A NOTE ON HARMONIC QUADRANGLE IN ISOTROPIC PLANE

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ABSTRACT: The harmonic quadrangle is a cyclic quadrangle $ABCD$ with the following property: the point of intersection of the tangents at the vertices $A$ and $C$ lies on the line $BD$, and the intersection point of the tangents at the vertices $B$ and $D$ lies on the line $AC$. If one of the requests is fulfilled, the other one automatically follows. In this paper we give some characterizations of harmonic quadrangles among the cyclic ones. We also study a new harmonic quadrangle associated to the given harmonic quadrangle $ABCD$.

Keywords: isotropic plane, cyclic quadrangle, harmonic quadrangle

1. INTRODUCTION
An isotropic plane is a real projective plane where the metric is induced by a real line $f$ and a real point $F$, incident with it. The ordered pair $(f, F)$ is called the absolute figure of the isotropic plane. All straight lines through the absolute point $F$ are called isotropic lines, and all points incident with $f$ are called isotropic points. Two lines are parallel if they intersect in an isotropic point, and two points are parallel if they lie on the same isotropic line. In the affine model of the isotropic plane where the coordinates of the points are defined by

$$x = \frac{x_1}{x_0}, \ y = \frac{x_2}{x_0}$$

the absolute line $f$ has the equation $x_0 = 0$ and the absolute point $F$ has the coordinates $(0,0,1)$. For two non-parallel points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ a distance is defined by $d(A, B) = x_B - x_A$, and for two parallel points $A = (x, y_A)$ and $B = (x, y_B)$ a span is defined by $s(A, B) = y_B - y_A$. Two non-parallel lines $p$ and $q$ given by the equations $y = k_px + l_p$ and $y = k_qx + l_q$ form an angle defined by $\angle(p,q) = k_q - k_p$, [5].

2. ON HARMONIC QUADRANGLE IN ISOTROPIC PLANE
In [6] it is shown that in order to prove geometric facts for each cyclic quadrangle, it is sufficient to give a proof for the standard cyclic quadrangle. We consider the standard cyclic quadrangle $ABCD$ with the circumscribed circle given by

$$y = x^2 \quad (1)$$

and vertices are chosen to be

$$A = (a, a^2), B = (b, b^2), C = (c, c^2), D = (d, d^2), \quad (2)$$

with $a, b, c, d$ being mutually different real numbers, where $a < b < c < d$. The following two lemmas characterize such a quadrangle.

Lemma 1 ([6], p. 267) For any cyclic quadrangle $ABCD$ there exist four distinct real numbers $a, b, c, d$ such that, in the defined canonical affine coordinate system, the vertices have the form (2), the circumscribed circle has the equation (1) and the sides are given by

$$AB...y = (a + b)x - ab, \quad DA...y = (a + d)x - ad,$$

$$BC...y = (b + c)x - bc, \quad AC...y = (a + c)x - ac,$$

$$CD...y = (c + d)x - cd, \quad BD...y = (b + d)x - bd. \quad (3)$$

Tangents $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ of the circle (1) at the points (2) are of the form

$$\mathcal{A}...y = 2ax - a^2, \quad \mathcal{B}...y = 2bx - b^2,$$

$$\mathcal{C}...y = 2cx - c^2, \quad \mathcal{D}...y = 2dx - d^2. \quad (4)$$
The points of intersection of the tangents in (4) are
\[
\begin{align*}
T_{AB} &= \mathcal{A} \cap \mathcal{B} = (\frac{a+b}{2}, ab), \\
T_{AC} &= \mathcal{A} \cap \mathcal{C} = (\frac{b+c}{2}, ac), \\
T_{AD} &= \mathcal{A} \cap \mathcal{D} = (\frac{a+d}{2}, ad), \\
T_{BC} &= \mathcal{B} \cap \mathcal{C} = (\frac{b+c}{2}, bc), \\
T_{BD} &= \mathcal{B} \cap \mathcal{D} = (\frac{b+d}{2}, bd), \\
T_{CD} &= \mathcal{C} \cap \mathcal{D} = (\frac{c+d}{2}, cd).
\end{align*}
\] (5)

An allowable triangle is a triangle whose sides are non isotropic lines, see [2]. According to (6) the allowable cyclic quadrangle is the cyclic quadrangle having the allowable diagonal triangle.

