

A. Matković, J. Pečarić and J. Perić

*A refinement of the Jessen-Mercer inequality and a
generalization on convex hulls in \mathbb{R}^k*

JOURNAL OF MATHEMATICAL INEQUALITIES

AIMS AND SCOPE

Journal of Mathematical Inequalities (**JMI**, J. Math. Inequal.) presents carefully selected original research articles from all areas of pure and applied mathematics, provided they are concerned with mathematical inequalities. **JMI** will also periodically publish invited survey articles with interesting results treating the theory of inequalities, as well as relevant book reviews.

JMI is published quarterly, in March, June, September, and December.

SUBMISSION OF MANUSCRIPTS

Authors are requested to submit their articles electronically as $\text{T}_{\text{E}}\text{X}$ / $\text{L}_{\text{A}}\text{T}_{\text{E}}\text{X}$ files, with enclosed Adobe Acrobat PDF format through the provided web interface on journal web page.

The author who submitted the article for publication will be denoted as a corresponding author. He/She manages all communication and correspondence with the **JMI** regarding the article, makes any revisions, and reviews and releases the author proofs. Authors may indicate a member of the Editorial Board whom they consider appropriate for the article. However, assignment to that particular editor is not assured.

COPYRIGHT

The acceptance of the article automatically implies the copyright transfer to the **JMI**. Manuscripts are accepted for review with the understanding that the same work has not been published (except in the form of an abstract), that it is not under consideration for publication elsewhere, that it will not be submitted to another journal while under review for the **JMI**, and that its submission for publication has been approved by all of the authors.

OPEN ACCESS

Journal of Mathematical Inequalities is published as open access journal. All interested readers are allowed to view, download, print, and redistribute any article without paying for subscription. Offprints of each article may be ordered from **JMI** prior to publication.

PREPARATION OF MANUSCRIPTS

Manuscripts should be written in English, using prepared journal LaTeX style.

The first page should contain the article title, authors' names (complete first names and family names), complete affiliations (name of institution, address, city, state, and zip code), e-mail addresses, proposed running head (not more than 40 characters), a short abstract (not more than 150 words), a list of key words and phrases, and the AMS 2010 Mathematics Subject Classification primary (and secondary) codes. Avoid abbreviations, mathematical symbols and formulas, diagrams, and reference to the text of the article.

Figures should be prepared in a digital form suitable for direct reproduction, at resolution of 300 dpi or higher, and in EPS, TIFF or JPEG format.

Bibliographic references should be listed alphabetically at the end of the article, each numbered by an Arabic number between square brackets. The following information should be provided for references to journals: names of authors, full title of the article, abbreviated name of the journal, volume, year of publication, and range of page numbers. For standard abbreviations of journal names, the authors should consult the latest Abbreviations of Names and Series reviewed in Mathematical Reviews.

SOURCE FILE, PROOFS

Upon acceptance of the article, the authors will be asked to send the related LaTeX source file to the Editorial Office, in accordance to the journal style. In order to accelerate the publication process, the AMS LaTeX package is strongly preferred. PDF proofs will be sent by e-mail to the corresponding author.

FORTHCOMING PAPERS

Papers accepted and prepared for publication will appear in the forthcoming section of Journal Web page. They are identical in form as final printed papers, except volume, issue and page numbers.

JMI is published by Publishing House **ELEMENT**, Zagreb, Croatia.

All correspondence and subscription orders should be addressed to the Editorial Office:

www.ele-math.com
e-mail: jmi@ele-math.com

Journal of Mathematical Inequalities
Editorial Office
Menceticeva 2, 10000 Zagreb, Croatia
Fax: +385 1 6008799

A REFINEMENT OF THE JESSEN–MERCER INEQUALITY AND A GENERALIZATION ON CONVEX HULLS IN \mathbb{R}^k

A. MATKOVIĆ, J. PEČARIĆ AND J. PERIĆ

(Communicated by S. Varošanec)

Abstract. A refinement of the Jessen–Mercer inequality is obtained and shown to be an improvement of the upper bound for the Jessen’s difference given in [12]. Also a generalization of the Jessen–Mercer inequality for convex functions on convex hulls in \mathbb{R}^k is given and demonstrated to be an improvement of the inequalities obtained in [3]. An elegant method of producing n -exponentially convex and exponentially convex functions is applied using the Jessen–Mercer differences. Lagrange and Cauchy mean value type theorems are proved and shown to be useful in studying Stolarsky type means defined by using the Jessen–Mercer differences.

1. Introduction

Let E be a nonempty set and L be a subspace of the vector space \mathbb{R}^E over \mathbb{R} which contains $\mathbf{1}$, that is L having following properties

$L1$: $f, g \in L \Rightarrow (\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;

$L2$: $\mathbf{1} \in L$, i.e., if $f(t) = 1$ for $t \in E$, then $f \in L$.

If L additionally has property

$L3$: $(\forall f, g \in L) (\min\{f, g\} \in L \wedge \max\{f, g\} \in L)$,

then it is a *lattice*. Obviously, (\mathbb{R}^E, \leq) (with standard ordering) is a lattice. It can also be easily verified that a subspace $X \subseteq \mathbb{R}^E$ is a lattice if and only if $x \in X$ implies $|x| \in X$. This is a simple consequence of the fact that for every $x \in X$ the functions $|x|$, x^- and x^+ can be defined by

$$|x|(t) = |x(t)|, \quad x^+(t) = \max\{0, x(t)\}, \quad x^-(t) = -\min\{0, x(t)\}, \quad t \in E,$$

and $x^+ + x^- = |x|$, $x^+ - x^- = x$,

$$\min\{x, y\} = \frac{1}{2}(x + y - |x - y|), \quad \max\{x, y\} = \frac{1}{2}(x + y + |x - y|). \quad (1.1)$$

We consider *positive linear functionals* $A: L \rightarrow \mathbb{R}$, i.e., functionals having properties

$A1$: $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for $f, g \in L$, $\alpha, \beta \in \mathbb{R}$ (linearity);

$A2$: $f \in L$, $f(t) \geq 0$ on $E \Rightarrow A(f) \geq 0$ (positivity).

Mathematics subject classification (2010): 39B62, 26D15.

Keywords and phrases: Jessen–Mercer inequality, convex functions, convex hulls, n -exponential convexity, Stolarsky type means.

This work has been fully supported by Croatian Science Foundation under the project 5435.

If A additionally satisfies the condition $A(\mathbf{1}) = 1$, then it is *normalized*.

Throughout the paper when using interval $[m, M]$ we assume that $-\infty < m < M < \infty$.

The following result is a variant of the well known Jensen's inequality [6] (see also [14, p. 47]) of Mercer's type [10] proved in [3]. We call it the Jensen-Mercer inequality.

THEOREM A. *Let L satisfy L_1, L_2 on a nonempty set E , and let A be a positive normalized linear functional. If φ is a continuous convex function on $[m, M]$, then for all $f \in L$ such that $\varphi(f), \varphi(m+M-f) \in L$ (so that $m \leq f(t) \leq M$ for all $t \in E$), we have*

$$\varphi(m+M-A(f)) \leq \varphi(m) + \varphi(M) - A(\varphi(f)). \quad (1.2)$$

REMARK 1. In fact, to be more specific, the following series of inequalities was proved

$$\begin{aligned} \varphi(m+M-A(f)) &\leq A(\varphi(m+M-f)) \\ &\leq \frac{M-A(f)}{M-m}\varphi(M) + \frac{A(f)-m}{M-m}\varphi(m) \\ &\leq \varphi(m) + \varphi(M) - A(\varphi(f)). \end{aligned} \quad (1.3)$$

Furthermore, if the function φ is concave, inequalities (1.2) and (1.3) are reversed.

The reversed Jensen inequality follows easily from Jensen's inequality (see [14]).

THEOREM B. *Let \mathbf{p} be a real n -tuple such that*

$$p_1 > 0, \quad p_i \leq 0 (i = 2, \dots, n), \quad P_n > 0$$

where $P_n = \sum_{i=1}^n p_i$. Let U be a convex set in a real vector space M , $\mathbf{x}_i \in U$ ($i = 1, \dots, n$)

and $\frac{1}{P_n} \sum_{i=1}^n p_i \mathbf{x}_i \in U$. If $f: U \rightarrow \mathbb{R}$ is a convex function, then

$$f\left(\frac{1}{P_n} \sum_{i=1}^n p_i \mathbf{x}_i\right) \geq \frac{1}{P_n} \sum_{i=1}^n p_i f(\mathbf{x}_i). \quad (1.4)$$

Recently, M. Klaričić Bakula *et al.* [8] obtained the following refinement of the converse Jensen's inequality [1] (see also [14, p. 98]).

THEOREM C. *Let L satisfy L_1, L_2, L_3 on a nonempty set E , and let A be a positive normalized linear functional. If φ is a convex function on $[m, M]$ then for all $g \in L$ such that $\varphi(g) \in L$ we have $A(g) \in [m, M]$ and*

$$A(\varphi(g)) \leq \frac{M-A(g)}{M-m}\varphi(m) + \frac{A(g)-m}{M-m}\varphi(M) - A(\tilde{g})\delta_\varphi, \quad (1.5)$$

where

$$\tilde{g} = \frac{1}{2} - \frac{1}{M-m} \left| g - \frac{m+M}{2} \right|, \quad \delta_\varphi = \varphi(m) + \varphi(M) - 2\varphi\left(\frac{m+M}{2}\right).$$

Using inequality (1.5) and the following Lemma we will refine the series of inequalities (1.3).

