Four-node Plate Elements with Assumed Shear Strain that Satisfy the Patch Test

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Abstract. We present the method of deriving mixed finite elements for Mindlin plates from a pure displacement approach, by reducing the general expression for the shear strain, while retaining its ability to pass the standard constant curvature path test. Two types of finite elements are constructed and presented on some standard benchmark problems with comparable elements from literature.

1 Introduction

The linked interpolation concept in developing finite elements for modeling Mindlin plate problems was introduced as a 2D expansion of the beam elements used for solving beam problems according to the equivalent Timoshenko beam theory [1]. While the beam elements based on the linked interpolations of the certain order for the transverse displacements and the section rotations, give exact solution of the differential equations, the equivalent plate elements suffer of the shear locking and slow convergence, especially when the low order elements are used on the coarse thin model meshes. Nevertheless, this concept has some advantages: it is a pure displacement approach that requires only $C^0$ continuity and it passes the standard patch tests exactly [2] (for example, the four-node element passes the constant bending test on the arbitrary element patch geometry).

For “over-stiffening” of the elements are responsible higher order modes of shear strains, when derived directly from the linked interpolations for displacement and rotations [3]. To overcome this problem many remedies have been presented by the various authors. Hughes and Tezduyar [4] used the “Kirchhoff mode” as a conceptual aid to produce simple shear modes. MacNeal [5] and Bathe and Dvorkin [6] assumed directly shear expressions in linear form between average values on opposite element sides. They differed only in transformation forms when extending the procedure to general quadrilaterals. MacNeal used the element diagonal intersection as a metric coordinate origin, while Bathe and Dvorkin applied the linear shear change to the covariant tensor components. Their four-node elements are known as QUAD4 and MITC4 respectively.

Ibrahimbegović [7] used bilinear distribution for the assumed shear strain calculated from the constant shears along the element sides and upgraded it with the cubic linked interpolation forms. Auricchio and Taylor [8] used shear degrees of freedom expressing linear change over the element and combined it with linked interpolations for displacement and rotations together with additional higher order bubble parameters. By static condensation all internal degrees of freedom are eliminated when forming final element stiffness matrix of the element named Q4-LIM.
Here we are presenting assumed shear strain modeled elements, similar to MITC4, but derived from shear expressions consistent with the linked interpolation concept, showing how a mixed model and a pure displacement model can complement each other. Also, here assumed shear is less sensitive to element distortion.

2 Mindlin plate theory and sign conventions

We consider a plate of a uniform thickness \( t \), a plate mid-surface lying in the horizontal co-ordinate plane and a distributed loading \( q \) assumed to act on the plate mid-surface in the direction perpendicular to it. In the Mindlin theory, the shear angles, the changes of the angles which the vertical fibers close with the mid-surface, are the differences between the total plate section rotations and the rotations due to plate displacement derivatives [1, 2] in the positive global coordinate system:

\[
\Gamma = \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \theta_x + \frac{\partial w}{\partial x} \\ \theta_y + \frac{\partial w}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} + \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} w = \mathbf{e}\mathbf{\theta} + \nabla w, \quad (1)
\]

while the curvatures (the fiber’s changes of rotations) are

\[
\kappa = \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 \theta_y}{\partial x^2} - \frac{\partial^2 \theta_x}{\partial y^2} \\ -\frac{\partial^2 \theta_x}{\partial y^2} \\ \frac{\partial^2 \theta_x}{\partial x^2} - \frac{\partial^2 \theta_y}{\partial x \partial y} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & 0 \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} = \mathbf{L}\mathbf{\theta}, \quad (2)
\]