**Lemma 2** The diagonal points $U, V, W$ of the allowable cyclic quadrangle $ABCD$ are of the form
\[
\begin{align*}
U &= \left(\frac{ac-bd}{a+c-b-d}, \frac{ac(b+d)-bd(a+c)}{a+c-b-d}\right), \\
V &= \left(\frac{ab-cd}{a+b-c-d}, \frac{ab(c+d)-cd(a+b)}{a+b-c-d}\right), \\
W &= \left(\frac{ad-bc}{a+d-b-c}, \frac{ad(b+c)-bc(a+d)}{a+d-b-c}\right),
\end{align*}
\] (6)

and the sides of the diagonal triangle are given with
\[
\begin{align*}
UV...y &= \frac{2(ad-bc)}{a+d-b-c} x - \frac{ad(b+c)-bc(a+d)}{a+d-b-c}, \\
UW...y &= \frac{2(ab-cd)}{a+b-c-d} x - \frac{ab(c+d)-cd(a+b)}{a+b-c-d}, \\
VW...y &= \frac{2(ac-bd)}{a+c-b-d} x - \frac{ac(b+d)-bd(a+c)}{a+c-b-d},
\end{align*}
\] (7)

where $a+c-b-d \neq 0$, $a+b-c-d \neq 0$, $a+d-b-c \neq 0$.

Apparently, conditions $a+c-b-d \neq 0$, $a+b-c-d \neq 0$, and $a+d-b-c \neq 0$ are the conditions for the cyclic quadrangle $ABCD$ to be allowable.

The geometry of harmonic quadrangle firstly has been discussed in [11]. This paper presents the sequel to such an investigation. The **harmonic quadrangle** is a cyclic quadrangle $ABCD$ with the following property: the point of intersection of the tangents at the vertices $A$ and $C$ lies on the line $BD$, and the intersection point of the tangents at the vertices $B$ and $D$ lies on the line $AC$. If one of the requests is fulfilled, the other one automatically follows. Due to Theorem 3 there are many other characterizations of harmonic quadrangles among the cyclic ones.

**Lemma 3** [11] Let $ABCD$ be an allowable cyclic quadrangle with vertices given by (2), sides by (3) and tangents of its circumscribed circle (4) at its vertices given by (5). These are the equivalent statements:

1. the point $T_{AC} = \mathcal{A} \cap \mathcal{C}$ is incident with the diagonal $BD$;
2. the point $T_{BD} = \mathcal{B} \cap \mathcal{D}$ is incident with the diagonal $AC$;
3. the equality
$$d(A,B) \cdot d(C,D) = -d(B,C) \cdot d(D,A)$$ (8)
holds;
4. the equality
$$2(ac+bd) = (a+c)(b+d)$$ (9)
holds.

Since the properties 1-3 have completely geometrical sense, the property 4 does not depend on the choice of the affine coordinate system. Choosing the $y$-axis to be incident with the diagonal point $U$, because of (6) $ac = bd$ follows. Since $ac < 0$ and $bd < 0$, we can use the notation
$$ac = bd = -k^2.$$ (10)
Thus the diagonal point \( U \) turns into
\[
U = (0, k^2), \quad V = \left( \frac{ad - bc}{a + d - b - c}, -k^2 \right),
\]
\[
W = \left( \frac{ad - bc}{a + d - b - c}, -k^2 \right).
\]
Such a harmonic quadrangle is said to be a standard harmonic quadrangle. As every harmonic quadrangle can be represented in the standard position, in order to prove geometric facts for each harmonic quadrangle, it is sufficient to give a proof for the standard harmonic quadrangle. Figure 1 presents one standard harmonic quadrangle.

3. FEW NEW PROPERTIES OF HARMONIC QUADRANGLE IN ISOTROPIC PLANE

Hereby, we present some interesting results that characterize the harmonic quadrangle among cyclic ones. For the Euclidean counterparts see [4].

**Theorem 1** Let \( ABCD \) be an allowable cyclic quadrangle. \( ABCD \) is the harmonic quadrangle if and only if the isotropic line through the point \( D \) intersects sides \( AB \), \( BC \), \( CA \) in the points \( P \), \( Q \), \( R \), respectively, such that \( s(Q,R) = s(R,P) \). The analogous statements for the other three vertices are valid as well.