LEMMA 1. Let ϕ be a convex function on U where U is a convex set in \mathbb{R}^k , $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in U^n$ and $\mathbf{p} = (p_1, \dots, p_n)$ is nonnegative n -tuple such that $\sum_{i=1}^n p_i = 1$.

Then

$$\begin{aligned} & \min\{p_1, \dots, p_n\} \left[\sum_{i=1}^n \phi(\mathbf{x}_i) - n\phi\left(\frac{1}{n} \sum_{i=1}^n \mathbf{x}_i\right) \right] \\ & \leq \sum_{i=1}^n p_i \phi(\mathbf{x}_i) - \phi\left(\sum_{i=1}^n p_i \mathbf{x}_i\right) \end{aligned} \tag{1.6}$$

$$\leq \max\{p_1, \dots, p_n\} \left[\sum_{i=1}^n \phi(\mathbf{x}_i) - n\phi\left(\frac{1}{n} \sum_{i=1}^n \mathbf{x}_i\right) \right]. \tag{1.7}$$

Proof. This is a simple consequence of [11, p. 717, Theorem 1]. \square

2. Refinement of the Jessen-Mercer inequality

Next two theorems are our main results.

THEOREM 1. Let L satisfy $L1, L2, L3$ on a nonempty set E , and let A be a positive normalized linear functional. If φ is a continuous convex function on $[m, M]$, then for all $f \in L$ such that $\varphi(f), \varphi(m+M-f) \in L$, we have

$$\begin{aligned} \varphi(m+M-A(f)) & \leq A(\varphi(m+M-f)) \\ & \leq \frac{M-A(f)}{M-m} \varphi(M) + \frac{A(f)-m}{M-m} \varphi(m) - A\left(\frac{1}{2} - \frac{1}{M-m} \left| f - \frac{m+M}{2} \right|\right) \delta_\varphi \\ & \leq \varphi(m) + \varphi(M) - A(\varphi(f)) - \left[1 - \frac{2}{M-m} A\left(\left| f - \frac{m+M}{2} \right|\right) \right] \delta_\varphi \\ & \leq \varphi(m) + \varphi(M) - A(\varphi(f)), \end{aligned}$$

where

$$\delta_\varphi = \varphi(M) + \varphi(m) - 2\varphi\left(\frac{M+m}{2}\right). \tag{2.1}$$

Proof. Using the first inequality from the series (1.3) and applying inequality (1.5) first to the function $m + M - f$, and then to the function f , we obtain

$$\begin{aligned} & \varphi(m + M - A(f)) \\ & \leq A(\varphi(m + M - f)) \\ & \leq \frac{M - A(f)}{M - m} \varphi(M) + \frac{A(f) - m}{M - m} \varphi(m) - A\left(\frac{1}{2} - \frac{1}{M - m} \left|f - \frac{m + M}{2}\right|\right) \delta_\varphi \\ & = \varphi(m) + \varphi(M) - \left[\frac{M - A(f)}{M - m} \varphi(m) + \frac{A(f) - m}{M - m} \varphi(M)\right] - A\left(\frac{1}{2} - \frac{1}{M - m} \left|f - \frac{m + M}{2}\right|\right) \delta_\varphi \\ & \leq \varphi(m) + \varphi(M) - A(\varphi(f)) - 2A\left(\frac{1}{2} - \frac{1}{M - m} \left|f - \frac{m + M}{2}\right|\right) \delta_\varphi \\ & = \varphi(m) + \varphi(M) - A(\varphi(f)) - \left[1 - \frac{2}{M - m} A\left(\left|f - \frac{m + M}{2}\right|\right)\right] \delta_\varphi, \\ & \leq \varphi(m) + \varphi(M) - A(\varphi(f)). \end{aligned}$$

The last inequality is a simple consequence of the easily provable facts that $\delta_\varphi = \varphi(M) + \varphi(m) - 2\varphi\left(\frac{M+m}{2}\right) \geq 0$ and $1 - \frac{2}{M-m} A\left(\left|f - \frac{m+M}{2}\right|\right) \geq 0$. \square

THEOREM 2. *Let L satisfy L1, L2, L3 on a nonempty set E , and let A be a positive normalized linear functional. If φ is a continuous convex function on $[m, M]$, then for all $f \in L$ such that $\varphi(f), \varphi(m + M - f) \in L$, we have*

$$\begin{aligned} & \varphi(m + M - A(f)) \\ & \leq \frac{M - A(f)}{M - m} \varphi(M) + \frac{A(f) - m}{M - m} \varphi(m) - \left(\frac{1}{2} - \frac{1}{M - m} \left|A(f) - \frac{m + M}{2}\right|\right) \delta_\varphi \\ & \leq \varphi(m) + \varphi(M) - A(\varphi(f)) - \left[1 - \frac{1}{M - m} \left(A\left(\left|f - \frac{m + M}{2}\right|\right) + \left|A(f) - \frac{m + M}{2}\right|\right)\right] \delta_\varphi \\ & \leq \varphi(m) + \varphi(M) - A(\varphi(f)) - \left[1 - \frac{2}{M - m} A\left(\left|f - \frac{m + M}{2}\right|\right)\right] \delta_\varphi \\ & \leq \varphi(m) + \varphi(M) - A(\varphi(f)), \end{aligned}$$

where δ_φ is defined as in (2.1).

Proof. Theorem C for the function f gives us

$$A(\varphi(f)) \leq \frac{M - A(f)}{M - m} \varphi(m) + \frac{A(f) - m}{M - m} \varphi(M) - A\left(\frac{1}{2} - \frac{1}{M - m} \left|f - \frac{m + M}{2}\right|\right) \delta_\varphi. \tag{2.2}$$

Let the functions $p, q: [m, M] \rightarrow [0, 1]$ be defined by

$$p(t) = \frac{M - t}{M - m}, \quad q(t) = \frac{t - m}{M - m}.$$

For any $t \in [m, M]$ we can write

$$\varphi(t) = \varphi\left(\frac{M-t}{M-m}m + \frac{t-m}{M-m}M\right) = \varphi(p(t)m + q(t)M).$$

By Lemma 1 for $n = 2$ it follows

$$\varphi(t) \leq p(t)\varphi(m) + q(t)\varphi(M) - \min\{p(t), q(t)\}\delta_\varphi,$$

where $\delta_\varphi = \varphi(M) + \varphi(m) - 2\varphi\left(\frac{M+m}{2}\right)$. Using (1.1) we can write it in the form

$$\varphi(t) \leq \frac{M-t}{M-m}\varphi(m) + \frac{t-m}{M-m}\varphi(M) - \left(\frac{1}{2} - \frac{1}{M-m}\left|t - \frac{m+M}{2}\right|\right)\delta_\varphi.$$

Substituting $t \leftrightarrow A(g)$, where $g \in L$ such that $A(g) \in [m, M]$, we get

$$\varphi(A(g)) \leq \frac{M-A(g)}{M-m}\varphi(m) + \frac{A(g)-m}{M-m}\varphi(M) - \left(\frac{1}{2} - \frac{1}{M-m}\left|A(g) - \frac{m+M}{2}\right|\right)\delta_\varphi. \tag{2.3}$$

Now, applying inequality (2.3) on $g = m + M - f$ (and using linearity and normality of A), and then using inequality (2.2), we have

$$\begin{aligned} & \varphi(m + M - A(f)) \\ & \leq \frac{M-A(f)}{M-m}\varphi(M) + \frac{A(f)-m}{M-m}\varphi(m) - \left(\frac{1}{2} - \frac{1}{M-m}\left|A(f) - \frac{m+M}{2}\right|\right)\delta_\varphi \\ & = \varphi(m) + \varphi(M) - \left[\frac{M-A(f)}{M-m}\varphi(m) + \frac{A(f)-m}{M-m}\varphi(M)\right] \\ & \quad - \left(\frac{1}{2} - \frac{1}{M-m}\left|A(f) - \frac{m+M}{2}\right|\right)\delta_\varphi \\ & \leq \varphi(m) + \varphi(M) - A(\varphi(f)) \\ & \quad - A\left(\frac{1}{2} - \frac{1}{M-m}\left|f - \frac{m+M}{2}\right|\right)\delta_\varphi - \left(\frac{1}{2} - \frac{1}{M-m}\left|A(f) - \frac{m+M}{2}\right|\right)\delta_\varphi \\ & = \varphi(m) + \varphi(M) \\ & \quad - A(\varphi(f)) - \left[1 - \frac{1}{M-m}\left(A\left(\left|f - \frac{m+M}{2}\right|\right) + \left|A(f) - \frac{m+M}{2}\right|\right)\right]\delta_\varphi \\ & \leq \varphi(m) + \varphi(M) - A(\varphi(f)) - \left[1 - \frac{2}{M-m}A\left(\left|f - \frac{m+M}{2}\right|\right)\right]\delta_\varphi. \end{aligned}$$

The last inequality is obtained applying Jessen’s inequality to the continuous and convex function $|x|$ so that

$$\left|A(f) - \frac{m+M}{2}\right| = \left|A\left(f - \frac{m+M}{2}\right)\right| \leq A\left(\left|f - \frac{m+M}{2}\right|\right). \quad \square$$

Using Theorem 2 we can get an upper bound for the difference $A(\varphi(f)) - \varphi(A(f))$ obtained in [12].