Here \( \mathbf{\theta} \) is the rotation vector with components \( \theta_x \) and \( \theta_y \) around the respective horizontal global co-ordinate axes, \( w \) is the transverse displacement field, \( \Gamma \) is the shear strain vector and \( \kappa \) is the curvature vector, \( \nabla w \) is gradient of the displacement field and \( \mathbf{L} \) is a differential operator on the rotation field. For linear elastic material, we have the following stress-resultant vectors:

\[
\mathbf{M} = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \frac{E t^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \frac{\partial \theta_y}{\partial x} \\ -\frac{\partial \theta_x}{\partial y} \\ \frac{\partial \theta_x}{\partial y} - \frac{\partial \theta_y}{\partial x} \end{bmatrix} = \mathbf{D}\mathbf{\kappa} = \mathbf{D}\mathbf{L}\mathbf{\theta}, \quad (3)
\]

\[
\mathbf{S} = \begin{bmatrix} S_x \\ S_y \end{bmatrix} = k G t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \mathbf{D}_s \Gamma = \mathbf{D}_s (\mathbf{e}\mathbf{\theta} + \nabla w), \quad (4)
\]
where $M_x$, $M_y$, and $M_{xy}$ are the bending and twisting moments around the respective co-ordinate axes, $S_x$ and $S_y$ are the shear-stress resultants, $E$ and $G$ are the Young and shear moduli, while $\nu$ and $k$ are Poisson’s coefficient and the shear correction factor usually set to 5/6 for plates with constant thickness.

3. **Linked interpolation for a four-node quadrilateral plate element**

With the linked interpolation concept for a four-node quadrilateral element, the complete second-order polynomial expansion is involved in interpolating displacement over the element domain (9 items in Pascal’s triangle). All existing nodal parameters $w_1, \ldots, w_4$, $\theta_{x1}, \ldots, \theta_{x4}$, $\theta_{y1}, \ldots, \theta_{y4}$, are involved together with one extra internal parameter $w_b$ [2]:

\[
\begin{align*}
 w &= \frac{1-\xi}{2} \frac{1-\eta}{2} w_1 + \frac{1+\xi}{2} \frac{1-\eta}{2} w_2 + \frac{1+\xi}{2} \frac{1+\eta}{2} w_3 + \frac{1-\xi}{2} \frac{1+\eta}{2} w_4 \\
 &+ \frac{1-\xi^2}{4} \frac{1-\eta}{2} \left[ (\theta_{x2} - \theta_{x1})(x_2 - x_1) - (\theta_{x2} - \theta_{x4})(y_2 - y_1) \right] \\
 &+ \frac{1-\xi^2}{4} \frac{1+\eta}{2} \left[ (\theta_{y3} - \theta_{y4})(x_3 - x_4) - (\theta_{y3} - \theta_{y4})(y_3 - y_4) \right] \\
 &+ \frac{1+\xi}{2} \frac{1-\eta^2}{4} \frac{1}{2} \left[ (\theta_{y3} - \theta_{y2})(x_3 - x_2) - (\theta_{y3} - \theta_{y2})(y_3 - y_2) \right] \\
 &+ \frac{1-\xi}{2} \frac{1-\eta^2}{4} \frac{1}{2} \left[ (\theta_{y4} - \theta_{y1})(x_4 - x_1) - (\theta_{y4} - \theta_{y1})(y_4 - y_1) \right] \\
 &+ \frac{1-\xi^2}{4} \frac{1-\eta^2}{4} w_b. \quad (5)
\end{align*}
\]

At the same time, the interpolations for the rotation fields have no need to be expanded and remain in the Lagrangian form (4 items in Pascal’s triangle):

\[
\begin{align*}
 \theta_x &= \sum_{i=1}^{4} N_i \theta_{xi}, \\
 \theta_y &= \sum_{i=1}^{4} N_i \theta_{yi}, \quad (6, 7)
\end{align*}
\]

where $N_i$ are the standard bilinear interpolation functions associated with nodal rotational parameters $\theta_{xi}$ and $\theta_{yi}$. In this form the above interpolations can fully cover any cylindrical bending for constant moment distribution over the element domain [9].