**Proof:** The isotropic line through \( D \) has the equation \( x = d \). From (1) we get
\[
P(d, d(a + b) - ab),
Q(d, d(b + c) - bc),
R(d, d(a + c) - ac).
\]

Now we have
\[
s(Q,R) = d(a + c) - ac - [d(b + c) - bc] = (a - b)(d - c),
\]
\[
s(R,P) = d(a + b) - ab - [d(a + c) - ac] = (b - c)(d - a).
\]

Obviously, \( s(Q,R) = s(R,P) \) precisely when \( ab + ad + bc + cd = 2ac + 2bd \), i.e. when \( ABCD \) is a harmonic quadrangle.

**Theorem 2** Let \( ABCD \) be an allowable cyclic quadrangle. If \( C_{AB} = AB \cap C, \quad D_{AB} = AB \cap D, \quad A_{BC} = BC \cap A, \quad D_{BC} = BC \cap D, \quad A_{CD} = CD \cap A, \quad B_{CD} = CD \cap B, \quad B_{AD} = AD \cap B, \quad C_{AD} = AD \cap C \) are considered, then the following statements are equivalent:

1. \( ABCD \) is a harmonic quadrangle.
2. The lines \( B_{CD}, \quad D_{BC}, \quad AC \) are concurrent.
3. The lines \( A_{CD}, \quad C_{AB}, \quad BD \) are concurrent.
4. The lines \( B_{BC}, \quad D_{AC}, \quad AC \) are concurrent.
5. The lines \( A_{DB}, \quad D_{BC}, \quad BD \) are concurrent.

**Proof:** We will prove that statement 1 is equivalent to statement 5. Used calculations are simple, but long, so hereby we present only the final results of them.

The points \( D_{AB} \) and \( D_{BC} \) have the following coordinates
\[
D_{AB} = \left( \frac{ab - d^2}{a + b - 2d} \frac{2abd - d^2(a + b)}{a + b - 2d} \right),
\]
\[
D_{BC} = \left( \frac{bc - d^2}{b + c - 2d} \frac{2bcd - d^2(b + c)}{b + c - 2d} \right).
\]
Therefore, the lines $AD_{BC}$ and $CD_{AB}$ have the equations
\[
AD_{BC} \cdots y = \frac{(a^2 + d^2)(b + c) - 2(a^2 + bc)}{d(d - 2a) + a(b + c) - bc} x - \frac{d^2(b + c - a) + bc(a - 2d)}{d(d - 2a) + a(b + c) - bc} a,
\]
\[
CD_{AB} \cdots y = \frac{(c^2 + d^2)(a + b) - 2(c^2 + ab)}{d(d - 2c) + c(a + b) - ab} x - \frac{d^2(a + b - c) + ab(c - 2d)}{d(d - 2c) + c(a + b) - ab} c.
\]
These two lines intersect in the point $(x_0, y_0)$:
\[
x_0 = \frac{N_x}{D_x},
\]
\[
y_0 = \frac{N_y}{D_y}
\]
where
\[
N_x = abc(ab - ac + bc) - 4ab^2cd - d^2(ab^2 - a^2c - ac^2 + b^2c - 4abc) - 4acd^3 + ad^4(a - b + c),
\]
\[
D_x = a^2b^2 - a^2c^2 + b^2c^2 - 2d(a^2b + ab^2 - a^2c - ac^2 + b^2c + bc^2) + 2d^2(2ab - 2ac + 2bc + b^2) - 4bd^3 + d^4,
\]
\[
N_y = a^2b^2c^2 - 4a^2bc^2d + 2abcd^2(2ab + ac + 2bc - b^2) - 2d^3(ab^2 - a^2c - ac^2 + b^2c + bc^2) + ad^4(a - b + c),
\]
\[
D_y = a^2b^2 - a^2c^2 + b^2c^2 - 2d(a^2b + ab^2 - a^2c - ac^2 + b^2c + bc^2) + 2d^2(2ab - 2ac + 2bc + b^2) - 4bd^3 + d^4.
\]
which is incident with the line $BD$ having the equation
\[
y = (b + d)x - bd
\]
if and only if $ab - 2ac + bc + ad - 2bd + cd = 0$. According to Theorem 3, that happens precisely when $ABCD$ is a harmonic quadrangle.

