A Proof Of All Three Euclidean Four Point Atiyah–Sutcliffe Conjectures

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Outline

Introduction 1/3

In 2001. Sir Michael Atiyah, inspired by physics (Berry–Robbins problem related to spin statistics theorem of quantum mechanics) associated a remarkable determinant to any n distinct points in Euclidean 3–space, via elementary construction.
More generally, let $(x_1, x_2, \ldots, x_n)$ be n distinct points inside the ball of radius R in Euclidean 3–space. Let the oriented line $x_i x_j$ meet the boundary 2–sphere in a point u_{ij} regarded as a point of the complex Riemann sphere $(\mathbb{C} \cup \{\infty\})$.

Form a complex polynomial p_i of degree $n-1$ whose roots are $u_{ij}, i \neq j$ (p_i is determined up to a scalar factor). The Atiyah's conjecture C_1 now reads

Conjecture C_1

For all $(x_1, x_2, \ldots, x_n)$ the n polynomials p_i are linearly independent.

Conjecture C_1 $\Leftrightarrow$ nonvanishing of the determinant D of the matrix of coefficients of the polynomials p_i.
The determinant D can be normalized so that D becomes a continuous function of $(x_1, x_2, \ldots, x_n)$ which is $SL(2, \mathbb{C})$–invariant (using the ball model or upper half space model of hyperbolic 3–space).

The more refined conjectures of Atiyah and Sutcliffe C_2 and C_3 relate D to products of 2 and $n-1$–subsequences of points $x_1, x_2, \ldots, x_n$.

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The conjecture C_1 is proved for $n = 3, 4$ and for general n only for some special configurations (M.F. Atiyah, M. Eastwood and P. Norbury, D. Ðoković).

In a lengthy preprint [5] we have verified the conjectures C_2 and C_3 for parallelograms, cyclic quadrilaterals and some infinite families of tetrahedra.

Also we proved C_2 for Ðoković's dihedral configurations. In [8] a proof of C_1 is given for convex planar quadrilaterals. We have also proposed a strengthening of the conjecture C_3 for configurations of four points (Four Points Conjectures, stronger then some new conjectures in [8]) and a number of conjectures for almost collinear configurations, and proved them for n up to 10.

In [3] Eastwood and Norbury found an intrinsic formula for the four point Atiyah determinant (a polynomial of sixth degree in six distances having several hundreds of terms) and gave a proof of C_1.

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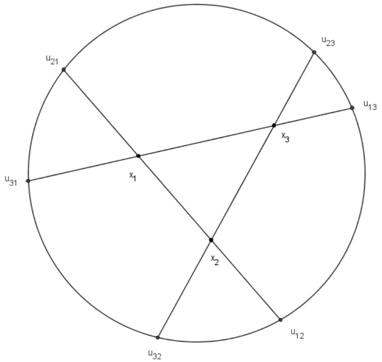
The present author found a new geometric fact for arbitrary tetrahedra which leads to a proof of C_2 and C_3 for arbitrary four points in the euclidean three space (and also a proof of stronger Four Points Conjecture of Svrtan and Urbiha). Later we obtain another intrinsic polynomial formula a la Eastwood and Norbury for four points (and for five "planar" points – having one hundred thousand terms) and have an existence proof of a polynomial formula for all planar configurations what was conjectured in [3].

This approach produces also trigonometric formulas for four points Atiyah determinants (not involving so called Crelle angles which are used in [8]). Some work is done in the hyperbolic case by finding a hyperbolic analogue of the Eastwood and Norbury formula (in the planar case- spacial case is quite a challenge!).

We also introduced Atiyah type energies associated to any graph and can prove that Conjecture C_1 is true , for arbitrary n, for some of these energies (work in progress).

3 points inside circle



- Three points x_1, x_2, x_3 inside disk $(|z| \leq R)$
- Three point-pairs on circle
- $P_1 \quad (u_{12})(u_{13})$
- $P_2 \quad (u_{21})(u_{23})$
- $P_3 \quad (u_{31})(u_{32})$
- Point–pair u_{12}, u_{13} define quadratic with roots
  $p_1 = (z - u_{12})(z - u_{13})$
- 3 point–pairs $\rightarrow 3$ quadratics
- $P_1, P_2, P_3 \rightarrow \{p_1, p_2, p_3\}$

Theorem (Atiyah 2001.)