If, in particular, a line “s” passing through two arbitrary points with constant $\xi$ or $\eta$, is taken into consideration (Fig. 1), a shear strain $\gamma_{s,\xi}$ (for a constant $\eta$) expressed with respect to the global shear strains is
Figure 1: Arbitrary 4-node quadrilateral plate element – relations between the global and the natural coordinates and the shear strain transformation

\[ \gamma_{x,\xi} = \gamma_x \cos \alpha \xi + \gamma_y \sin \alpha \xi = \left( \frac{\partial w}{\partial x} + \theta \right) \cos \alpha \xi + \left( \frac{\partial w}{\partial y} - \theta \right) \sin \alpha \xi \]

\[ = \frac{\partial w}{\partial x} \cos \alpha \xi + \frac{\partial w}{\partial y} \sin \alpha \xi + \theta \cos \alpha \xi - \theta \sin \alpha \xi = \frac{\partial w}{\partial s} + \theta \alpha , \]

where \( \theta \alpha \) is a rotation component acting around the normal vector to the “s” line (\( \eta = \text{const} \)) across the element. Since the values for \( \sin \alpha \xi \) and \( \cos \alpha \xi \) are constant for \( \xi \), they can be expressed as (see Fig. 1)

\[ \sin \alpha \xi = \frac{\partial y}{\partial \xi} \quad \text{and} \quad \cos \alpha \xi = \frac{\partial x}{\partial \xi} \]

eventually giving

\[ \gamma_{x,\xi} = \frac{\partial w}{\partial \xi} \frac{\partial \xi}{\partial s} - \theta \frac{\partial y}{\partial \xi} \frac{\partial \xi}{\partial s} + \theta \frac{\partial x}{\partial \xi} \frac{\partial \xi}{\partial s} + \theta \left( \frac{\partial w}{\partial \xi} - \theta \frac{\partial y}{\partial \xi} + \theta \frac{\partial x}{\partial \xi} \right) \frac{1}{\frac{\partial \xi}{\partial s}} . \]
Since \[
\frac{\partial s}{\partial \xi} = \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2} = \sqrt{\left(x_a + x_b\right)^2 + \left(y_a + y_b\right)^2} = \frac{L_{ab}}{2} = \text{const}
\]
and \[
\frac{\partial y}{\partial \xi} = \frac{y_a + y_b}{2} = \text{const}
\]
and \[
\frac{\partial x}{\partial \xi} = \frac{x_a + x_b}{2} = \text{const}
\]
the expression for \(\gamma_{s,\xi}\) will be at most linear in \(\xi\) for every \(\eta=\text{const}\). The shear strain in \(\xi\)-direction over the element domain follows as
\[
\gamma_{s,\xi}(\xi, \eta) = \left[1 - \eta \left(\frac{1}{2} \left(\frac{w_x - w_i}{2} + \frac{\theta_{s,1} + \theta_{s,4}}{2}, \frac{x_i - x_1}{2} - \frac{\theta_{s,1} + \theta_{s,4}}{2}, \frac{y_i - y_1}{2}\right) + \frac{1 + \eta}{2} \left(\frac{w_x - w_i}{2} + \frac{\theta_{s,1} + \theta_{s,4}}{2}, \frac{x_i - x_1}{2} - \frac{\theta_{s,1} + \theta_{s,4}}{2}, \frac{y_i - y_1}{2}\right)\right)\right] \frac{1}{\frac{\partial s}{\partial \xi}}.
\]
(11)