**Theorem 3** Let $ABCD$ be an allowable quadrangle and $C$ its circumscribed circle. If $C_{AB} = AB \cap \mathcal{C}$, $D_{AB} = AB \cap \mathbf{D}$, $A_{BC} = BC \cap \mathcal{C}$, $D_{BC} = BC \cap \mathbf{D}$, $A_{CD} = CD \cap \mathcal{C}$, $B_{CD} = CD \cap \mathbf{D}$, $A_{AD} = AD \cap \mathcal{C}$, $C_{AD} = AD \cap \mathbf{C}$ are considered, then the following statements are equivalent:

1. $ABCD$ is a harmonic quadrangle.
2. The second tangents to $C$ at $A_{BC}$ and $A_{CD}$ intersect on $AC$.
3. The second tangents to $C$ at $B_{AD}$ and $B_{CD}$ intersect on $BD$.
4. The second tangents to $C$ at $C_{AB}$ and $C_{AD}$ intersect on $AC$.
5. The second tangents to $C$ at $D_{BC}$ and $D_{AB}$ intersect on $BD$.

**Proof:** Let us prove the equivalency of statements 1 and 5.

The lines
\[
t_{D_{AB}} \cdots y = 2 \frac{2ab - ad - bd}{a + b - 2d} x - \left( \frac{2ab - ad - bd}{a + b - 2d} \right)^2,
\]
\[
t_{D_{BC}} \cdots y = 2 \frac{2bc - bd - cd}{b + c - 2d} x - \left( \frac{2bc - bd - cd}{b + c - 2d} \right)^2
\]
are the tangents to $C$ at $D_{AB}$ and $D_{BC}$ (different from $\mathbf{D}$). They intersect at the point
\[
(12)
\]
By doing some calculations, we obtain
\[
\frac{1}{2}(b + d) \left( \frac{2ab - ad - bd}{a + b - 2d} \right) - \frac{1}{2}(b + d) \left( \frac{2bc - cd - bd}{b + c - 2d} \right) = \frac{1}{2} \left( \frac{2ab - ad - bd}{a + b - 2d} \right) - \frac{1}{2} \left( \frac{2bc - cd - bd}{b + c - 2d} \right) - \left( \frac{2ab - ad - bd}{a + b - 2d} \right) + \left( \frac{2bc - cd - bd}{b + c - 2d} \right)
\]
Therefore, the point with the coordinates $(12)$ lies on the line $BD$ from (3) when $ab + ad + bd + cd - 2ac - 2bd = 0$, i.e. when $ABCD$ is a harmonic quadrangle.
The theorem given above provides the following geometric construction of the harmonic quadrangle: Let $c$ be a circle inscribed to given triangle $PQR$. $D$ is a point of contact of a tangent $QR$ and the circle $c$. The line $DP$ intersects the circle $c$ in the point $B$. $A$ and $C$ are points of intersection of the lines $BR$ and $BQ$ with the circle $c$, respectively. The quadrangle $ABCD$ which the construction results with is the harmonic quadrangle. This claim follows from Theorem 3 by using the notation in a way: $D_{QR} = QR, D_{BC} = Q$ and $D_{AB} = R$. Figure 2 illustrates the described construction.

Figure 2: A construction of the harmonic quadrangle $ABCD$ from the given triangle $PQR$

There is an interesting result discussed in [1]: whole family of harmonic quadrangles can be obtained out of given harmonic quadrangle. Namely, let $ABCD$ be a harmonic quadrangle. Lines $a', b', c', d'$ are taken in a way that are incident to vertices $A, B, C, D$, respectively, and make equal angles to sides $AB, BC, CD, DA$, respectively. The quadrangle formed by lines $a', b', c', d'$ is a harmonic quadrangle as well. Furthermore, denoting obtained quadrangle by $A'B'C'D'$, the ratio of the corresponding sides of given quadrangle $ABCD$ and obtained quadrangle $A'B'C'D'$ is equal, Figure 3. Only in one case points $A', B', C', D'$ coincide with one point $P_1$, the first Brocard point. In similar manner, the second Brocard point $P_2$ is obtained as well. In the latter case lines $P_2A, P_2B, P_2C,$ and $P_2D$ form the equal angles with the sides $AD, DC, CB,$ and $BA$, respectively. Brocard points are of the form

$$P_1 = (k, 3k^2), P_2 = (-k, 3k^2).$$

Figure 3: A harmonic quadrangle $A'B'C'D'$ obtained from the given harmonic quadrangle $ABCD$

In the sequel, we will also study on the harmonic quadrangle joint to given harmonic quadrangle. Similar investigations in the Euclidean case can be found in [3]. First let us prove lemma that follows:

**Lemma 4** Let $ABCD$ be an allowable cyclic quadrangle. There exist unique point $Q_1$ such that triangles $Q_1AB$ and $Q_1CD$ have equal angles (i.e. $\angle(Q_1A, AB) = \angle(Q_1C, CD), \angle(AB, BQ_1) = \angle(CD, Q_1D)$). Similarly, there exist unique point $Q_2$ such that triangles $Q_2BC$ and $Q_2DA$ have equal angles.

