}{2}\frac{d}{dt}\|\partial_t\eta\|^2_{L^2(0,1)} + 2\int_{\Omega_F(t)}|\mathbf{D}\mathbf{u}|^2 + \frac{1}{2}\frac{d}{dt}\|\partial_z\eta\|^2_{L^2(0,1)}$$
$$+\frac{1}{2}\frac{d}{dt}\Big(\|\partial_t\mathbf{d}\|^2_{L^2(\Omega_S)} + 2\|\mathbf{D}(\mathbf{d})\|^2_{L^2(\Omega_S)} + \|\nabla\cdot\mathbf{d}\|^2_{L^2(\Omega_S)}\Big) = \pm P_{in/out}(t)\int_{\Gamma_{in/out}}u_z$$

By using the trace inequality and Korn inequality one can estimate:

$$|P_{in/out}(t)\int_{\Gamma_{in/out}}u_z| \le C|P_{in/out}|\|\mathbf{u}\|_{H^1(\Omega_F(t))} \le \frac{C}{2\varepsilon}|P_{in/out}|^2 + \frac{\varepsilon C}{2}\|\mathbf{D}(\mathbf{u})\|^2_{L^2(\Omega_F(t))}.$$

By choosing ε such that $\frac{\varepsilon C}{2} \leq 1$ we get the energy inequality (12). □

3. **Weak Solutions.** To define weak solutions of the moving-bounday problem (5)-(11) we introduce the following notation. We use a_S to denote the following bilinear form associated with the elastic properties of the thick structure:

$$a_S(\mathbf{d},\boldsymbol{\psi}) := \int_{\Omega_S} 2\mathbf{D}(\mathbf{d}) : \mathbf{D}(\boldsymbol{\psi}) + (\nabla\cdot\mathbf{d})\ (\nabla\cdot\boldsymbol{\psi}). \tag{19}$$

Here $A : B := \text{tr}\left[AB^T\right]$. Furthermore, we use b to denote the following trilinear form corresponding to the (symmetrized) nonlinear advection term in the Navier-Stokes equations:

$$b(t,\mathbf{u},\mathbf{v},\mathbf{w}) := \frac{1}{2}\int_{\Omega_F(t)} (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} - \frac{1}{2}\int_{\Omega_F(t)} (\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}. \tag{20}$$

Finally, we define a linear functional which associates the inlet and outlet dynamic pressure boundary data to a test function $\mathbf{v}$ in the following way:

$$\langle F(t),\mathbf{v}\rangle_{\Gamma_{in/out}} = P_{in}(t)\int_{\Gamma_{in}} v_z - P_{out}(t)\int_{\Gamma_{out}} v_z.$$

To define a weak solution to problem (5)-(11) we introduce the following function spaces. For the fluid velocity we would like to work with the classical function space. However, due to the moving fluid-structure interface which is modeled by the wave equation, the lateral boundary of the fluid domain is not necessarily a Lipshitz function. Namely, from the energy inequality (12) we see that $\eta \in H^1(0,1)$. The Sobolev embedding then implies that $\eta \in C^{0,1/2}(0,1)$, which means that $\Omega_F(t)$ is not necessarily a Lipshitz domain. However, $\Omega_F(t)$ is locally a sub-graph of a Hölder continuous function. In that case one can define a"Lagrangian" trace

$$\begin{aligned} &\gamma_{\Gamma(t)} : C^1(\overline{\Omega_F(t)}) \to C(\Gamma),\\ &\gamma_{\Gamma(t)} : v \mapsto v(t,z,r+\eta(t,z)). \end{aligned} \tag{21}$$