COROLLARY 1. Let L satisfy $L1$, $L2$, $L3$ on a nonempty set E , and let A be a positive normalized linear functional. If φ is a continuous convex function on $[m, M]$, then for all $f \in L$ such that $\varphi(f), \varphi(m+M-f) \in L$, we have

$$A(\varphi(f)) - \varphi(A(f)) \leq \frac{1}{M-m} \left(A \left(\left| f - \frac{m+M}{2} \right| \right) + \left| A(f) - \frac{m+M}{2} \right| \right) \delta_\varphi,$$

where δ_φ is defined as in (2.1).

Proof. Theorem 2 gives us

$$A(\varphi(f)) \leq \varphi(m) + \varphi(M) - \varphi(A(f) - m + M) - \left[1 - \frac{1}{M-m} \left(A \left(\left| f - \frac{m+M}{2} \right| \right) + \left| A(f) - \frac{m+M}{2} \right| \right) \right] \delta_\varphi. \quad (2.4)$$

Since the function φ is convex, it follows

$$\varphi(m+M-A(f)) + \varphi(A(f)) \geq 2\varphi\left(\frac{M+m}{2}\right). \quad (2.5)$$

Combining inequalities (2.4) and (2.5) we obtain

$$\begin{aligned} & A(\varphi(f)) - \varphi(A(f)) \\ & \leq \varphi(m) + \varphi(M) - [\varphi(m+M-A(f)) + \varphi(A(f))] \\ & \quad - \left[1 - \frac{1}{M-m} \left(A \left(\left| f - \frac{m+M}{2} \right| \right) + \left| A(f) - \frac{m+M}{2} \right| \right) \right] \delta_\varphi \\ & \leq \varphi(m) + \varphi(M) - 2\varphi\left(\frac{M+m}{2}\right) \\ & \quad - \left[1 - \frac{1}{M-m} \left(A \left(\left| f - \frac{m+M}{2} \right| \right) + \left| A(f) - \frac{m+M}{2} \right| \right) \right] \delta_\varphi \\ & = \frac{1}{M-m} \left(A \left(\left| f - \frac{m+M}{2} \right| \right) + \left| A(f) - \frac{m+M}{2} \right| \right) \delta_\varphi. \quad \square \end{aligned}$$

3. Generalization on convex hulls in \mathbb{R}^k

We present a generalization of the Jessen-Mercer inequality for convex functions on convex hulls in \mathbb{R}^k .

Convex hull of vectors $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^k$ is the set

$$\left\{ \sum_{i=1}^n \alpha_i \mathbf{x}_i \mid \alpha_i \in \mathbb{R}, \alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1 \right\}$$

and it is represented by $K = \text{co}(\{\mathbf{x}_1, \dots, \mathbf{x}_n\})$.

Barycentric coordinates over K are continuous real functions $\lambda_1, \dots, \lambda_n$ on K with the following properties:

$$\begin{aligned} \lambda_i(\mathbf{x}) &\geq 0, \quad i = 1, \dots, n \\ \sum_{i=1}^n \lambda_i(\mathbf{x}) &= 1 \\ \mathbf{x} &= \sum_{i=1}^n \lambda_i(\mathbf{x})\mathbf{x}_i. \end{aligned} \tag{3.1}$$

If $\mathbf{x}_2 - \mathbf{x}_1, \dots, \mathbf{x}_n - \mathbf{x}_1$ are linearly independent vectors, then each $\mathbf{x} \in K$ can be written in the unique way as a convex combination of $\mathbf{x}_1, \dots, \mathbf{x}_n$ in the form (3.1).

We also consider k -simplex $S = \text{co}(\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k+1}\})$ in \mathbb{R}^k which is a convex hull of its vertices $\mathbf{v}_1, \dots, \mathbf{v}_{k+1} \in \mathbb{R}^k$, where vertices $\mathbf{v}_2 - \mathbf{v}_1, \dots, \mathbf{v}_{k+1} - \mathbf{v}_1 \in \mathbb{R}^k$ are linearly independent. In this case we denote k -simplex by $S = [\mathbf{v}_1, \dots, \mathbf{v}_{k+1}]$. Barycentric coordinates $\lambda_1, \lambda_2, \dots, \lambda_{k+1}$ over S are nonnegative linear polynomials on S and have a special form (see [2]).

With L^k we denote the linear class of functions $\mathbf{g}: E \rightarrow \mathbb{R}^k$ defined by

$$\mathbf{g}(t) = (g_1(t), \dots, g_k(t)), \quad g_i \in L \quad (i = 1, \dots, k).$$

For a given linear functional A , we also consider linear operator $\tilde{A} = (A, \dots, A): L^k \rightarrow \mathbb{R}^k$ defined by

$$\tilde{A}(\mathbf{g}) = (A(g_1), \dots, A(g_k)). \tag{3.2}$$

If $A(\mathbf{1}) = 1$ is satisfied, then using (A1) we also have

$$A3: A(f(\mathbf{g})) = f(\tilde{A}(\mathbf{g})) \text{ for every linear function } f \text{ on } \mathbb{R}^k.$$

For $n \in \mathbb{N}$ we denote

$$\Delta_{n-1} = \left\{ (\mu_1, \dots, \mu_n) : \mu_i \geq 0, i \in \{1, \dots, n\}, \sum_{i=1}^n \mu_i = 1 \right\}.$$

Also, if φ is a function defined on a convex subset $U \subseteq \mathbb{R}^k$ and $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n \in U$, we denote

$$S_\varphi^n(\mathbf{x}_1, \dots, \mathbf{x}_n) = \sum_{i=1}^n \varphi(\mathbf{x}_i) - n\varphi\left(\frac{1}{n} \sum_{i=1}^n \mathbf{x}_i\right).$$

Let's notice that δ_φ from Theorem B is equal to $S_\varphi^2(m, M)$.

Obviously, if φ is convex, $S_\varphi^n(\mathbf{x}_1, \dots, \mathbf{x}_n) \geq 0$.

The next variant of Jensen's inequality was proved by A. Matković and J. Pečarić in [9].

THEOREM 3. *Let U be a convex subset in \mathbb{R}^k , $\mathbf{x}_1, \dots, \mathbf{x}_n \in U$ and $\mathbf{y}_1, \dots, \mathbf{y}_m \in \text{co}(\{\mathbf{x}_1, \dots, \mathbf{x}_n\})$. If φ is a convex function on U , then the inequality*

$$\varphi\left(\frac{\sum_{i=1}^n P_i \mathbf{x}_i - \sum_{j=1}^m W_j \mathbf{y}_j}{P_n - W_m}\right) \leq \frac{\sum_{i=1}^n P_i \varphi(\mathbf{x}_i) - \sum_{j=1}^m W_j \varphi(\mathbf{y}_j)}{P_n - W_m} \tag{3.3}$$

holds for all positive real numbers p_1, \dots, p_n and w_1, \dots, w_m satisfying the condition

$$p_i \geq W_m \text{ for all } i = 1, \dots, n,$$

where $P_n = \sum_{i=1}^n p_i$ and $W_m = \sum_{j=1}^m w_j$.

Next theorem generalizes and improves Theorem 3.

THEOREM 4. *Let L satisfy properties L1, L2, L3 on a nonempty set E , A be a positive linear functional on L and \tilde{A} defined as in (3.2). Let $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^k$ and $K = \text{co}(\{\mathbf{x}_1, \dots, \mathbf{x}_n\})$. Let φ be a convex function on K and $\lambda_1, \dots, \lambda_n$ barycentric coordinates over K . Then for all $\mathbf{g} \in L^k$ such that $\mathbf{g}(E) \subset K$ and $\varphi(\mathbf{g}), \lambda_i(\mathbf{g}) \in L$, $i = 1, \dots, n$ and positive real numbers p_1, \dots, p_n , with $P_n = \sum_{i=1}^n p_i$, satisfying the condition*

$$p_i \geq A(\mathbf{1}) \text{ for all } i = 1, \dots, n, \tag{3.4}$$

we have

$$\begin{aligned} & \varphi\left(\frac{\sum_{i=1}^n p_i \mathbf{x}_i - \tilde{A}(\mathbf{g})}{P_n - A(\mathbf{1})}\right) \\ & \leq \frac{\sum_{i=1}^n p_i \varphi(\mathbf{x}_i) - \sum_{i=1}^n A(\lambda_i(\mathbf{g})) \varphi(\mathbf{x}_i) - \min_i \{p_i - A(\lambda_i(\mathbf{g}))\} S_\varphi^n(\mathbf{x}_1, \dots, \mathbf{x}_n)}{P_n - A(\mathbf{1})} \\ & \leq \frac{\sum_{i=1}^n p_i \varphi(\mathbf{x}_i) - A(\varphi(\mathbf{g})) - S_\varphi^n(\mathbf{x}_1, \dots, \mathbf{x}_n) [\min_i \{p_i - A(\lambda_i(\mathbf{g}))\} + A(\min_i \{\lambda_i(\mathbf{g})\})]}{P_n - A(\mathbf{1})}. \end{aligned} \tag{3.5}$$

Proof. For each $t \in E$ we have $\mathbf{g}(t) \in K$. Using barycentric coordinates we have $\lambda_i(\mathbf{g}(t)) \geq 0$, $i = 1, \dots, n$, $\sum_{i=1}^n \lambda_i(\mathbf{g}(t)) = 1$ and

$$\mathbf{g}(t) = \sum_{i=1}^n \lambda_i(\mathbf{g}(t)) \mathbf{x}_i.$$

Since φ is convex on K , then

$$\varphi(\mathbf{g}(t)) \leq \sum_{i=1}^n \lambda_i(\mathbf{g}(t)) \varphi(\mathbf{x}_i) - \min_i \{\lambda_i(\mathbf{g}(t))\} S_\varphi^n(\mathbf{x}_1, \dots, \mathbf{x}_n). \tag{3.6}$$