For any triple x_1, x_2, x_3 of distinct points inside the disk the three quadratics $\{p_1, p_2, p_3\}$ are linearly independent.

Remark: Atiyah's proof, which is synthetic, does not generalize to more than three points.

Normalized determinant D_3

Theorem 1.

3-by-3 determinant of the coefficient matrix:

$$|M_3| = \begin{vmatrix} 1 & -u_{12} - u_{13} & u_{12}u_{13} \\ 1 & -u_{21} - u_{23} & u_{21}u_{23} \\ 1 & -u_{31} - u_{32} & u_{31}u_{32} \end{vmatrix} \neq 0, \quad D_3 = \frac{|M_3|}{(u_{12}-u_{21})(u_{13}-u_{31})(u_{23}-u_{32})}$$

Remark: $D_3 = 1$ only for collinear points.

Theorem 2.

$D_3 \geq 1$.

Remark: Theorem 2. $\Leftrightarrow$ Theorem 1.

Points on the "circle at ∞" are directions on a plane.

Remark: Theorem 1. and Theorem 2. are also true for $R = \infty$.

Explicit formulas for D_3

Extrinsic formula: $$D_3 = 1 + \frac{(u_{21}-u_{31})(u_{13}-u_{23})(u_{12}-u_{32})}{(u_{12}-u_{21})(u_{13}-u_{31})(u_{23}-u_{32})}$$

Intrinsic formula for hyperbolic triangles $(0 < A+B+C < \pi)$:

$$D_3 = \frac{1}{2}(\cos^2(A/2)+\cos^2(B/2)+\cos^2(C/2)) - \frac{1}{4}\Phi ,$$

where $\Phi^2 = 4\cos\left(\frac{A+B+C}{2}\right)\cos\left(\frac{-A+B+C}{2}\right)\cos\left(\frac{A-B+C}{2}\right)\cos\left(\frac{A+B-C}{2}\right)$

$$= -1+\cos^2(A)+\cos^2(B)+\cos^2(C)+2\cos(A)\cos(B)\cos(C)$$

Intrinsic formula involving side lengths

$a,\ b,\ c,\ p=(a+b+c)/2,\ p_a=p-a,\ p_b=p-b,\ p_c=p-c$:

$$D_3 = 1+e^{-p}\,\frac{\sinh(p_a)\sinh(p_b)\sinh(p_c)}{\sinh(a)\sinh(b)\sinh(c)}\ \left(\to 1+\frac{(-a+b+c)(a-b+c)(a+b-c)}{8abc}\text{ Euclidean case}\right)$$

$$= 1+e^{-(p_a+p_b+p_c)}\frac{(e^{p_a}-e^{-p_a})(e^{p_b}-e^{-p_b})(e^{p_c}-e^{-p_c})}{(e^{p_a+p_b}-e^{-(p_a+p_b)})(e^{p_a+p_c}-e^{-(p_a+p_c)})(e^{p_b+p_c}-e^{-(p_b+p_c)})}$$

$$= 1+\frac{(e^{2p_a}-1)(e^{2p_b}-1)(e^{2p_c}-1)}{(e^{2(p_a+p_b)}-1)(e^{2(p_a+p_c)}-1)(e^{2(p_b+p_c)}-1)}$$

$$D_3 = 1 + \frac{(e^{2p_a}-1)(e^{2p_b}-1)(e^{2p_c}-1)}{(e^{2(p_a+p_b)}-1)(e^{2(p_a+p_c)}-1)(e^{2(p_b+p_c)}-1)}$$

Lemma.

For $0 < a < b$ the function $f(x) = \dfrac{e^{\frac{a}{x}}-1}{e^{\frac{b}{x}}-1}$ $(0 < x < \infty)$ is strictly increasing and $\lim_{x\to\infty} f(x) = \dfrac{a}{b}$.