Furthermore, it was shown in [4, 11, 16] that the trace operator $\gamma_{\Gamma(t)}$ can be extended by continuity to a linear operator from $H^1(\Omega_F(t))$ to $H^s(\Gamma)$, $0 \leq s < \frac{1}{4}$. Therefore, we define the fluid velocity solution space to be the closure in $H^1(\Omega_F(t))$ of the set $\{\mathbf{u} = (u_z,u_r) \in C^1(\overline{\Omega_F(t)})^2 : \nabla\cdot\mathbf{u} = 0, u_z = 0 \text{ on } \Gamma(t),\ u_r = 0 \text{ on } \Omega_F(t)\setminus\Gamma(t)\}$. Using the fact that $\Omega_F(t)$ is locally a sub-graph of a Hölder continuous function we can get the following characterization of the velocity solution space $\mathcal{V}_F(t)$: (see [4, 11])

$$\begin{aligned} \mathcal{V}_F(t) \quad = \quad &\{\mathbf{u} = (u_z,u_r) \in H^1(\Omega_\eta(t))^2 : \nabla\cdot\mathbf{u} = 0,\\ &u_z = 0 \text{ on } \Gamma(t),\ u_r = 0 \text{ on } \Omega_\eta(t)\setminus\Gamma(t)\}. \end{aligned} \tag{22}$$

The function space associated displacement of the thin structural layer is

$$\mathcal{V}_K = H^1_0(\Gamma), \tag{23}$$

and the function space associated with displacement of the thick structural layer is

$$\mathcal{V}_S = \{\boldsymbol{d} = (d_z,d_r) \in H^1(\Omega_S)^2 : d_z = 0 \text{ on } \Gamma,\ \boldsymbol{d} = 0 \text{ on } \Gamma^s_{in/out}\cup\Gamma_{ext}\}. \tag{24}$$

Motivated by the energy inequality (12) we also define the corresponding evolution spaces for the fluid and structure sub-problems, respectively:

$$\mathcal{W}_F(0,T) = L^\infty(0,T;L^2(\Omega_F(t)) \cap L^2(0,T;\mathcal{V}_F(t)), \tag{25}$$

$$\mathcal{W}_K(0,T) = W^{1,\infty}(0,T;L^2(\Gamma)) \cap L^2(0,T;\mathcal{V}_K), \tag{26}$$

$$\mathcal{W}_S(0,T) = W^{1,\infty}(0,T;L^2(\Omega_S)) \cap L^2(0,T;\mathcal{V}_S). \tag{27}$$

Finally, we are in a position to define the solution space for the coupled fluid-multi-layered-structure interaction problem. This space must involve the kinematic coupling condition. The dynamic coupling condition will be enforced in a weak sense, through integration by parts in the weak formulation of the problem. Thus, we define

$$\begin{array}{c}\mathcal{W}(0,T)=\{(\mathbf{u},\eta,\boldsymbol{d})\in\mathcal{W}_F(0,T)\times\mathcal{W}_K(0,T)\times\mathcal{W}_S(0,T):\\ \mathbf{u}(t,z,R+\eta(t,z))=\partial_t\eta(t,z)\mathbf{e}_r,\ \boldsymbol{d}(t,z,R)=\eta(t,z)\mathbf{e}_r\}.\end{array}\tag{28}$$

The equality $\mathbf{u}(t,z,R+\eta(t,z))=\partial_t\eta(t,z)\mathbf{e}_r$ is taken in a sense of operator $\gamma_{\Gamma(t)}$, defined in (21). The corresponding test space will be denoted by

$$\begin{array}{c}\mathcal{Q}(0,T)=\{(\mathbf{q},\psi,\boldsymbol{\psi})\in C_c^1([0,T);\mathcal{V}_F\times\mathcal{V}_K\times\mathcal{V}_S):\\ \mathbf{q}(t,z,R+\eta(t,z))=\psi(t,z)\mathbf{e}_r=\boldsymbol{\psi}(t,z,R)\}.\end{array}\tag{29}$$