Similarly,
\[
\gamma_{s,\eta}(\xi, \eta) = \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial s} - \frac{\partial y}{\partial \eta} \frac{\partial \eta}{\partial s} + \theta_{s} \frac{\partial x}{\partial \eta} \frac{\partial \eta}{\partial s} = \left(\frac{\partial w}{\partial \eta} - \theta_{s} \frac{\partial y}{\partial \eta} + \theta_{s} \frac{\partial x}{\partial \eta}\right) \frac{1}{\frac{\partial \eta}{\partial s}},
\]
(12)
for line with \(\xi=\text{const}\) and the shear strain in \(\eta\)-direction has the expression
\[
\gamma_{s,\eta}(\xi, \eta) = \left[1 - \xi \left(\frac{1}{2} \left(\frac{w_x - w_i}{2} + \frac{\theta_{s,1} + \theta_{s,4}}{2}, \frac{x_i - x_1}{2} - \frac{\theta_{s,1} + \theta_{s,4}}{2}, \frac{y_i - y_1}{2}\right) + \frac{1 + \xi}{2} \left(\frac{w_x - w_i}{2} + \frac{\theta_{s,1} + \theta_{s,4}}{2}, \frac{x_i - x_1}{2} - \frac{\theta_{s,1} + \theta_{s,4}}{2}, \frac{y_i - y_1}{2}\right)\right)\right] \frac{1}{\frac{\partial s}{\partial \eta}}.
\]
(13)
When variable $\eta$ in (11) reaches $\eta=-1$ and $\eta=+1$, the well-known expressions for the constant shear strain $\gamma_{12,\xi}$ and $\gamma_{43,\xi}$ on the opposite element sides are derived as in the beam linked interpolation. Similarly the constant $\gamma_{14,\eta}$ and $\gamma_{23,\eta}$ on opposite element sides can be derived from (13).

In (11) and (13) the terms within parentheses multiplying shape functions, must equal zero for the case of constant bending and shear strains in both directions should vanish independently of the bending vector orientation.

The third (second-order) terms in (11) and (13) which are derived exclusively from quadratic interpolation part for the displacement $w$ are the main cause of the element matrix over-stiffness [2] since they are in sharp contrast with the equilibrium requirements relating linear expressions for bending moments from (3) with shear forces as its derivatives. For this reason, one would wish to eliminate them from the formulation. Of course, the same must be also done for the fourth (third-order) terms in (11) and (13). It is interesting to notice that this fourth term is actually the same in both expressions and that is connected with distortion geometry parameters $\Delta$:

$$
\Delta x = x_1 - x_2 + x_3 - x_4 \text{ and } \Delta y = y_1 - y_2 + y_3 - y_4.
$$

This parameters will be zero for regular element mesh (parallelogram shaped elements). In contrast to the third terms, however, the fourth term can be eliminated from the expressions (11) and (13) by proper choice of bubble term $w_b$:

$$
w_b = \left( \theta_{x1} - \theta_{x2} + \theta_{x3} - \theta_{x4} \right) \frac{\Delta y}{2} - \left( \theta_{y1} - \theta_{y2} + \theta_{y3} - \theta_{y4} \right) \frac{\Delta x}{2}, \quad (15)
$$

Here, we impose a linear shear distribution by deliberate choice of the shear strains without the second-order terms and by appropriate expression for $w_b$ (16) in case of distorted element geometry. The result is linear change of $\gamma_{s\xi}$ with respect to $\eta$-direction and constant in $\xi$-direction over the whole element domain despite the element geometry. The proposed plate element constructed in this way is here denoted as $Q4$-AS1.

But the bubble term $w_b$ can be left as internal degree of freedom and condensed statically within the element matrix, leaving the assumed shear strain expressions in $\xi$-direction (16) and in $\eta$-direction (17) with the distorted part respectively.

$$
\gamma_{s\xi}(\xi, \eta) = \left[ 1 - \frac{\eta}{2} \gamma_{12,\xi} + \frac{1 + \eta}{2} \gamma_{43,\xi} + \frac{\xi}{4} \left( \theta_{x1} - \theta_{x2} + \theta_{x3} - \theta_{x4} \right) \frac{\Delta y}{2} - \left( \theta_{y1} - \theta_{y2} + \theta_{y3} - \theta_{y4} \right) \frac{\Delta x}{2} - w_b \right] \frac{1}{\partial \xi}, \quad (16)
$$