By using this lemma the recent monotonicity conjecture of Atiyah (in case $n = 3$) follows immediately (if a is replaced by a/R etc... in previous formulas).

$$D_3 = 1 + e^{-p}\frac{\sinh(p_a)\sinh(p_b)\sinh(p_c)}{\sinh(a)\sinh(b)\sinh(c)} = 1 + \frac{e^{-p_a-p_b-p_c}\sinh(p_a)\sinh(p_b)\sinh(p_c)}{\sinh(p_a+p_b)\sinh(p_a+p_c)\sinh(p_b+p_c)}$$

$$= 1 + \frac{(\cosh(p_a+p_b+p_c)-\sinh(p_a+p_b+p_c))\sinh(p_a)\sinh(p_b)\sinh(p_c)}{\sinh(p_a+p_b)\sinh(p_a+p_c)\sinh(p_b+p_c)}$$

$$= 1 + \frac{(1-\tanh(p_a))(1-\tanh(p_b))(1-\tanh(p_c))\tanh(p_a)\tanh(p_b)\tanh(p_c)}{(\tanh(p_a)+\tanh(p_b))(\tanh(p_a)+\tanh(p_c))(\tanh(p_b)+\tanh(p_c))}$$

4 points inside a ball

- Four points x_1, x_2, x_3, x_4 in a ball $(|z| \leq R)$
- 4 point-triples on the boundary 2–sphere
- $P_1 \quad (u_{12})(u_{13})(u_{14})$
- $P_2 \quad (u_{21})(u_{23})(u_{24})$
- $P_3 \quad (u_{31})(u_{32})(u_{34})$
- $P_4 \quad (u_{41})(u_{42})(u_{43})$
- point-triple u_{12}, u_{13}, u_{14} defines a cubic (polynomial):

  $p_1 := (z - u_{12})(z - u_{13})(z - u_{14})$

  $$p_1 = z^3 - (u_{12} + u_{13} + u_{14})z^2 + (u_{12}u_{13} + u_{12}u_{14} + u_{13}u_{14})z - \\ -u_{12}u_{13}u_{14}$$
- 4 point–triples $\to$ 4 cubics
- $P_1, P_2, P_3, P_4 \to \{p_1, p_2, p_3, p_4\}$

Normalized 4–point Atiyah's determinant D_4

Determinant of the coefficient matrix of polynomials:

$$|M_4| = \begin{vmatrix} 1 & -u_{12} - u_{13} - u_{14} & u_{12}u_{13} + u_{12}u_{14} + u_{13}u_{14} & -u_{12}u_{13}u_{14} \\ 1 & -u_{21} - u_{23} - u_{24} & u_{21}u_{23} + u_{21}u_{24} + u_{23}u_{24} & -u_{21}u_{23}u_{24} \\ 1 & -u_{31} - u_{32} - u_{34} & u_{31}u_{32} + u_{31}u_{34} + u_{32}u_{34} & -u_{31}u_{32}u_{34} \\ 1 & -u_{41} - u_{42} - u_{43} & u_{41}u_{42} + u_{41}u_{43} + u_{42}u_{43} & -u_{41}u_{42}u_{43} \end{vmatrix},$$

$$D_4 = \frac{|M_4|}{(u_{12} - u_{21})(u_{13} - u_{31})(u_{14} - u_{41})(u_{23} - u_{32})(u_{24} - u_{42})(u_{34} - u_{43})}$$

Conjectures ($n = 4$)

C_1 (Atiyah): $D_4 \neq 0 \quad (\Leftrightarrow p_1, p_2, p_3, p_4$ lin. indep.)

C_2 (Atiyah–Sutcliffe): $|D_4| \geq 1$

C_3 (Atiyah–Sutcliffe): $|D_4|^2 \geq D_3(1,2,3) \cdot D_3(1,2,4) \cdot D_3(1,3,4) \cdot D_3(2,3,4)$

New proof of the Eastwood–Norbury formula

The four points:

$P_i : x_i = (z_i, r_i),\ z_i \in \mathbb{C},\ r_i \in \mathbb{R}$

$R_{ij} := r_{ij} + r_i - r_j,\ z_{ij} := z_i - z_j$

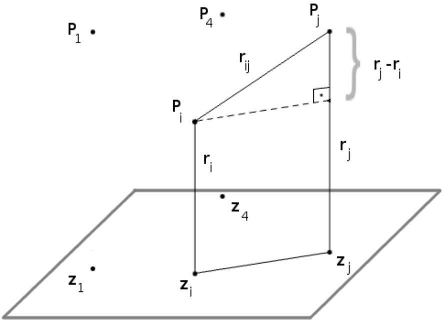
$R_{ij}R_{ji} = r_{ij}^2 - (r_i - r_j)^2 = |z_{ij}|^2 = -z_{ij}z_{ji}$

$$p_1 = \left(z + \frac{\overline{z_{12}}}{R_{12}}\right)\left(z + \frac{\overline{z_{13}}}{R_{13}}\right)\left(z + \frac{\overline{z_{14}}}{R_{14}}\right)$$

$$p_2 = \left(z + \frac{\overline{z_{21}}}{R_{21}}\right)\left(z + \frac{\overline{z_{23}}}{R_{23}}\right)\left(z + \frac{\overline{z_{24}}}{R_{24}}\right)$$

$$p_3 = \left(z + \frac{\overline{z_{31}}}{R_{31}}\right)\left(z + \frac{\overline{z_{32}}}{R_{32}}\right)\left(z + \frac{\overline{z_{34}}}{R_{34}}\right)$$

$$p_4 = \left(z + \frac{\overline{z_{41}}}{R_{41}}\right)\left(z + \frac{\overline{z_{42}}}{R_{42}}\right)\left(z + \frac{\overline{z_{43}}}{R_{43}}\right)$$



Matrix of coefficients of $\{p_1, p_2, p_3, p_4\}$

$$M_4 = \begin{vmatrix} \boxed{1} & \cdot & \cdot & \frac{\bar{z}_{12}}{R_{12}}\frac{\bar{z}_{13}}{R_{13}}\frac{\bar{z}_{14}}{R_{14}} \\ 1 & \boxed{\frac{\bar{z}_{21}}{R_{21}} + \frac{\bar{z}_{23}}{R_{23}} + \frac{\bar{z}_{24}}{R_{24}}} & \cdot & \cdot \\ 1 & \cdot & \boxed{\frac{\bar{z}_{31}}{R_{31}}\frac{\bar{z}_{32}}{R_{32}} + \frac{\bar{z}_{31}}{R_{31}}\frac{\bar{z}_{34}}{R_{34}} + \frac{\bar{z}_{32}}{R_{32}}\frac{\bar{z}_{34}}{R_{34}}} & \cdot \\ 1 & \cdot & \cdot & \boxed{\frac{\bar{z}_{41}}{R_{41}}\frac{\bar{z}_{42}}{R_{42}}\frac{\bar{z}_{43}}{R_{43}}} \end{vmatrix} \begin{matrix} \\ \cdot A \\ \cdot B \\ \cdot C \end{matrix}$$

$$A = z_{21}, \ B = z_{31}z_{32}, \ C = z_{41}z_{42}z_{43}$$

Normalized Atiyah determinant

$$D_4 = \underbrace{\det(M_4)}_{antisym.} \cdot \underbrace{z_{21} \cdot z_{31}z_{32} \cdot z_{41}z_{42}z_{43}}_{antisym.} = \sum 1 \cdot \left(\frac{z_{21}\bar{z}_{21}}{R_{21}} + \frac{z_{21}\bar{z}_{23}}{R_{23}} + \frac{z_{21}\bar{z}_{24}}{R_{24}} \right) \cdot$$

$$\cdot \left(\frac{z_{31}\bar{z}_{31}z_{32}\bar{z}_{32}}{R_{31}R_{32}} + \frac{z_{31}\bar{z}_{31}z_{32}\bar{z}_{34}}{R_{31}R_{34}} + \frac{z_{32}\bar{z}_{32}z_{32}\bar{z}_{34}}{R_{32}R_{34}} \right) R_{14}R_{24}R_{34} =$$

$$= \sum (R_{12}R_{24} + \underbrace{z_{21}z_{24}})(R_{13}R_{23}R_{34} + R_{13}\underbrace{z_{32}\bar{z}_{34}} + R_{23}\underbrace{z_{31}\bar{z}_{34}})R_{14} +$$

$$+ (R_{13}R_{24}R_{34}\underbrace{z_{21}\bar{z}_{23}} + R_{13}R_{24}R_{32}\underbrace{z_{21}\bar{z}_{34}} + R_{24}\underbrace{z_{21}\bar{z}_{23}z_{31}\bar{z}_{34}})R_{14}$$

(where summations are over all permutations of indices).