Applying positive linear functional A on (3.6) we get

$$A(\varphi(\mathbf{g})) \leq \sum_{i=1}^n A(\lambda_i(\mathbf{g})) \varphi(\mathbf{x}_i) - A\left(\min_i \{\lambda_i(\mathbf{g})\}\right) S_\varphi^n(\mathbf{x}_1, \dots, \mathbf{x}_n),$$

where

$$\sum_{i=1}^n A(\lambda_i(\mathbf{g})) = A\left(\sum_{i=1}^n \lambda_i(\mathbf{g})\right) = A(\mathbf{1})$$

and

$$A(\mathbf{1}) \geq A(\lambda_i(\mathbf{g})) \geq 0 \quad \text{for all } i = 1, \dots, n.$$

Also we have

$$\tilde{A}(\mathbf{g}) = \sum_{i=1}^n A(\lambda_i(\mathbf{g})) \mathbf{x}_i.$$

Now we can write

$$\begin{aligned} \frac{\sum_{i=1}^n p_i \mathbf{x}_i - \tilde{A}(\mathbf{g})}{P_n - A(\mathbf{1})} &= \frac{1}{P_n - A(\mathbf{1})} \left(\sum_{i=1}^n p_i \mathbf{x}_i - \sum_{i=1}^n A(\lambda_i(\mathbf{g})) \mathbf{x}_i \right) \\ &= \frac{1}{P_n - A(\mathbf{1})} \sum_{i=1}^n (p_i - A(\lambda_i(\mathbf{g}))) \mathbf{x}_i. \end{aligned}$$

We have

$$\frac{1}{P_n - A(\mathbf{1})} \sum_{i=1}^n (p_i - A(\lambda_i(\mathbf{g}))) = 1$$

and

$$\frac{1}{P_n - A(\mathbf{1})} (p_i - A(\lambda_i(\mathbf{g}))) \geq 0 \quad \text{for all } i = 1, \dots, n,$$

since

$$p_i \geq A(\mathbf{1}) \geq A(\lambda_i(\mathbf{g})) \quad \text{for all } i = 1, \dots, n.$$

Therefore, expression $\frac{\sum_{i=1}^n p_i \mathbf{x}_i - \tilde{A}(\mathbf{g})}{P_n - A(\mathbf{1})}$ is a convex combination of vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ and belongs to K .

Since φ is convex on K , we have

$$\begin{aligned} &\varphi \left(\frac{\sum_{i=1}^n p_i \mathbf{x}_i - \tilde{A}(\mathbf{g})}{P_n - A(\mathbf{1})} \right) \\ &= \varphi \left(\frac{1}{P_n - A(\mathbf{1})} \sum_{i=1}^n (p_i - A(\lambda_i(\mathbf{g}))) \mathbf{x}_i \right) \\ &\leq \frac{1}{P_n - A(\mathbf{1})} \sum_{i=1}^n (p_i - A(\lambda_i(\mathbf{g}))) \varphi(\mathbf{x}_i) - \min_i \left\{ \frac{p_i - A(\lambda_i(\mathbf{g}))}{P_n - A(\mathbf{1})} \right\} S_\varphi^n(\mathbf{x}_1, \dots, \mathbf{x}_n) \\ &= \frac{\sum_{i=1}^n p_i \varphi(\mathbf{x}_i) - \sum_{i=1}^n A(\lambda_i(\mathbf{g})) \varphi(\mathbf{x}_i) - \min_i \{p_i - A(\lambda_i(\mathbf{g}))\} S_\varphi^n(\mathbf{x}_1, \dots, \mathbf{x}_n)}{P_n - A(\mathbf{1})} \\ &\leq \frac{\sum_{i=1}^n p_i \varphi(\mathbf{x}_i) - A(\varphi(\mathbf{g})) - S_\varphi^n(\mathbf{x}_1, \dots, \mathbf{x}_n) [\min_i \{p_i - A(\lambda_i(\mathbf{g}))\} + A(\min_i \{\lambda_i(\mathbf{g})\})]}{P_n - A(\mathbf{1})}. \end{aligned}$$

□

Next Corollary shows that Theorem 4 is a generalization of Theorem 2 for convex hulls.

COROLLARY 2. Let L satisfy properties $L1$, $L2$, $L3$ on a nonempty set E and A be a positive normalized linear functional on L . Let φ be a convex function on an interval $I = [m, M] \subset \mathbb{R}$. Then for all $g \in L$ such that $g(E) \subset I$ and $\varphi(g) \in L$, we have

$$\begin{aligned} & \varphi(m+M-A(g)) \\ & \leq \frac{A(g)-m}{M-m} \varphi(m) + \frac{M-A(g)}{M-m} \varphi(M) - \left(\frac{1}{2} - \frac{1}{M-m} \left| A(g) - \frac{m+M}{2} \right| \right) S_{\varphi}^2(m, M) \\ & \leq \varphi(m) + \varphi(M) - A(\varphi(g)) \\ & \quad - \left[1 - \frac{1}{M-m} \left(\left| A(g) - \frac{m+M}{2} \right| + A \left(\left| g - \frac{m+M}{2} \right| \right) \right) \right] S_{\varphi}^2(m, M). \end{aligned} \quad (3.7)$$

Proof. For each $t \in E$ we have $g(t) \in I$.

Since interval $I = [m, M]$ is 1-simplex with vertices m and M , then the barycentric coordinates have the special form:

$$\lambda_1(g(t)) = \frac{M-g(t)}{M-m} \quad \text{and} \quad \lambda_2(g(t)) = \frac{g(t)-m}{M-m}.$$

Applying functional A we have

$$A(\lambda_1(g)) = \frac{M-A(g)}{M-m} \quad \text{and} \quad A(\lambda_2(g)) = \frac{A(g)-m}{M-m}. \quad (3.8)$$

Choosing $n = 2$, $p_1 = p_2 = 1$, $x_1 = m$, $x_2 = M$ from (3.5) it follows

$$\begin{aligned} & \varphi(m+M-A(g)) \\ & \leq \varphi(m) + \varphi(M) - \left[\frac{M-A(g)}{M-m} \varphi(m) + \frac{A(g)-m}{M-m} \varphi(M) \right] \\ & \quad - \left(\frac{1}{2} - \frac{1}{M-m} \left| A(g) - \frac{m+M}{2} \right| \right) \left[\varphi(m) + \varphi(M) - 2\varphi \left(\frac{m+M}{2} \right) \right] \\ & = \frac{A(g)-m}{M-m} \varphi(m) + \frac{M-A(g)}{M-m} \varphi(M) - \left(\frac{1}{2} - \frac{1}{M-m} \left| A(g) - \frac{m+M}{2} \right| \right) S_{\varphi}^2(m, M) \\ & \leq \varphi(m) + \varphi(M) - A(\varphi(g)) \\ & \quad - \left[\frac{1}{2} - \frac{1}{M-m} \left| A(g) - \frac{m+M}{2} \right| + A \left(\frac{1}{2} - \frac{1}{M-m} \left| g - \frac{m+M}{2} \right| \right) \right] S_{\varphi}^2(m, M) \\ & = \varphi(m) + \varphi(M) - A(\varphi(g)) \\ & \quad - \left[1 - \frac{1}{M-m} \left(\left| A(g) - \frac{m+M}{2} \right| + A \left(\left| g - \frac{m+M}{2} \right| \right) \right) \right] S_{\varphi}^2(m, M). \quad \square \end{aligned}$$

REMARK 2. The inequalities in (3.7) are also improvements of the inequalities obtained in [3].

THEOREM 5. Let L satisfy properties $L1$, $L2$, $L3$ on a nonempty set E , A be a positive linear functional on L and \tilde{A} defined as in (3.2). Let $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^k$ and

$K = \text{co}(\{\mathbf{x}_1, \dots, \mathbf{x}_n\})$. Let φ be a convex function on K and $\lambda_1, \dots, \lambda_n$ barycentric coordinates over K . Then for all $\mathbf{g} \in L^k$ such that $\mathbf{g}(E) \subset K$ and $\varphi(\mathbf{g}), \lambda_i(\mathbf{g}) \in L$, $i = 1, \dots, n$ and positive real numbers p_1, \dots, p_n satisfying the conditions $P_n - A(\mathbf{1}) > 0$, where $P_n = \sum_{i=1}^n p_i$, and

$$\frac{\sum_{i=1}^n p_i \mathbf{x}_i - \tilde{A}(\mathbf{g})}{P_n - A(\mathbf{1})} \in K, \tag{3.9}$$

we have

$$\begin{aligned} \varphi \left(\frac{\sum_{i=1}^n p_i \mathbf{x}_i - \tilde{A}(\mathbf{g})}{P_n - A(\mathbf{1})} \right) &\geq \frac{P_n \varphi \left(\frac{1}{P_n} \sum_{i=1}^n p_i \mathbf{x}_i \right) - A(\mathbf{1}) \varphi \left(\frac{1}{A(\mathbf{1})} \tilde{A}(\mathbf{g}) \right)}{P_n - A(\mathbf{1})} \\ &\geq \frac{P_n \varphi \left(\frac{1}{P_n} \sum_{i=1}^n p_i \mathbf{x}_i \right) - \sum_{i=1}^n A(\lambda_i(\mathbf{g})) \varphi(\mathbf{x}_i) + \min_i \{A(\lambda_i(\mathbf{g}))\} S_\varphi^n(\mathbf{x}_1, \dots, \mathbf{x}_n)}{P_n - A(\mathbf{1})}. \end{aligned} \tag{3.10}$$