$^1$ Note that in [3] a related development initially led to dual bubble parameters, which were then made equal in order to eliminate the cubic shear distribution in the case of constant curvature with zero shear strain.
\[
\gamma_{\xi\eta}(\xi, \eta) = \left[ \frac{1-\xi}{2} \gamma_{(14)\eta} + \frac{1+\xi}{2} \gamma_{(23)\eta} + \frac{1-\xi^2}{4} \eta \left( \frac{\theta_{x_1} - \theta_{x_2} + \theta_{x_5} - \theta_{x_4}}{2} - \left( \frac{\theta_{y_1} - \theta_{y_2} + \theta_{y_5} - \theta_{y_4}}{2} \right) \Delta y \right) \right] \frac{1}{\partial \xi \partial \eta}.
\]

(17)

The plate element constructed in this way is here denoted as \(Q_{4-AS2}\). The new elements are compared with the element developed on pure linked interpolation \(Q_{4-U2}\) from [2] and with \(MITC4\) element from [6] with their results from literature. As it is known, the \(MITC4\) allows linear shear strain over the element domain as covariant tensor components.

4. The patch test

Consistency of the new developed elements is tested for the constant strain conditions on the patch examples with five elements, covering a rectangular domain of a plate as proposed in [10]. The displacements and rotations for all internal nodes within the patch are checked for the specific displacements and rotations given at all external nodes.

The new proposed elements \(Q_{4-AS1}\) and \(Q_{4-AS2}\), without and with distorted part respectively in the assumed shear strain expressions (16) and (17) pass the constant bending patch test exactly, as does the element developed on pure linked interpolation \(Q_{4-U2}\) [2] and the \(MITC4\) element from [6].

5. Clamped square plate

In this example a quadratic plate with clamped edges is considered. Only upper right quarter of the plate is modeled with symmetric boundary conditions imposed on the symmetry lines. Two ratios of span versus thickness are analyzed, \(L/t=10\) representing a relatively thick plate and \(L/t=1000\) representing its thin counterpart. The loading on the plate is uniformly distributed of magnitude \(q = 1\). The plate material properties are \(E = 10.92\) and \(\nu = 0.3\).

In this model example, regular meshes are used and the present elements \(Q_{4-AS1}\) and \(Q_{4-AS2}\) behave identically since the distorted parts in (16) and (17) are zero, so they are denoted just as \(Q_{4-AS}\). Numerical results are given in Tables 1 and 2 and are compared to the elements \(Q_{4-U2}\) and \(MITC4\) presented in [2, 6] and with the reference solutions taken from [8]. The number of elements per mesh in these tables relates to one quarter of the structure as shown in Fig. 2 for the mesh of 4x4 elements.
Figure 2: A quarter of the square plate under uniform load (16-element regular mesh)

The dimensionless results $w^* = w / (qL^4/100D)$ and $M^* = M / (qL^2/100)$, where $D = E t^3 / (12(1-\nu^2))$ and $L$ is the plate span, given in these tables are related to the central displacement of the plate and the bending moment at the integration point nearest to the center of the plate.

The number of Gauss integration points used for calculation of the stiffness matrix is 2x2 for the presented elements, unlike the element $Q4-U2$ with the linked interpolation which needed 3x3 integration points.

Table 1: Clamped square plate: displacement and moment at the center with regular meshes, $L/t=10$