By writing $z_{ij}\bar{z}_{kl} = C[i,j,k,l] + \sqrt{-1}\, S[i,j,k,l]$ and using a Lagrange identity (involving the dot product of two cross products; a fact mentioned by N. Wildberger to the author) we have
$S[i,j,k,l]S[p,q,r,s] = C[i,j,p,q]C[k,l,r,s] - C[i,j,r,s]C[k,l,p,q]$
(we have discovered this identity independently) and using the formula

$$C[i,j,k,l] = \mathrm{Re}(z_{ij}\bar{z}_{kl}) = \frac{1}{2}[|z_{il}|^2 + |z_{jk}|^2 - |z_{ik}|^2 - |z_{jl}|^2] = \\ = \frac{1}{2}[r_{il}^2 + r_{jk}^2 - r_{ik}^2 - r_{jl}^2] - (r_i - r_j)(r_k - r_l)$$

we obtain our derivation of the Eastwood–Norbury formula.

By this new method we obtained a polynomial formula for the planar configurations of 5 points (by S_5–symmetrization of a "one page" expression) and a rational formula for the spatial 5 point configuration (this last formula has almost 100000 terms).

This settles one of the Eastwood–Norbury conjectures. We do not yet have definite geometric interpretations for the "nonplanar" part of the formula involving heights r_i, $i = 1, \ldots, 5$.

Our trigonometric (euclidean) Eastwood–Norbury formula (where $c_{i\_jk} := \cos(ij, ik)$ and $c_{ij\_kl} := \cos(ij, kl)$):

$$
\begin{aligned}
16Re(D_4) = \quad & (1 + c_{3\_12} + c_{2\_34})(1 + c_{1\_24} + c_{4\_13}) + \\
& (1 + c_{2\_13} + c_{3\_24})(1 + c_{4\_12} + c_{1\_34}) + \\
& (1 + c_{3\_12} + c_{1\_34})(1 + c_{2\_14} + c_{4\_23}) + \\
& (1 + c_{1\_23} + c_{3\_14})(1 + c_{2\_34} + c_{4\_12}) + \\
& (1 + c_{2\_13} + c_{1\_24})(1 + c_{3\_14} + c_{4\_23}) + \\
& (1 + c_{1\_23} + c_{2\_14})(1 + c_{3\_24} + c_{4\_13}) + \\
& 2(c_{14\_23}c_{13\_24} - c_{14\_23}c_{12\_34} + c_{13\_24}c_{12\_34}) + \\
& 72(\text{normalized volume})^2.
\end{aligned}
$$

Open problems: Hyperbolic (euclidean) version for $n \geq 4$ ($n \geq 5$) points in terms of distances, or in terms of angles.

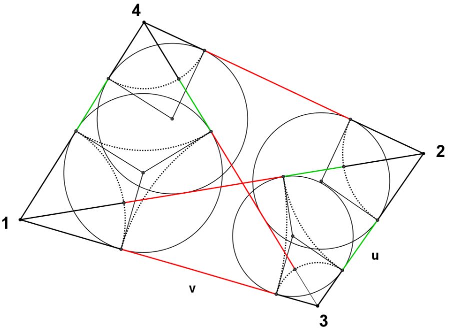
Positive parametrization of distances between 4 points

Key Lemma. (Shear coordinates of a tetrahedron)

In any tetrahedron (degenerate or not) one has the following type of nonnegative splitting of edge lengths:

$$r_{12} = t_1 + u + v + t_2, r_{13} = t_1 + v + t_3,$$
$$r_{23} = t_2 + u + t_3, r_{14} = t_1 + u + t_4,$$
$$r_{24} = t_2 + v + t_4, r_{34} = t_3 + u + v + t_4$$

if and only if $r_{12} + r_{34} = \max\{r_{12} + r_{34}, r_{13} + r_{24}, r_{14} + r_{23}\}$.



Proof.