Definition 3.1. (Weak Solution) We say that $(\mathbf{u},\eta,\boldsymbol{d})\in\mathcal{W}(0,T)$ is a weak solution of problem (5)-(11) if for every $(\mathbf{q},\psi,\boldsymbol{\psi})\in\mathcal{Q}(0,T)$ the following holds:

$$\begin{array}{c}-\int_0^T\int_{\Omega_F(t)}\mathbf{u}\cdot\partial_t\mathbf{q}+\int_0^T b(t,\mathbf{u},\mathbf{u},\mathbf{q})+2\int_0^T\int_{\Omega_F(t)}\mathbf{D}(\mathbf{u}):\mathbf{D}(\mathbf{q})\\ -\frac{1}{2}\int_0^T\int_\Gamma(\partial_t\eta)^2\psi-\int_0^T\int_\Gamma\partial_t\eta\partial_t\psi+\int_0^T\int_\Gamma\partial_z\eta\partial_z\psi\\ -\int_0^T\int_{\Omega_S}\partial_t\boldsymbol{d}\cdot\partial_t\boldsymbol{\psi}+\int_0^T a_s(\boldsymbol{d},\boldsymbol{\psi})=\int_0^T\langle F(t),\mathbf{q}\rangle_{\Gamma_{in/out}}\\ +\int_{\Omega_F(0)}\mathbf{u}_0\cdot\mathbf{q}(0)+\int_\Gamma v_0\psi(0)+\int_{\Omega_S}\mathbf{V}_0\cdot\boldsymbol{\psi}(0).\end{array}\tag{30}$$

In deriving the weak formulation we used integration by parts in a classical way, and the following equalities which hold for smooth functions:

$$\begin{aligned}\int_{\Omega_F(t)}(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{q}= &\ \frac{1}{2}\int_{\Omega_F(t)}(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{q}-\frac{1}{2}\int_{\Omega_F(t)}(\mathbf{u}\cdot\nabla)\mathbf{q}\cdot\mathbf{u}\\ &+\frac{1}{2}\int_\Gamma(\partial_t\eta)^2\psi\pm\frac{1}{2}\int_{\Gamma_{out/in}}|u_r|^2v_r,\end{aligned}$$

$$\int_0^T\int_{\Omega_F(t)}\partial_t\mathbf{u}\cdot\mathbf{q}=-\int_0^T\int_{\Omega_F(t)}\mathbf{u}\cdot\partial_t\mathbf{q}-\int_{\Omega_F(0)}\mathbf{u}_0\cdot\mathbf{q}(0)-\int_0^T\int_\Gamma(\partial_t\eta)^2\psi.$$

4. **Conclusions.** The energy estimate (12) and the spaces of weak solutions show that the presence of a fluid-structure interface with mass regularizes the solution of this fluid-structure interaction problem. If we had a FSI problem between an incompressible, viscous fluid and a thick structure only, the trace of the displacement of the structure would not have been defined at the fluid-structure interface, and the evolution of the fluid-structure interface could not be controlled by the energy estimates. In problem (5)-(11) not only that the trace of the displacement and the axial derivative of the displacement of the fluid-structure interface are well defined, but the time-derivative of the displacement of the fluid-structure interface is controlled by the energy estimate. The kinetic energy term $\|\partial_t\eta\|^2$ in the energy

estimate (12), which is responsible for the control of the evolution of the fluid-structure interface, appears in (12) due to the inertia of the fluid-structure interface with mass. Our preliminary results indicate that this will play a crucial role in proving existence of a weak solution to this fluid-multi-structure interaction problem [17]. Namely, in a problem in which viscoelasticity of the structure is lacking, the inertia of the fluid-structure interface with mass provides a new regularizing mechanism for a weak solution to exist. This is reminiscent of the results by Hansen and Zuazua [12] in which the presence of a point mass at the interface between two linearly elastic strings with solutions in asymmetric spaces (different regularity on each side) allowed the proof of well-posedness due to the regularizing effects by the point mass. Further research in this direction for problem (5)-(11) is under way [17].

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