Proof. For each $t \in E$ we have $\mathbf{g}(t) \in K$. Using barycentric coordinates we have $\lambda_i(\mathbf{g}(t)) \geq 0$, $i = 1, \dots, n$, $\sum_{i=1}^n \lambda_i(\mathbf{g}(t)) = 1$ and

$$\mathbf{g}(t) = \sum_{i=1}^n \lambda_i(\mathbf{g}(t)) \mathbf{x}_i.$$

Also we have

$$\tilde{A}(\mathbf{g}) = \sum_{i=1}^n A(\lambda_i(\mathbf{g})) \mathbf{x}_i.$$

We can easily see that

$$\frac{1}{A(\mathbf{1})} \tilde{A}(\mathbf{g}) = \frac{1}{A(\mathbf{1})} \sum_{i=1}^n A(\lambda_i(\mathbf{g})) \mathbf{x}_i \in K,$$

since

$$\frac{1}{A(\mathbf{1})} \sum_{i=1}^n A(\lambda_i(\mathbf{g})) = 1 \quad \text{and} \quad \frac{1}{A(\mathbf{1})} A(\lambda_i(\mathbf{g})) \geq 0, \quad i = 1, \dots, n.$$

Since φ is convex on K , then using Lemma 1

$$\varphi \left(\frac{1}{A(\mathbf{1})} \tilde{A}(\mathbf{g}) \right) \leq \frac{1}{A(\mathbf{1})} \sum_{i=1}^n A(\lambda_i(\mathbf{g})) \varphi(\mathbf{x}_i) - \min_i \left\{ \frac{A(\lambda_i(\mathbf{g}))}{A(\mathbf{1})} \right\} S_\varphi^n(\mathbf{x}_1, \dots, \mathbf{x}_n). \tag{3.11}$$

Using Theorem B and (3.11) we have

$$\begin{aligned} \varphi \left(\frac{P_n \left(\frac{1}{P_n} \sum_{i=1}^n p_i \mathbf{x}_i \right) - A(\mathbf{1}) \left(\frac{1}{A(\mathbf{1})} \tilde{A}(\mathbf{g}) \right)}{P_n - A(\mathbf{1})} \right) &\geq \frac{P_n \varphi \left(\frac{1}{P_n} \sum_{i=1}^n p_i \mathbf{x}_i \right) - A(\mathbf{1}) \varphi \left(\frac{1}{A(\mathbf{1})} \tilde{A}(\mathbf{g}) \right)}{P_n - A(\mathbf{1})} \\ &\geq \frac{P_n \varphi \left(\frac{1}{P_n} \sum_{i=1}^n p_i \mathbf{x}_i \right) - A(\mathbf{1}) \frac{1}{A(\mathbf{1})} \sum_{i=1}^n A(\lambda_i(\mathbf{g})) \varphi(\mathbf{x}_i) + \min_i \{A(\lambda_i(\mathbf{g}))\} S_\varphi^n(\mathbf{x}_1, \dots, \mathbf{x}_n)}{P_n - A(\mathbf{1})}. \end{aligned}$$

□

REMARK 3. If positive real numbers p_1, \dots, p_n satisfy condition (3.4), then condition (3.9) is also satisfied since K is convex set. Hence (3.5) can be extended as follows

$$\begin{aligned} & \frac{P_n \varphi \left(\frac{1}{P_n} \sum_{i=1}^n p_i \mathbf{x}_i \right) - \sum_{i=1}^n A(\lambda_i(\mathbf{g})) \varphi(\mathbf{x}_i) + \min_i \{A(\lambda_i(\mathbf{g}))\} S_\varphi^n(\mathbf{x}_1, \dots, \mathbf{x}_n)}{P_n - A(\mathbf{1})} \\ & \leq \frac{P_n \varphi \left(\frac{1}{P_n} \sum_{i=1}^n p_i \mathbf{x}_i \right) - A(\mathbf{1}) \varphi \left(\frac{1}{A(\mathbf{1})} \tilde{A}(\mathbf{g}) \right)}{P_n - A(\mathbf{1})} \\ & \leq \varphi \left(\frac{\sum_{i=1}^n p_i \mathbf{x}_i - \tilde{A}(\mathbf{g})}{P_n - A(\mathbf{1})} \right) \\ & \leq \frac{\sum_{i=1}^n p_i \varphi(\mathbf{x}_i) - \sum_{i=1}^n A(\lambda_i(\mathbf{g})) \varphi(\mathbf{x}_i) - \min_i \{p_i - A(\lambda_i(\mathbf{g}))\} S_\varphi^n(\mathbf{x}_1, \dots, \mathbf{x}_n)}{P_n - A(\mathbf{1})} \\ & \leq \frac{\sum_{i=1}^n p_i \varphi(\mathbf{x}_i) - A(\varphi(\mathbf{g})) - S_\varphi^n(\mathbf{x}_1, \dots, \mathbf{x}_n) [\min_i \{p_i - A(\lambda_i(\mathbf{g}))\} + A(\min_i \{\lambda_i(\mathbf{g})\})]}{P_n - A(\mathbf{1})}. \end{aligned}$$

COROLLARY 3. Let L satisfy properties L1, L2, L3 on a nonempty set E and A be a positive normalized linear functional on L . Let φ be a convex function on an interval $I = [m, M] \subset \mathbb{R}$. Then for all $g \in L$ such that $g(E) \subset I$ and $\varphi(g) \in L$, we have

$$\begin{aligned} \varphi(m + M - A(g)) & \geq 2\varphi\left(\frac{m + M}{2}\right) - \varphi(A(g)) \\ & \geq 2\varphi\left(\frac{m + M}{2}\right) - \left[\frac{M - A(g)}{M - m} \varphi(m) + \frac{A(g) - m}{M - m} \varphi(M) \right] \\ & \quad + \left(\frac{1}{2} - \frac{1}{M - m} \left| A(g) - \frac{m + M}{2} \right| \right) S_\varphi^2(m, M). \end{aligned} \tag{3.12}$$

Proof. Choosing $n = 2$, $x_1 = m$, $x_2 = M$, $p_1 = p_2 = 1$ and using (3.8), the inequalities in (3.12) easily follow from (3.10). \square

Next we give generalizations of Corollary 2 and Corollary 3 for convex functions defined on k -simplices in \mathbb{R}^k .

Let S be a k -simplex in \mathbb{R}^k with vertices $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k+1} \in \mathbb{R}^k$. The barycentric coordinates $\lambda_1, \dots, \lambda_{k+1}$ over S are nonnegative linear polynomials which satisfy Lagrange’s property

$$\lambda_i(\mathbf{v}_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

It is known (see [2]) that for each $\mathbf{x} \in S$ barycentric coordinates $\lambda_1(\mathbf{x}), \dots, \lambda_{k+1}(\mathbf{x})$ have the form

$$\lambda_1(\mathbf{x}) = \frac{\text{Vol}_k([\mathbf{x}, \mathbf{v}_2, \dots, \mathbf{v}_{k+1}])}{\text{Vol}_k(S)},$$

$$\begin{aligned} \lambda_2(\mathbf{x}) &= \frac{\text{Vol}_k([\mathbf{v}_1, \mathbf{x}, \mathbf{v}_3, \dots, \mathbf{v}_{k+1}])}{\text{Vol}_k(S)}, \\ &\vdots \\ \lambda_{k+1}(\mathbf{x}) &= \frac{\text{Vol}_k([\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{x}])}{\text{Vol}_k(S)}, \end{aligned} \tag{3.13}$$

where $\text{Vol}_k(F)$ denotes the k -dimensional Lebesgue measure of a measurable set $F \subset \mathbb{R}^k$. Here, for example, $[\mathbf{v}_1, \mathbf{x}, \dots, \mathbf{v}_{k+1}]$ denotes the subsimplex obtained by replacing \mathbf{v}_2 by \mathbf{x} , i.e., the subsimplex opposite to \mathbf{v}_2 , when adding \mathbf{x} as a new vertex.

COROLLARY 4. *Let L satisfy properties $L1, L2, L3$ on a nonempty set E , A be a positive normalized linear functional on L and \tilde{A} defined as in (3.2). Let φ be a convex function on k -simplex $S = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k+1}]$ in \mathbb{R}^k and $\lambda_1, \dots, \lambda_{k+1}$ barycentric coordinates over S . Then for all $\mathbf{g} \in L^k$ such that $\mathbf{g}(E) \subset S$ and $\varphi(\mathbf{g}) \in L$ we have*

$$\begin{aligned} &\frac{(k+1)\varphi\left(\frac{1}{k+1}\sum_{i=1}^{k+1}\mathbf{v}_i\right) - \sum_{i=1}^{k+1}\lambda_i(\tilde{A}(\mathbf{g}))\varphi(\mathbf{v}_i) + \min_i\{\lambda_i(\tilde{A}(\mathbf{g}))\}S_\varphi^{k+1}(\mathbf{v}_1, \dots, \mathbf{v}_{k+1})}{k} \\ &\leq \frac{(k+1)\varphi\left(\frac{1}{k+1}\sum_{i=1}^{k+1}\mathbf{v}_i\right) - \varphi(\tilde{A}(\mathbf{g}))}{k} \\ &\leq \varphi\left(\frac{\sum_{i=1}^{k+1}\mathbf{v}_i - \tilde{A}(\mathbf{g})}{k}\right) \\ &\leq \frac{\sum_{i=1}^{k+1}\varphi(\mathbf{v}_i) - \sum_{i=1}^{k+1}\lambda_i(\tilde{A}(\mathbf{g}))\varphi(\mathbf{v}_i) - \min_i\{1 - \lambda_i(\tilde{A}(\mathbf{g}))\}S_\varphi^{k+1}(\mathbf{v}_1, \dots, \mathbf{v}_{k+1})}{k} \\ &\leq \frac{\sum_{i=1}^{k+1}\varphi(\mathbf{v}_i) - A(\varphi(\mathbf{g})) - S_\varphi^{k+1}(\mathbf{v}_1, \dots, \mathbf{v}_{k+1})\left[\min_i\{1 - \lambda_i(\tilde{A}(\mathbf{g}))\} + A(\min_i\{\lambda_i(\mathbf{g})\})\right]}{k}. \end{aligned} \tag{3.14}$$

Proof. Since barycentric coordinates $\lambda_1, \dots, \lambda_{k+1}$ over k -simplex S in \mathbb{R}^k are nonnegative linear polynomials, then $A(\lambda_i(\mathbf{g})) = \lambda_i(A(\mathbf{g}))$ for all $i = 1, \dots, k+1$.