The form of the solution:

$$t_1 = \frac{r_{13} + r_{14} - r_{34}}{2}, t_2 = \frac{r_{23} + r_{24} - r_{34}}{2}, t_3 = \frac{r_{13} + r_{23} - r_{12}}{2},$$
$$t_4 = \frac{r_{14} + r_{24} - r_{12}}{2}, u = \frac{r_{12} + r_{34} - (r_{13} + r_{24})}{2}, v = \frac{r_{12} + r_{34} - (r_{14} + r_{23})}{2}$$

proves the Lemma immediately.

Verification of the Atiyah–Sutcliffe four–point conjectures

Let us recall the original Eastwood–Norbury formula for the real part of the Atiyah's determinat D_4 of a tetrahedron:

$$Re(D_4) := prod - 4d_3(r_{12}r_{34}, r_{13}r_{24}, r_{23}r_{14}) + A_4 + vols;$$

where $d_3(a,b,c) := (-a+b+c)(a-b+c)(a+b-c)$;

$$A_4 =$$
$$(r_{14}((r_{24}+r_{34})^2 - r_{23}^2) + r_{24}((r_{14}+r_{34})^2 - r_{13}^2) + r_{34}((r_{24}+r_{14})^2 - r_{12}^2))d_3(r_{12}, r_{13}, r_{23}) +$$
$$+(r_{13}((r_{23}+r_{34})^2 - r_{24}^2) + r_{23}((r_{13}+r_{34})^2 - r_{14}^2) + r_{34}((r_{23}+r_{13})^2 - r_{12}^2))d_3(r_{12}, r_{14}, r_{24}) +$$
$$+(r_{12}((r_{23}+r_{24})^2 - r_{34}^2) + r_{23}((r_{12}+r_{24})^2 - r_{14}^2) + r_{24}((r_{23}+r_{12})^2 - r_{13}^2))d_3(r_{13}, r_{14}, r_{34}) +$$
$$+(r_{12}((r_{13}+r_{14})^2 - r_{34}^2) + r_{13}((r_{12}+r_{14})^2 - r_{24}^2) + r_{14}((r_{13}+r_{12})^2 - r_{23}^2))d_3(r_{23}, r_{24}, r_{34});$$

$$prod := 64r_{12}r_{13}r_{23}r_{14}r_{24}r_{34};$$

$$vols := 2(r_{12}^2r_{34}^2(r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2-r_{12}^2-r_{34}^2)+r_{13}^2r_{24}^2(-r_{13}^2+r_{14}^2+r_{23}^2-r_{24}^2+r_{12}^2+r_{34}^2)+$$
$$r_{14}^2r_{23}^2(r_{13}^2-r_{14}^2-r_{23}^2+r_{24}^2+r_{12}^2+r_{34}^2)-r_{12}^2r_{13}^2r_{23}^2-r_{12}^2r_{14}^2r_{24}^2-r_{13}^2r_{14}^2r_{34}^2-r_{23}^2r_{24}^2r_{34}^2);$$

($vols = 288 volume^2$) and normalized Atiyah determinant of face triangles:

$$\delta_1 := 1 + \frac{1}{8}\frac{d_3(r_{23}, r_{24}, r_{34})}{r_{23}r_{24}r_{34}}, \delta_2 := 1 + \frac{1}{8}\frac{d_3(r_{13}, r_{14}, r_{34})}{r_{13}r_{14}r_{34}},$$

$$\delta_3 := 1 + \frac{1}{8}\frac{d_3(r_{12}, r_{14}, r_{24})}{r_{12}r_{14}r_{24}}, \delta_4 := 1 + \frac{1}{8}\frac{d_3(r_{12}, r_{13}, r_{23})}{r_{12}r_{13}r_{23}}.$$

We first prove a stronger four–point conjecture of Svrtan – Urbiha (arXiv:math0609174v1 (Conjecture 2.1 (weak version)) which implies (c.f. Proposition 2.2 in loc.cit) all three four–point conjectures C_1, C_2, C_3 of Atiyah – Sutcliffe).