<table>
<thead>
<tr>
<th>Element mesh</th>
<th>$Q4-U2$ (GP:3x3) [2]</th>
<th>$Q4-AS$ (GP:2x2)</th>
<th>MITC4 [6]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$w^<em>$ $M^</em>$</td>
<td>$w^<em>$ $M^</em>$</td>
<td>$w^<em>$ $M^</em>$</td>
</tr>
<tr>
<td>1x1</td>
<td>0.02679 -</td>
<td>0.0267857 -</td>
<td></td>
</tr>
<tr>
<td>2x2</td>
<td>0.1192017 2.02221</td>
<td>0.1430755 2.25919</td>
<td>0.1430755</td>
</tr>
<tr>
<td>4x4</td>
<td>0.1436084 2.25778</td>
<td>0.1487936 2.31504</td>
<td>0.1487936</td>
</tr>
<tr>
<td>8x8</td>
<td>0.1487596 2.30471</td>
<td>0.1500372 2.31868</td>
<td>0.1500372</td>
</tr>
<tr>
<td>16x16</td>
<td>0.1500353 2.31616</td>
<td>0.1503550 2.31964</td>
<td>0.1503550</td>
</tr>
<tr>
<td>32x32</td>
<td>0.1503556 2.31903</td>
<td>0.1504356 2.31990</td>
<td>0.1504356</td>
</tr>
<tr>
<td>64x64</td>
<td>0.1504358 2.31975</td>
<td>0.1504558 2.31996</td>
<td>0.1504558</td>
</tr>
<tr>
<td>Ref. sol. [8]</td>
<td>0.1499 2.31</td>
<td>0.1499 2.31</td>
<td>0.1499 2.31</td>
</tr>
</tbody>
</table>
Table 2: Clamped square plate: displacement and moment at the center with regular meshes, L/t=1000

<table>
<thead>
<tr>
<th>Element mesh</th>
<th>$Q4-U2$ (GP:3x3) [2]</th>
<th>$Q4-AS$ (GP:2x2)</th>
<th>MITC4 [6]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$w^*$</td>
<td>$M^*$</td>
<td>$w^*$</td>
</tr>
<tr>
<td>1x1</td>
<td>0.0000027</td>
<td>-</td>
<td>0.0000027</td>
</tr>
<tr>
<td>2x2</td>
<td>0.0001284</td>
<td>-</td>
<td>0.1211262</td>
</tr>
<tr>
<td>4x4</td>
<td>0.0046933 0.10731</td>
<td>0.1250713 2.30025</td>
<td>0.1250713</td>
</tr>
<tr>
<td>8x8</td>
<td>0.0598813 1.18496</td>
<td>0.1261671 2.29290</td>
<td>0.1261671</td>
</tr>
<tr>
<td>16x16</td>
<td>0.1189899 2.17415</td>
<td>0.1264424 2.29109</td>
<td>0.1264424</td>
</tr>
<tr>
<td>32x32</td>
<td>0.1260024 2.28275</td>
<td>0.1265114 2.29066</td>
<td>0.1265114</td>
</tr>
<tr>
<td>64x64</td>
<td>0.1264813 2.28988</td>
<td>0.1265287 2.29055</td>
<td>0.1265287</td>
</tr>
<tr>
<td>Ref. sol. [8]</td>
<td>0.1265319 2.29051</td>
<td>0.1265319 2.29051</td>
<td>0.1265319</td>
</tr>
</tbody>
</table>

According to the above results $Q4-AS$ elements are equal to MITC4 element. In the same time they are resistant to the locking phenomena notable when $Q4-U2$ linked-interpolation element is used on the coarse meshes modeling thin plates with the full integration scheme.

6. Clamped circular plate

Next, the circular plate loaded by a uniform pressure and with the clamped edges is analyzed. The element mesh is obviously irregular with non-parallel element sides so the influence of such irregularity is observed on the behavior of the three elements: $Q4-AS1$ (with $w_b$ given by (15)), $Q4-AS2$ (with $w_b$ as an independent degree of freedom) and MITC4.

![Clamped circular plate](image)

Figure 3: A clamped circular plate under uniform load – a 12-element mesh is shown.
The results are given in Tables 3 to 5 for various thickness of the plate. The problem geometry and material properties are given in Fig. 3 (only one quarter of the plate is analyzed), where an example of a 12-element mesh is also shown.