Choosing $\mathbf{x}_i = \mathbf{v}_i$ for all $i = 1, \dots, k+1$ and $p_1 = p_2 = \dots = p_{k+1} = 1$, the inequalities in (3.14) easily follow from (3.5) and (3.10). \square

REMARK 4. As a special case of Corollary 4 for $k = 1$ and if we take p and q nonnegative real numbers such that $A(g) = \frac{pm + qM}{p + q}$ we get right hand side of the inequality (2.3) in [7].

REMARK 5. Using the same technique and the same special case as in Example 1 and Remark 8 in [13], from (3.14) we get the same results, that is, the k -dimensional version of the Hammer-Bullen inequality, namely

$$\frac{1}{|S|} \int_S f(t) dt - f(\mathbf{v}^*) \leq \frac{k}{k+1} \sum_{i=1}^{k+1} f(\mathbf{v}_i) - \frac{k}{|S|} \int_S f(t) dt,$$

and, as a special case in one dimension, an improvement of classical Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2} - \frac{1}{4}S_f^2(a,b).$$

4. n -exponential convexity and exponential convexity of the Jessen-Mercer differences, applications to Stolarsky type means

Motivated by Theorems 1 and 2, we define two functionals $\Phi_i: L_f \rightarrow \mathbb{R}, i = 1, 2$, by

$$\begin{aligned} \Phi_1(\varphi) &= \varphi(m) + \varphi(M) - \varphi(m+M-A(f)) - A(\varphi(f)) \\ &\quad - \left[1 - \frac{2}{M-m}A\left(\left|f - \frac{m+M}{2}\right|\right)\right] \delta_\varphi \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \Phi_2(\varphi) &= \varphi(m) + \varphi(M) - \varphi(m+M-A(f)) - A(\varphi(f)) \\ &\quad - \left[1 - \frac{1}{M-m} \left(A\left(\left|f - \frac{m+M}{2}\right|\right) + \left|\frac{m+M}{2} - A(f)\right|\right)\right] \delta_\varphi, \end{aligned} \tag{4.2}$$

where A, f and δ_φ are as in Theorem 1, $L_f = \{\varphi: I \rightarrow \mathbb{R}: \varphi(f), \varphi(m+M-f) \in L\}, [m, M] \subseteq I$. Obviously, Φ_1 and Φ_2 are linear.

If φ is additionally continuous and convex then Theorems 1 and 2 imply $\Phi_i(f) \geq 0, i = 1, 2$.

In the following with φ_0 we denote the function defined by $\varphi_0(x) = x^2$ on whatever domain we need.

Now, we give Lagrange and Cauchy mean value type theorems for the functionals $\Phi_i, i = 1, 2$.

THEOREM 6. *Let L satisfy L1, L2 and L3 on a nonempty set E and let A be a positive normalized linear functional on L . Let $f \in L$ be such that $\varphi_0 \in L_f, f(E) \in [m, M], [m, M] \subseteq I$ and let $\varphi \in C^2(I)$ be such that $\varphi \in L_f$. If Φ_1 and Φ_2 are linear functionals defined as in (4.1) and (4.2) then there exist $\xi_i \in [m, M], i = 1, 2$ such that*

$$\Phi_i(\varphi) = \frac{\varphi''(\xi_i)}{2} \Phi_i(\varphi_0), \quad i = 1, 2.$$

Proof. We give a proof for the functional Φ_1 . Since $\varphi \in C^2(I)$ there exist real numbers $a = \min_{x \in [m, M]} \varphi''(x)$ and $b = \max_{x \in [m, M]} \varphi''(x)$. It is easy to show that the functions φ_1, φ_2 defined by

$$\varphi_1(x) = \frac{b}{2}x^2 - \varphi(x), \quad \varphi_2(x) = f(x) - \frac{a}{2}x^2$$

are continuous and convex, therefore $\Phi_1(\varphi_1) \geq 0, \Phi_1(\varphi_2) \geq 0$. This implies

$$\frac{a}{2}\Phi_1(\varphi_0) \leq \Phi_1(\varphi) \leq \frac{b}{2}\Phi_1(\varphi_0).$$

If $\Phi_1(\varphi_0) = 0$, there is nothing to prove. Suppose $\Phi_1(\varphi_0) > 0$. We have

$$a \leq \frac{2\Phi_1(\varphi)}{\Phi_1(\varphi_0)} \leq b.$$

Hence, there exists $\xi_1 \in [m, M]$ such that

$$\Phi_1(\varphi) = \frac{\varphi''(\xi_1)}{2}\Phi_1(\varphi_0). \quad \square$$

THEOREM 7. *Let L satisfy L1, L2 and L3 on a non-empty set E and let A be a positive normalized linear functional on L . Let $f \in L$ be such that $\varphi_0 \in L_f, f(E) \in [m, M], [m, M] \subseteq I$ and $\varphi_1, \varphi_2 \in C^2(I)$ such that $\varphi_1, \varphi_2 \in L_f$. If Φ_1 and Φ_2 are linear functionals defined as in (4.1) and (4.2) then there exist $\xi_i \in [m, M], i = 1, 2$ such that*

$$\frac{\Phi_i(\varphi_1)}{\Phi_i(\varphi_2)} = \frac{\varphi_1''(\xi_i)}{\varphi_2''(\xi_i)}, \quad i = 1, 2$$

provided that the denominators are non-zero.

Proof. We give a proof for the functional Φ_1 . Define $\varphi_3 \in C^2([m, M])$ by

$$\varphi_3 = c_1\varphi_1 - c_2\varphi_2, \quad \text{where } c_1 = \Phi_1(\varphi_2), \quad c_2 = \Phi_1(\varphi_1).$$

Using Theorem 6 we get that there exists $\xi_1 \in [m, M]$ such that

$$\left(c_1 \frac{\varphi_1''(\xi_1)}{2} - c_2 \frac{\varphi_2''(\xi_1)}{2} \right) \Phi_1(\varphi_0) = 0.$$

Since $\Phi_1(\varphi_0) \neq 0$, (otherwise we have a contradiction with $\Phi_1(\varphi_2) \neq 0$, by Theorem 6), we obtain

$$\frac{\Phi_1(\varphi_1)}{\Phi_1(\varphi_2)} = \frac{\varphi_1''(\xi_1)}{\varphi_2''(\xi_1)}. \quad \square$$

Next we introduce some function properties which are going to be explored here and immediately after that we give some characterizations of these properties.

DEFINITION 1. A function $\psi: I \rightarrow \mathbb{R}$ is n -exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^n \xi_i \xi_j \psi \left(\frac{x_i + x_j}{2} \right) \geq 0$$

holds for all choices $\xi_i \in \mathbb{R}$ and $x_i \in I, i = 1, \dots, n$.

A function $\psi: I \rightarrow \mathbb{R}$ is n -exponentially convex if it is n -exponentially convex in the Jensen sense and continuous on I .

REMARK 6. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact non-negative functions. Also, n -exponentially convex functions in the Jensen sense are k -exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leq n$.

By definition of positive semi-definite matrices and some basic linear algebra we have the following proposition.

PROPOSITION 1. *If ψ is an n -exponentially convex in the Jensen sense, then the matrix $\left[\psi \left(\frac{x_i + x_j}{2} \right) \right]_{i,j=1}^k$ is positive semi-definite for all $k \in \mathbb{N}$, $k \leq n$. Particularly, $\det \left[\psi \left(\frac{x_i + x_j}{2} \right) \right]_{i,j=1}^k \geq 0$ for all $k \in \mathbb{N}$, $k \leq n$.*

DEFINITION 2. A function $\psi : I \rightarrow \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n -exponentially convex in the Jensen sense for all $n \in \mathbb{R}$.

A function $\psi : I \rightarrow \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

REMARK 7. It is known (and easy to show) that a function $\psi : I \rightarrow \mathbb{R}^+$ is log-convex in the Jensen sense if and only if

$$\alpha^2 \psi(x) + 2\alpha\beta \psi\left(\frac{x+y}{2}\right) + \beta^2 \psi(y) \geq 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen-sense if and only if it is 2-exponentially convex in the Jensen sense. Also, using basic convexity theory, it follows that a positive function is log-convex if and only if it is 2-exponentially convex.