**Proof:** Let $p$ and $q$ be lines through $A$ and $B$, respectively, with $\angle(p, AB) = m$ and $\angle(AB, q) = n$.
The equations of \( p \) and \( q \) are
\[
\begin{align*}
p &\quad y = (a + b - m)x + am - ab, \\
q &\quad y = (a + b + n)x - bn - ab.
\end{align*}
\]
Thus, \( Q_1 = p \cap q \) has coordinates
\[
\left( \frac{am + bn}{m + n}, \frac{a^2m + b^2n + mn(a - b)}{m + n} \right)
\]
(14)

Analogously, let \( p' \) and \( q' \) be lines through \( C \) and \( D \), respectively, with \( \zeta(p', CD) = m \) and \( \zeta(CD, q') = n \). Then the point \( Q'_1 = p' \cap q' \) is given with
\[
\left( \frac{cm + dn}{m + n}, \frac{c^2m + d^2n + mn(c - d)}{m + n} \right)
\]
(15)

Looking for \( m \) and \( n \) such that \( Q_1 \) and \( Q'_1 \) coincide, we obtain unique solution
\[
\begin{align*}
m &= \frac{(b - d)(a - b + c - d)}{a - b - c + d}, \\
n &= \frac{(a - c)(-a + b - c + d)}{a - b - c + d},
\end{align*}
\]
(16)

which turns (14) into
\[
Q_1 = \left( \frac{ad - bc}{a - b - c + d}, 2\left( \frac{ad - bc}{a - b - c + d} \right)^2 + \frac{ac(b - d) - bd(a - c)}{a - b - c + d} \right).
\]
(17)

In similar manner, we get
\[
Q_2 = \left( \frac{ab - cd}{a + b - c - d}, 2\left( \frac{ab - cd}{a + b - c - d} \right)^2 + \frac{bd(c - a) - ac(b - d)}{a + b - c - d} \right),
\]
(18)
as well.

**Theorem 4** Let \( ABCD \) be a harmonic quadrangle, and let \( Q_1, Q_2 \) be the points, such that \( \zeta(Q_1A, AB) = \zeta(Q_1C, CD) \), \( \zeta(AB, BQ_1) = \zeta(CD, DQ_1) \), and \( \zeta(Q_2B, BC) = \zeta(Q_2D, DA) \), \( \zeta(BC, CQ_2) = \zeta(DA, AQ_2) \), respectively. The quadrangle \( Q_1M_{AC}Q_2M_{BD} \) is also a harmonic quadrangle.

**Proof:** Due to \( ac = bd = -k^2 \), (17) and (18) turn into
\[
\begin{align*}
Q_1 &= \left( \frac{ad - bc}{a - b - c + d}, 2\left( \frac{ad - bc}{a - b - c + d} \right)^2 + k^2 \right), \\
Q_2 &= \left( \frac{ab - cd}{a + b - c - d}, 2\left( \frac{ab - cd}{a + b - c - d} \right)^2 + k^2 \right).
\end{align*}
\]

These two points are incident with the circle
\[\text{y} = 2x^2 + k^2,\]
which is also passing through \( M_{AC}, M_{BD}, U, P_1 \) and \( P_2 \). [1]

It is left to prove \( d(Q_1, M_{AC}) \cdot d(Q_2, M_{BD}) = -d(M_{AC}, Q_2) \cdot d(M_{BD}, Q_1) \). That can be easily checked from:
\[
\begin{align*}
d(Q_1, M_{AC}) &= \frac{a + c}{2} - \frac{ad - bc}{a - b - c + d} = \frac{(a - c)(a - b + c - d)}{2(a - b - c + d)}, \\
d(Q_2, M_{BD}) &= \frac{b + d}{2} - \frac{ab - cd}{a + b - c - d} = \frac{(b - d)(-a + b - c + d)}{2(a + b - c - d)}, \\
d(M_{AC}, Q_2) &= \frac{ab - cd}{a + b - c - d} - \frac{a + c}{2} = \frac{(a - c)(-a + b - c + d)}{2(a + b - c - d)}, \\
d(M_{BD}, Q_1) &= \frac{ad - bc}{a - b - c + d} - \frac{b + d}{2} = \frac{(b - d)(-a + b - c + d)}{2(a - b - c + d)}.
\end{align*}
\]

Let us notice that due to (11) and (3), points \( W, Q_1 \) and \( V, Q_2 \) are parallel ones. Therefore, \( Q_1 \) and \( Q_2 \) can be obtained as the points of intersection of the circle \( y = 2x^2 + k^2 \) and isotropic lines through diagonal points \( W \) and \( V \), respectively.