Table 3: Clamped circular plate: displacement and moment at the center for Q4-AS1 element, for various thicknesses

<table>
<thead>
<tr>
<th>Element</th>
<th>mesh</th>
<th>Q4-AS1 t=1</th>
<th>Q4-AS1 t=0.1</th>
<th>Q4-AS1 t=0.01</th>
<th>Q4-AS1 t=0.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>w_c</td>
<td>M_c</td>
<td>w_c</td>
<td>M_c</td>
<td>w_c</td>
<td>M_c</td>
</tr>
<tr>
<td>3</td>
<td>0.10910</td>
<td>1.93655</td>
<td>0.092093</td>
<td>1.91232</td>
<td>0.091921</td>
</tr>
<tr>
<td>12</td>
<td>0.11448</td>
<td>2.03000</td>
<td>0.097091</td>
<td>2.03976</td>
<td>0.096920</td>
</tr>
<tr>
<td>48</td>
<td>0.11527</td>
<td>2.02984</td>
<td>0.097655</td>
<td>2.02911</td>
<td>0.097479</td>
</tr>
<tr>
<td>Ref. [9]</td>
<td>0.11551</td>
<td>2.03125</td>
<td>0.097838</td>
<td>2.03125</td>
<td>0.097658</td>
</tr>
</tbody>
</table>

Table 4: Clamped circular plate: displacement and moment at the center for Q4-AS2 element, for various thicknesses

<table>
<thead>
<tr>
<th>Element</th>
<th>mesh</th>
<th>Q4-AS2 t=1</th>
<th>Q4-AS2 t=0.1</th>
<th>Q4-AS2 t=0.01</th>
<th>Q4-AS2 t=0.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>w_c</td>
<td>M_c</td>
<td>w_c</td>
<td>M_c</td>
<td>w_c</td>
<td>M_c</td>
</tr>
<tr>
<td>3</td>
<td>0.10910</td>
<td>1.93659</td>
<td>0.092096</td>
<td>1.91236</td>
<td>0.091923</td>
</tr>
<tr>
<td>12</td>
<td>0.11448</td>
<td>2.03000</td>
<td>0.097091</td>
<td>2.03976</td>
<td>0.096920</td>
</tr>
<tr>
<td>48</td>
<td>0.11527</td>
<td>2.02984</td>
<td>0.097655</td>
<td>2.02911</td>
<td>0.097479</td>
</tr>
<tr>
<td>Ref. [9]</td>
<td>0.11551</td>
<td>2.03125</td>
<td>0.097835</td>
<td>2.03125</td>
<td>0.097658</td>
</tr>
</tbody>
</table>

Table 5: Clamped circular plate: displacement and moment at the center for MITC4 element [6], for various thicknesses

<table>
<thead>
<tr>
<th>Element</th>
<th>mesh</th>
<th>MITC4 t=1</th>
<th>MITC4 t=0.1</th>
<th>MITC4 t=0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>w_c</td>
<td>M_c</td>
<td>w_c</td>
<td>M_c</td>
<td>w_c</td>
</tr>
<tr>
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<td>0.10755</td>
<td>1.93</td>
<td>0.09068</td>
<td>1.88</td>
</tr>
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<td>12</td>
<td>0.11431</td>
<td>2.04</td>
<td>0.09699</td>
<td>2.05</td>
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<tr>
<td>48</td>
<td>0.11525</td>
<td>2.04</td>
<td>0.09766</td>
<td>2.04</td>
</tr>
<tr>
<td>Ref. [9]</td>
<td>0.11551</td>
<td>2.03125</td>
<td>0.097835</td>
<td>2.03125</td>
</tr>
</tbody>
</table>

Unlike the rectangular clamped plate example with regular meshes, the results of the above example show slight differences in the convergence. The best results are obtained by Q4-AS2 element with the bubble parameter, but even Q4-AS1 with the ordinary assumed shear strains is slightly better than MITC4 with the same assumption of strains but with the covariant tensor transformation to the global coordinates.
8. Conclusions

We have demonstrated that the assumed strain formulation in modeling Mindlin plate elements can be derived from the linked interpolation concept by reducing the general shear strain expression, retaining its capability to pass the standard constant curvature patch test. New developed element $Q4-AS1$ is of the mixed type and is almost identical to the well-known $MITC4$ element. Slight improvement is found when the distortion term is included in the $Q4-AS2$ new element and better results are found only on the distorted model meshes.

References