The substitution from the Key Lemma

$Sub := \{r_{12} = t_1 + u + v + t_2, r_{13} = t_1 + v + t_3, r_{23} = t_2 + u + t_3,$
$r_{14} = t_1 + u + t_4, r_{24} = t_2 + v + t_4, r_{34} = t_3 + u + v + t_4\};$

in the Maple code DifferSU :=

$$\left\{\text{coeffs}\left(\text{expand}\left(\text{subs}\left(\text{Sub}, \frac{1}{64}\text{numer}\left(\frac{Re(D_4) - 4vols}{prod} - \frac{\delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2}{4}\right)\right)\right)\right)\right\};$$

gives the output DifferSU $= \{2, 3, 4, \ldots, 5328, 5564, 6036\}$ which proves the conjecture coefficientwise.

The Maple code for the strongest Atiyah – Sutcliffe conjecture DifferAS :=

$$\left\{\text{coeffs}\left(\text{expand}\left(\text{subs}\left(\text{Sub}, \frac{1}{64}\text{numer}\left(\left(\frac{Re(D_4) - 4vols}{prod}\right)^2 - \delta_1\delta_2\delta_3\delta_4\right)\right)\right)\right)\right\};$$

gives the output DifferAS $= \{64, 128, 192, \ldots, 233472, 246720, 261888\}$

(coefficients of a 4512 terms inequality of degree 12 in 6 distances).

Remark 1. Similarly to DifferSU one can check the upper estimate with the additional coefficient equal to $37/27$.

Remark 2. Recently we also proved Atiyah – Sutcliffe conjecture C_2 directly from the following new formula:

$$Re(D_4) = 64\prod_{1\leq i<j\leq 4} r_{ij} + 8d_3(r_{12}r_{34}, r_{13}r_{24}, r_{14}r_{23}) + 4vols + 32R_4,$$

where

$$\begin{aligned} R_4 = {} & 4m_{2211} + (s_{13}p_{24}^2 + s_{24}p_{13}^2)u + (s_{14}p_{23}^2 + s_{23}p_{14}^2)v + (m_{221} + 8m_{2111})w + \\ & + 2(\tau_{13}^2 + \tau_{14}^2 + \tau_{13}\tau_{14})uv + (2m_{211} + 8m_{1111})(2u^2 + uv + 2v^2) \\ & + 4m_{111}(u^3 + v^3) + (3m_{21} + 14m_{111} + 3m_{11}w)uvw + [(s_{14}p_{14} + s_{23}p_{23})(u + w) + \\ & + (s_{13}p_{13} + s_{24}p_{24})(v + w)]\, uv + \left[(\tau_{13} + \tau_{14})(u^2 + uv + v^2) + \right. \\ & \left. + \tau_{14}u^2 + \tau_{13}v^2\right] uv + 2(m_1 + w)(4m_1 + 3w)u^2v^2 \end{aligned}$$

and where

$$u = \frac{r_{12} + r_{34} - r_{13} - r_{24}}{2},\ v = \frac{r_{12} + r_{34} - r_{14} - r_{23}}{2},\ w = u + v, \tau_{13} = t_1t_3 + t_2t_4,$$
$$\tau_{14} = t_1t_4 + t_2t_3, t_1 = \frac{r_{13} + r_{14} - r_{34}}{2},\ t_2 = \frac{r_{23} + r_{24} - r_{34}}{2},\ t_3 = \frac{r_{13} + r_{23} - r_{12}}{2},$$
$$t_4 = \frac{r_{14} + r_{24} - r_{12}}{2},\ s_{ij} = t_i + t_j,\ p_{ij} = t_it_j, m_1 = t_1 + t_2 + t_3 + t_4,$$
$$m_{11} = t_1t_2 + \cdots,\ m_{21} = t_1^2t_2 + \cdots,\ m_{111} = t_1t_2t_3 + \cdots, m_{1111} = t_1t_2t_3t_4,$$
$$m_{2111} = t_1^2t_2t_3t_4 + \cdots,\ m_{221} = t_1^2t_2^2t_3 + \cdots,\ m_{2211} = t_1^2t_2^2t_3t_4 + \cdots$$

Mixed Atiyah determinants

We further generalize Atiyah normalized determinant $D(x_1, \ldots, x_n)$ to $D^\Gamma(x_1, \ldots, x_n)$, where Γ is any (simple) graph with the vertex set $\{x_1, \ldots, x_n\}$.

Definition.