**Theorem 5** A diagonal point \( Q_1Q_2 \cap M_{AC}M_{BD} \) of the quadrangle \( Q_1M_{AC}Q_2M_{BD} \) is a parallel point to the diagonal point \( U \) of the quadrangle \( ABCD \). This point is incident with a line \( P_1P_2 \),
Figure 4: The harmonic quadrangle $Q_1M_{AC}Q_2M_{BD}$ associated to the harmonic quadrangle $ABCD$

as well, the connection line of Brocard points of $ABCD$. The other two diagonal points of the quadrangle $Q_1M_{AC}Q_2M_{BD}$ are incident to the line VW from (11).

**Proof:** Let us prove the first claim. We will show that the point of intersection of the lines $Q_1Q_2$ and $M_{AC}M_{BD}$ lies on the y-axis, more precisely that the point $(0, 3k^2)$ lies both on $Q_1Q_2$ and $M_{AC}M_{BD}$. By direct calculation, we get the equation of the line $Q_1Q_2$:

$$y = 2 \left( \frac{ad - bc}{a - b + c - d} + \frac{ab - cd}{a + b - c - d} \right)x - 2 \frac{(ad - bc)(ab - cd)}{(a - b + c + d)(a + b - c - d)} + k^2,$$

i.e.

$$y = 2 \left( \frac{ad - bc}{a - b + c - d} + \frac{ab - cd}{a + b - c - d} \right)x - 2 \frac{a^2bd - acd^2 - ab^2c + bc^2d}{a^2 - b^2 + c^2 - d^2} + k^2. \tag{20}$$

Due to $ac = bd = -k^2$ we get that $Q_1Q_2$ has the equation of the form

$$y = 2 \left( \frac{ad - bc}{a - b + c - d} + \frac{ab - cd}{a + b - c - d} \right)x + 3k^2. \tag{21}$$

On the other hand, the equation of the line $M_{AC}M_{BD}$ is

$$y = \frac{b^2 + d^2 - a^2 - c^2}{b + d - a - c} \left( x - \frac{a + c}{2} \right) + \frac{a^2 + c^2}{2},$$

i.e.

$$y = (a + b + c + d) \left( x - \frac{a + c}{2} \right) + \frac{a^2 + c^2}{2}. \tag{19}$$
Thus, the line $M_{AC}M_{BD}$ has the equation of the form
\[ y = (a + b + c + d)x + 3k^2. \] (22)

Obviously, the coordinates of the point $(0, 3k^2)$ satisfy both equations (21) and (22).

Let us now prove the second claim. The connection line of the two diagonal points $Q_1M_{AC} \cap Q_2M_{BD}$, $Q_1M_{BD} \cap M_{AC}Q_2$ is the polar line of the diagonal point $Q_1Q_2 \cap M_{AC}M_{BD}$ with respect to the circle $y = 2x^2 + k^2$. Therefore, its projective coordinates are
\[
\begin{bmatrix}
    k^2 & 0 & -\frac{1}{2} \\
    0 & 2 & 0 \\
    -\frac{1}{2} & 0 & 0 \\
-\frac{1}{2} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    1 \\
    0 \\
    3k^2 \\
    k^2
\end{bmatrix}
= 
\begin{bmatrix}
    \frac{-1}{2}k^2 \\
    0 \\
    -\frac{1}{2}
\end{bmatrix}
= 
\begin{bmatrix}
    \frac{-1}{2} \\
    0 \\
    1
\end{bmatrix}.
\]

Thus, it is the line $y = -k^2$.

4. CONCLUSIONS

In this paper we studied the properties of the harmonic quadrangle in the isotropic plane. We gave a few characterizations of harmonic quadrangles among the cyclic ones, showed a construction of a such quadrangle, and studied a new harmonic quadrangle associated to a given one.

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