Lyapunov type equation for discrete exponential trichotomies

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Abstract

For a nonautonomous dynamics obtained by a sequence of linear operators acting on an arbitrary Hilbert space, we give a complete characterization of the notion of a uniform exponential trichotomy in terms of what can be considered to be a discrete version of the Lyapunov equation. We then use this characterization to study the stability of exponential trichotomies under small linear and nonlinear perturbations. ©2017 All rights reserved.

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1. Introduction

The notions of an exponential dichotomy and trichotomy play a central role in the qualitative theory of differential equations and dynamical systems. Indeed, under the assumption that a trajectory admits an exponential dichotomy, one is able to establish the existence of invariant stable and unstable manifolds as well as the nonautonomous version of the Grobman-Hartman theorem. On the other hand, under the assumption that the trajectory admits an exponential trichotomy, one can develop the center manifold theory (see [8, 12, 16–19, 29, 50, 51]). We refer to [10, 15, 31, 33, 49] for a detailed discussion, many historical comments and further references. Due to the importance of those notions, it is of considerable interest to obtain their useful characterizations and this is the main theme of the present paper.

More precisely, motivated by the characterization of hyperbolic operators on Hilbert spaces given in [20], we obtain a full characterization of the notion of a uniform exponential trichotomy for a nonautonomous dynamics defined by a sequence of linear operators on an arbitrary Hilbert space. The notion of an exponential trichotomy essentially corresponds to have a splitting of the phase space into three directions, one of which is contracting (with exponential rate) under the action of dynamics, the second is expanding under the action of the dynamics while the third direction (called the center direction) does not have to exhibit either contraction or expansion but one still requires a certain control on the growth along this direction. This notion includes the notion of an exponential dichotomy as a particular case when there is no center direction. We emphasize that the term trichotomy has been used in the literature for various
different notions. For example, the notion of trichotomy introduced in [25] essentially corresponds to have an exponential dichotomy on both positive and negative line but not necessarily on the whole line (see [1, 26, 36] for related works). On the other hand, the notion of trichotomy used in the present paper is motivated by the concept of the partial hyperbolicity introduced in [14] which has since then became an important part of the modern smooth ergodic theory (see [32] for a detailed exposition). A similar but weaker (in the sense that it requires less control on the central direction) concept of trichotomy has been extensively studied by Sasu and Sasu [43, 44, 46–48]. For some recent contributions which involve the study of nonuniform trichotomies and trichotomies with not necessarily exponential growth rates, we refer to [3, 5, 7–9, 11, 12, 34, 35].

Our characterization of exponential trichotomies is inspired by the related works which go back to Lyapunov who proved that an \( n \times n \) matrix \( A \) has all characteristic roots with negative real-parts, if and only if for every positive definite Hermitian matrix \( H \), there exists a unique positive definite Hermitian matrix \( B \) such that

\[
A^* B + BA = -H.
\]

In [20] this theorem was extended to the case of bounded linear operators acting on Hilbert spaces and applied to the asymptotic study of the linear differential equation with constant coefficients of the form \( x' = Ax \) where \( A \) is a bounded operator on an arbitrary Hilbert space. The case when \( A \) is not necessarily bounded was first considered by Datko [21] and consequently in a series of works [27, 30, 37, 38]. In the present paper we consider:

- the case of discrete time. The principal motivation for this is a general strategy of the discretization of dynamics which involves passing from continuous to discrete time since for many problems, it is easier to threat the case of discrete then continuous time. In the context of nonautonomous dynamics this strategy is outlined in detail for example in [33];

- the case of nonautonomous dynamics. Indeed, autonomous dynamics is not sufficient to threat numerous problems that arise in applications and thus the study of the time-dependent dynamics cannot be avoided. In the context of the present paper, one may consider that our results extend (in various directions) those in [20] which can be applied to the study of a linear difference equation with constant coefficients \( x_{n+1} = A x_n \) to results which enable us to consider a linear difference equations of the form \( x_{n+1} = A_n x_n \), where now the coefficients are allowed to depend on the time variable \( n \);

- the notion of an exponential trichotomy which as already emphasized includes the concepts of stability and dichotomy as very special cases.

We emphasize that some previous works did consider the problem of characterizing exponential behavior for the nonautonomous dynamics in terms of the appropriate Lyapunov equation (see [22, 39–41] and references therein), but that our results are the first one that deal with the case of exponential trichotomies. This can be regarded as a principal contribution of the present paper since the notion of an exponential trichotomy is the weakest one under which one is still able to obtain certain information about the qualitative behavior of the dynamics. Regul. Chaotic Dyn., Our strategy can be briefly outlined as follows: we carefully and in a nontrivial way reduce the study of the nonautonomous dynamics to the study of the autonomous dynamics (on a much larger space), apply appropriate results from [20] and return back to our original dynamics. The approach builds on that in [6, 23], where we considered only the case of exponential dichotomies and obtained their characterization in terms of Lyapunov functions rather then in terms of the Lyapunov equation.

We also apply our characterization of exponential trichotomy on the study the persistence of trichotomic behavior under sufficiently small linear and nonlinear perturbations. We refer to [3, 12, 24] for related results using different methods.