We will also need the following result (see for example [14]).

PROPOSITION 2. *If Ψ is a convex function on I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$ then the following inequality is valid*

$$\frac{\Psi(x_2) - \Psi(x_1)}{x_2 - x_1} \leq \frac{\Psi(y_2) - \Psi(y_1)}{y_2 - y_1}. \tag{4.3}$$

If Ψ is concave on I the inequality reverses.

When dealing with functions with different degree of smoothness divided differences are found to be very useful.

DEFINITION 3. The second order divided difference of a function $f : I \rightarrow \mathbb{R}$ at mutually different points $y_0, y_1, y_2 \in I$ is defined recursively by

$$[y_i; f] = f(y_i), \quad i = 0, 1, 2$$

$$\begin{aligned}
 [y_i, y_{i+1}; f] &= \frac{f(y_{i+1}) - f(y_i)}{y_{i+1} - y_i}, \quad i = 0, 1 \\
 [y_0, y_1, y_2; f] &= \frac{[y_1, y_2; f] - [y_0, y_1; f]}{y_2 - y_0}.
 \end{aligned}
 \tag{4.4}$$

REMARK 8. The value $[y_0, y_1, y_2; f]$ is independent of the order of the points y_0, y_1 and y_2 . This definition may be extended to include the case in which some or all the points coincide. Namely, taking the limit $y_1 \rightarrow y_0$ in (4.4), we get

$$\lim_{y_1 \rightarrow y_0} [y_0, y_1, y_2; f] = [y_0, y_0, y_2; f] = \frac{f(y_2) - f(y_0) - f'(y_0)(y_2 - y_0)}{(y_2 - y_0)^2}, \quad y_2 \neq y_0$$

provided that f' exists, and furthermore, taking the limits $y_i \rightarrow y_0, i = 1, 2$ in (4.4), we get

$$\lim_{y_2 \rightarrow y_0} \lim_{y_1 \rightarrow y_0} [y_0, y_1, y_2; f] = [y_0, y_0, y_0; f] = \frac{f''(y_0)}{2}$$

provided that f'' exists.

We use an idea from [5] to give an elegant method of producing an n -exponentially convex and exponentially convex functions applying the functionals Φ_1 and Φ_2 to a given family of functions with the same property.

THEOREM 8. Let $\Phi_i, i = 1, 2$, be linear functionals defined as in (4.1) and (4.2). Let $\Upsilon = \{\varphi_s : s \in J\}$, where J is an interval in \mathbb{R} , be a family of functions defined on an open interval I such that $\Upsilon \subseteq L_f$ and that the function $s \mapsto [y_0, y_1, y_2; \varphi_s]$ is n -exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Then $s \mapsto \Phi_i(\varphi_s)$ is an n -exponentially convex function in the Jensen sense on J . If the function $s \mapsto \Phi_i(\varphi_s)$ is also continuous on J then it is n -exponentially convex on J .

Proof. For $\xi_i \in \mathbb{R}, i = 1, \dots, n$ and $s_i \in J, i = 1, \dots, n$, we define the function $\chi : I \rightarrow \mathbb{R}$ by

$$\chi(y) = \sum_{i,j=1}^n \xi_i \xi_j \varphi_{\frac{s_i+s_j}{2}}(y).$$

Using the assumption that the function $s \mapsto [y_0, y_1, y_2; \varphi_s]$ is n -exponentially convex in the Jensen sense we obtain

$$[y_0, y_1, y_2; \chi] = \sum_{i,j=1}^n \xi_i \xi_j [y_0, y_1, y_2; \varphi_{\frac{s_i+s_j}{2}}] \geq 0,$$

which in turn implies that χ is a convex (and continuous) function on I , therefore $\Phi_i(\chi) \geq 0, i = 1, 2$. Hence

$$\sum_{i,j=1}^n \xi_i \xi_j \Phi_i(\varphi_{\frac{s_i+s_j}{2}}) \geq 0.$$

We conclude that the function $s \mapsto \Phi_i(\varphi_s)$ is n -exponentially convex on J in the Jensen sense. If the function $s \mapsto \Phi_i(\varphi_s)$ is also continuous on J , then $s \mapsto \Phi_i(\varphi_s)$ is n -exponentially convex by definition. \square

The following corollary is an immediate consequence of Theorem 8.

COROLLARY 5. *Let $\Phi_i, i = 1, 2$, be linear functionals defined as in (4.1) and (4.2). Let $Y = \{\varphi_s : s \in J\}$, where J is an interval in \mathbb{R} , be a family of functions defined on an open interval I such that $Y \subseteq L_f$ and that the function $s \mapsto [y_0, y_1, y_2; \varphi_s]$ is exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Then $s \mapsto \Phi_i(\varphi_s)$ is an exponentially convex function in the Jensen sense on J . If the function $s \mapsto \Phi_i(\varphi_s)$ is continuous on J then it is exponentially convex on J .*

COROLLARY 6. *Let $\Phi_i, i = 1, 2$, be linear functionals defined as in (4.1) and (4.2). Let $\Omega = \{\varphi_s : s \in J\}$, where J is an interval in \mathbb{R} , be a family of functions defined on an open interval I such that $\Omega \subseteq L_f$ and that the function $s \mapsto [y_0, y_1, y_2; \varphi_s]$ is 2-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Then the following statements hold:*

- (i) *If the function $s \mapsto \Phi_i(\varphi_s)$ is continuous on J then it is 2-exponentially convex function on J . If $s \mapsto \Phi_i(\varphi_s)$ is additionally strictly positive than it is also log-convex on J .*
- (ii) *If the function $s \mapsto \Phi_i(\varphi_s)$ is strictly positive and differentiable on J then for every $s, q, u, v \in J$, such that $s \leq u$ and $q \leq v$, we have*

$$\mu_{s,q}(\Phi_i, \Omega) \leq \mu_{u,v}(\Phi_i, \Omega), \quad i = 1, 2, \tag{4.5}$$

where

$$\mu_{s,q}(\Phi_i, \Omega) = \begin{cases} \left(\frac{\Phi_i(\varphi_s)}{\Phi_i(\varphi_q)} \right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp \left(\frac{\frac{d}{ds} \Phi_i(\varphi_s)}{\Phi_i(\varphi_s)} \right), & s = q. \end{cases} \tag{4.6}$$

for $\varphi_s, \varphi_q \in \Omega$ ($\mu_{s,q}(\Phi_i, \Omega), i = 1, 2$ are the Stolarsky type means).

Proof. (i) This is an immediate consequence of Theorem 8 and Remark 7.

(ii) Since by (i) the function $s \mapsto \Phi_i(\varphi_s)$ is log-convex on J , that is, the function $s \mapsto \log \Phi_i(\varphi_s)$ is convex on J . Applying Proposition 2 we get

$$\frac{\log \Phi_i(\varphi_s) - \log \Phi_i(\varphi_q)}{s - q} \leq \frac{\log \Phi_i(\varphi_u) - \log \Phi_i(\varphi_v)}{u - v} \tag{4.7}$$

for $s \leq u, q \leq v, s \neq q, u \neq v$, and therefrom conclude that

$$\mu_{s,q}(\Phi_i, \Omega) \leq \mu_{u,v}(\Phi_i, \Omega), \quad i = 1, 2.$$

Cases $s = q$ and $u = v$ follow from (4.7) as limit cases. \square

REMARK 9. Note that the results from Theorem 8, Corollary 5, Corollary 6 still hold when two of the points $y_0, y_1, y_2 \in I$ coincide, say $y_1 = y_0$, for a family of differentiable functions φ_s such that the function $s \mapsto [y_0, y_1, y_2; \varphi_s]$ is n -exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs are obtained by recalling Remark 8 and suitable characterization of convexity.

Now, we present several families of functions which fulfil the conditions of Theorem 8, Corollary 5 and Corollary 6 (and Remark 9). This enables us to construct a large family of functions which are exponentially convex. For a discussion related to this problem see [4].

In the rest of the section we consider only Φ_1 and Φ_2 defined as in (4.1) and (4.2) with A which is continuous and f such that compositions with any function from the chosen family Ω_i as well as with other functions which appear as arguments of Φ_1 and Φ_2 remain in L .

EXAMPLE 1. Consider a family of functions

$$\Omega_1 = \{g_s: \mathbb{R} \rightarrow [0, \infty): s \in \mathbb{R}\}$$

defined by

$$g_s(x) = \begin{cases} \frac{1}{s^2} e^{sx}, & s \neq 0, \\ \frac{1}{2} x^2, & s = 0. \end{cases}$$

We have $\frac{d^2 g_s}{dx^2}(x) = e^{sx} > 0$ which shows that g_s is convex on \mathbb{R} for every $s \in \mathbb{R}$ and $s \mapsto \frac{d^2 g_s}{dx^2}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 8 we also have that $s \mapsto [y_0, y_1, y_2; g_s]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Theorem 5 we conclude that $s \mapsto \Phi_i(g_s)$, $i = 1, 2$, are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous (although mapping $s \mapsto g_s$ is not continuous for $s = 0$), so they are exponentially convex.

For this family of functions, $\mu_{s,q}(\Phi_i, \Omega_1)$, $i = 1, 2$, from (4.6) become

$$\mu_{s,q}(\Phi_i, \Omega_1) = \begin{cases} \left(\frac{\Phi_i(g_s)}{\Phi_i(g_q)} \right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp \left(\frac{\Phi_i(id \cdot g_s)}{\Phi_i(g_s)} - \frac{2}{s} \right), & s = q \neq 0, \\ \exp \left(\frac{\Phi_i(id \cdot g_0)}{3\Phi_i(g_0)} \right), & s = q = 0, \end{cases}$$

and using (4.5) they are monotonous functions in parameters s and q .