We start with the normalized Atiyah determinant D viewed as a function of all directions u_{ij} $(1 \leq i \neq j \leq n)$. Then we define D^Γ by simultaneously switching the roles of directions (i.e. replacing u_{ij} by u_{ji} and also replacing u_{ji} by u_{ij}) for each pair ij such that $x_1 x_j$ is an edge of Γ.

For $n = 3$ we obtain eight mixed Atiyah's determinants (mixed energies) which we can label by binary sequences $D_3 = D_3^{000}, D_3^{001}, \ldots, D_3^{111}$ for which we also have simple explicit trigonometric formulas, which can be obtained from the original Atiyah determinant by suitable sign changes of the lengths of the sides of a triangle.

Observe that $D_3 = D_3^{000}$, $D_3^{111} = 1 + e^p \prod \sinh(p_a)/\sinh(a)$ are both ≥ 1 and all other mixed determinants are between 0 and 1
(eg. $D_3^{110} = 1 - e^{p_c} \sinh(p) \sinh(p_a) \sinh(p_b)/\prod \sinh(a)$) .

Main Theorem

Now we state our

Main Theorem.

We have $\sum_\Gamma D^\Gamma = n!$, where the summation extends over all simple graphs on n vertices.

The proof is obtained by our method of computing Atiyah's determinants.

Corollary.

For any configuration of points in a 3–space at least one of the mixed Atiyah determinants is nonzero.

Proof of the main Theorem

Proof of the Main Theorem.

In coordinates $B_{ij} = u_{ij} - u_{ji}$ (antisymmetric) and $A_{ij} = u_{ij} + u_{ji}$ (symmetric) $1 \le i \ne j \le n$, D^Γ differs from D in changing signs of B_{ij}'s for each edge $ij \in \Gamma$. Let us first observe that each nonconstant term in D (and in each D^Γ) is a square free Laurent monomial w.r.t. all variables B_{ij}'s, hence in the sum over Γ its contribution is zero.

Therefore, we have to compute the constant term (C.T.) of D (which is the same in all D^Γ). Since D is a symmetrization over S_n of its main diagonal term, we have $C.T.(D) = n!C.T.(diagonal\ term)$. But diagonal term of D is equal to

$$\frac{1 \cdot (-u_{21} + \cdots)((-u_{31})(-u_{32}) + \cdots) \cdots [(-u_{n,1})(-u_{n,1}) \cdots (-u_{n,n-1})]}{(u_{12} - u_{21})(u_{13} - u_{31})(u_{23} - u_{32}) \cdots (u_{1,n} - u_{n,1}) \cdots (u_{n-1,n} - u_{n,n-1})}$$

so $C.T.(diag.term) = C.T. \dfrac{\frac{B_{12}}{2}\frac{B_{13}}{2}\frac{B_{23}}{2}\cdots}{B_{12}B_{13}B_{23}\cdots} = \dfrac{1}{2^{\binom{n}{2}}}$ and $C.T.(D) = \dfrac{n!}{2^{\binom{n}{2}}}$ and

$$C.T.\left(\sum_{\Gamma} D^\Gamma\right) = n!.$$

□

New developments

- In 2011. M.Mazur and B.V.Petrenko restated the original Eastwood Norbury formula in trigonometric form which besides face angles of a tetrahedron uses also angles of so called Crelle triangle (associated to the tetrahedron). Our formula in [5] does not involve Crelles angles, but uses "skew" angles.
- C_2 proved for convex (planar) quadrilaterals
- C_3 proved for cyclic quadrilaterals (we have it proved already in [5])
- Three conjectures stated which are consequences of some of our conjectures in [5]. (Hence we have a proof of all three.)
- In a recent paper M.B.Khuzam and M.J.Johnson (arXiv:1401.2787v1) gave a verification (by linear programming) of both C_2 and C_3 four–point conjectures of Atiyah and Sutcliffe, by using symmetric functions of degree 12 in 12 variables $t_{il} = r_{ij} + r_{ik} - r_{jk}$, $\{i, j, k, l\} = \{1, 2, 3, 4\}$ (which are linearly dependent), so for C_2 (resp. C_3) they use 64 (resp. 114) huge monomial symmetric functions.

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