Using Theorem 7 it follows that for $i = 1, 2$

$$M_{s,q}(\Phi_i, \Omega_1) = \log \mu_{s,q}(\Phi_i, \Omega_1)$$

satisfy $m \leq M_{s,q}(\Phi_i, \Omega_1) \leq M$, which shows that $M_{s,q}(\Phi_i, \Omega_1)$ are means (of a function g). Notice that by (4.5) they are monotonous.

EXAMPLE 2. Consider a family of functions

$$\Omega_2 = \{f_s : (0, \infty) \rightarrow \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0, 1, \\ -\log x, & s = 0, \\ x \log x, & s = 1. \end{cases}$$

Here, $\frac{d^2 f_s}{dx^2}(x) = x^{s-2} = e^{(s-2)\ln x} > 0$ which shows that f_s is convex for $x > 0$ and $s \mapsto \frac{d^2 f_s}{dx^2}(x)$ is exponentially convex by definition. Arguing as in Example 1 we get that the mappings $s \mapsto \Phi_i(g_s)$, $i = 1, 2$ are exponentially convex. Functions (4.6) in this case are equal to:

$$\mu_{s,q}(\Phi_i, \Omega_2) = \begin{cases} \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{1-2s}{s(s-1)} - \frac{\Phi_i(f_s f_0)}{\Phi_i(f_s)}\right), & s = q \neq 0, 1, \\ \exp\left(1 - \frac{\Phi_i(f_0^2)}{2\Phi_i(f_0)}\right), & s = q = 0, \\ \exp\left(-1 - \frac{\Phi_i(f_0 f_1)}{2\Phi_i(f_1)}\right), & s = q = 1. \end{cases}$$

If Φ_i is positive, then Theorem 7 applied for $f = f_s \in \Omega_2$ and $g = f_q \in \Omega_2$ yields that there exists $\xi \in [m, M]$ such that

$$\xi^{s-q} = \frac{\Phi_i(f_s)}{\Phi_i(f_q)}.$$

Since the function $\xi \mapsto \xi^{s-q}$ is invertible for $s \neq q$, we then have

$$m \leq \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}} \leq M, \tag{4.8}$$

which together with the fact that $\mu_{s,q}(\Phi_i, \Omega_2)$ is continuous, symmetric and monotonous (by (4.5)), shows that $\mu_{s,q}(\Phi_i, \Omega_2)$ is a mean (of a function f).

EXAMPLE 3. Consider a family of functions

$$\Omega_3 = \{h_s : (0, \infty) \rightarrow (0, \infty) : s \in (0, \infty)\}$$

defined by

$$h_s(x) = \begin{cases} \frac{s^{-x}}{\ln^2 s}, & s \neq 1, \\ \frac{x^2}{2}, & s = 1. \end{cases}$$

Since $s \mapsto \frac{d^2 h_s}{dx^2}(x) = s^{-x}$ is the Laplace transform of a non-negative function (see [15]) it is exponentially convex. Obviously h_s are convex functions for every $s > 0$.

For this family of functions, $\mu_{s,q}(\Phi_i, \Omega_3)$, from (4.6) becomes

$$\mu_{s,q}(\Phi_i, \Omega_3) = \begin{cases} \left(\frac{\Phi_i(h_s)}{\Phi_i(h_q)} \right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp \left(-\frac{\Phi_i(id \cdot h_s)}{s\Phi_i(h_s)} - \frac{2}{s \ln s} \right), & s = q \neq 1, \\ \exp \left(-\frac{2\Phi_i(id \cdot h_1)}{3\Phi_i(h_1)} \right), & s = q = 1, \end{cases}$$

and it is monotonous in parameters s and q by (4.5).

Using Theorem 7, it follows that

$$M_{s,q}(\Phi_i, \Omega_3) = -L(s, q) \log \mu_{s,q}(\Phi_i, \Omega_3),$$

satisfies $m \leq M_{s,q}(\Phi_i, \Omega_3) \leq M$, which shows that $M_{s,q}(\Phi_i, \Omega_3)$ is a mean (of a function h). $L(s, q)$ is the logarithmic mean defined by $L(s, q) = \frac{s-q}{\log s - \log q}$, $s \neq q$, $L(s, s) = s$.

EXAMPLE 4. Consider a family of functions

$$\Omega_4 = \{k_s : (0, \infty) \rightarrow (0, \infty) : s \in (0, \infty)\}$$

defined by

$$k_s(x) = \frac{e^{-x\sqrt{s}}}{s}.$$

Since $s \mapsto \frac{d^2 k_s}{dx^2}(x) = e^{-x\sqrt{s}}$ is the Laplace transform of a non-negative function (see [15]) it is exponentially convex. Obviously k_s are convex functions for every $s > 0$.

For this family of functions, $\mu_{s,q}(\Phi_i, \Omega_4)$ from (4.6) becomes

$$\mu_{s,q}(\Phi_i, \Omega_4) = \begin{cases} \left(\frac{\Phi_i(k_s)}{\Phi_i(k_q)} \right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp \left(-\frac{\Phi_i(id \cdot k_s)}{2\sqrt{s}\Phi_i(k_s)} - \frac{1}{s} \right), & s = q, \end{cases}$$

and it is monotonous function in parameters s and q by (4.5).

Using Theorem 7, it follows that

$$M_{s,q}(\Phi_i, \Omega_4) = -(\sqrt{s} + \sqrt{q}) \log \mu_{s,q}(\Phi_i, \Omega_4)$$

satisfies $m \leq M_{s,q}(\Phi_i, \Omega_4) \leq M$, which shows that $M_{s,q}(\Phi_i, \Omega_4)$ is a mean (of a function k).

REFERENCES

[1] PAUL R. BEESACK, JOSIP E. PEČARIĆ, *On Jessen's inequality for convex functions*, J. Math. Anal. Appl. **110** (1985), no. 2, 536–552.
 [2] M. BESSENYEI, *The Hermite-Hadamard inequality on Simplices*, Amer. Math. Monthly **115** (4) (2008) 339–345.
 [3] W. S. CHEUNG, A. MATKOVIĆ, J. PEČARIĆ, *A variant of Jessen's inequality and generalized means*, JIPAM. J. Inequal. Pure Appl. Math. **7** (2006), no. 1, Article 10, 8 pp.

- [4] W. EHM, MARC G. GENTON, T. GNEITING, *Stationary covariances associated with exponentially convex functions*, Bernoulli **9** (2003), no. 4, 607–615.
- [5] J. JAKŠETIĆ, JOSIP E. PEČARIĆ, *Exponential convexity method*, J. Convex Anal. **20** (2013), no. 1, 181–197.
- [6] B. JESSEN, *Bemaerkinger om konvekse Funktioner og Uligheder imellem Middelveerdier I*, Mat. Tidsskrift B (1931), 17–29.
- [7] M. KLARIČIĆ BAKULA, J. PEČARIĆ, J. PERIĆ, *Extensions of the Hermite-Hadamard inequality with Applications*, Math. Inequal. Appl. **12** (2012), no. 4, 899–921.
- [8] M. KLARIČIĆ BAKULA, J. PEČARIĆ, J. PERIĆ, *On the converse Jensen inequality*, Appl. Math. Comput. **218** (2012), no. 11, 6566–6575.
- [9] A. MATKOVIĆ, J. PEČARIĆ, *A variant of Jensen's inequality for convex functions of several variables*, J. Math. Ineq. **1** (1) (2007) 45–51.
- [10] A. MCD. MERCER, *A variant of Jensen's inequality*, JIPAM. J. Inequal. Pure Appl. Math. **4** (2003), no. 4, Article 73, 2 pp.
- [11] D. S. MITRINOVIĆ, J. E. PEČARIĆ, A. M. FINK, *Classical and new inequalities in analysis. Mathematics and its Applications*, East European Series, 61. Kluwer Academic Publishers Group, Dordrecht, 1993.
- [12] J. PEČARIĆ, J. PERIĆ, *Remarks on the paper "Jensen's inequality and new entropy bounds" by S. Simić*, Math. Inequal. Appl. **6** (2012), no. 4, 631–636.
- [13] J. PEČARIĆ, J. PERIĆ, *Generalizations and improvements of converse Jensen's inequality for convex hulls in \mathbb{R}^k* , submitted to Math. Inequal. Appl.
- [14] J. E. PEČARIĆ, F. PROSCHAN, Y. L. TONG, *Convex functions, partial orderings, and statistical applications*, Mathematics in Science and Engineering, 187. Academic Press, Inc., Boston, MA, 1992.
- [15] D. V. WIDDER, *The Laplace Transform*, Princeton Mathematical Series, v. 6. Princeton University Press, Princeton, N. J., 1941.

(Received October 13, 2014)

A. Matković
 Faculty of Electrical Engineering, Mechanical Engineering
 and Naval Architecture
 University of Split
 Ruđera Boškovića 32, 21 000 Split, Croatia
 e-mail: anita@pmfst.hr

J. Pečarić
 Faculty of Textile Technology
 University of Zagreb
 Prilaz baruna Filipovića 30, 10 000 Zagreb, Croatia
 e-mail: pecaric@hazu.hr

J. Perić
 Faculty of Science, Department of Mathematics
 University of Split
 Teslina 12, 21 000 Split, Croatia
 e-mail: jperic@pmfst.hr