To summarize, in the present paper we for the first time:
• characterize trichotomic behavior in terms of the appropriate Lyapunov equation;
• show how this characterization can be used to study the stability of trichotomic behavior under small perturbations.

Finally, we refer to [28] and [52] for some interesting applications of the Lyapunov type techniques in the study of chaos synchronization and discrete fractional maps.

2. Preliminaries

2.1. Characterization of hyperbolic operators

Let \( X = (X, \langle \cdot, \cdot \rangle) \) be a Hilbert space over \( \mathbb{R} \) or \( \mathbb{C} \) and let \( B(X) \) denote the space of all bounded linear operators acting on \( X \). The identity operator on \( X \) will be denoted by \( I \). For two self-adjoint operators \( A, B \in B(X) \) we write \( A \geq B \), if \( \langle Ax, x \rangle \geq \langle Bx, x \rangle \) for each \( x \in X \). Furthermore, we say that the operator \( A \in B(X) \) is hyperbolic, if the spectrum of \( A \) does not intersect the unit circle \( S^1 = \{ \lambda : |\lambda| = 1 \} \).

We will now recall two important theorems concerning hyperbolic operators that will be used in the following sections. Both of those results are taken from [20].

**Theorem 2.1.** Let \( T \in B(X) \) be a hyperbolic operator. Then, every self-adjoint operator \( W \in B(X) \) with the property that there exists a self-adjoint operator \( H \in B(X) \) and \( \delta > 0 \) such that
\[
T^*WT - W = -H \quad \text{and} \quad H \geq \delta I, \tag{2.1}
\]

is necessarily invertible.

**Theorem 2.2.** Let \( T \in B(X) \) and assume that there exists a self-adjoint invertible operator \( W \in B(X) \) such that (2.1) holds with some self-adjoint operator \( H \in B(X) \) and \( \delta > 0 \). Then, \( T \) is hyperbolic, if and only if there exists a self-adjoint operator \( H' \in B(X) \) and \( \delta' > 0 \) such that
\[
TW^{-1}T^* - W^{-1} = -H' \quad \text{and} \quad H' \geq \delta'I.
\]

2.2. Exponential trichotomies and dichotomies

We recall the notions of a (uniform) exponential trichotomy and dichotomy. Let \( X \) be as in the previous subsection. Moreover, let \( J \in \{ \mathbb{Z}_0^+, \mathbb{Z}_0^-, \mathbb{Z} \} \) be an interval, where
\[
\mathbb{Z}_0^+ = \{ n \in \mathbb{Z} : n \geq 0 \} \quad \text{and} \quad \mathbb{Z}_0^- = \{ n \in \mathbb{Z} : n \leq 0 \}.
\]

Given a sequence \( (A_m)_{m \in J} \) of linear operators in \( B(X) \), we define the associated cocycle \( A(n, m) \) by
\[
A(n, m) = \begin{cases} A_{n-1} \cdots A_m & \text{if } n > m, \\ \text{Id} & \text{if } n = m, \end{cases}
\]
for \( n, m \in J, n \geq m \). We say that \( (A_m)_{m \in J} \) admits an exponential trichotomy, if there exist projections \( P^i_m : X \to X \) for \( i \in \{1, 2, 3\} \) and \( m \in J \) satisfying
\[
P^1_m + P^2_m + P^3_m = I, \quad A_m P^i_m = P^i_{m+1} A_m,
\]
for \( m \in \mathbb{Z} \) and \( i \in \{1, 2, 3\} \) such that the operator
\[
A_m | \text{Im} P^i_m : \text{Im} P^i_m \to \text{Im} P^i_{m+1},
\]
is invertible for each \( m \) such that \( m, m+1 \in J \) and \( i \in \{2, 3\} \) and there exist constants
\[
D > 0, \quad 0 \leq a < b, \quad \text{and} \quad 0 \leq c < d,
\]
such that
\[ ||A(m, n)P^1_n|| \leq De^{-d(m-n)}, \quad ||A(m, n)P^3_n|| \leq De^{a(m-n)}, \]
for \( m, n \in J \) with \( m \geq n \) and
\[ ||A(m, n)P^2_n|| \leq De^{-b(n-m)}, \quad ||A(m, n)P^3_n|| \leq De^{c(n-m)}, \]
for \( m, n \in J \) with \( m \leq n \), where \( A(m, n) \) in (2.3) denotes
\[ (A(n, m)||Im P^i_m|^{-1}: Im P^i_n \rightarrow Im P^i_m, \]
for \( i = 2, 3 \) respectively. Moreover, we say that the sequence \( (A_m)_{m \in J} \) admits an exponential dichotomy, if it admits an exponential trichotomy with \( P^3_m = 0 \) for \( m \in \mathbb{Z} \).

It turns out that in the particular case of dichotomies on \( Z \) the projections are uniquely determined (see [2, 42] for example).

**Proposition 2.3.** Assume that the sequence \( (A_m)_{m \in \mathbb{Z}} \) admits an exponential dichotomy. Then,
\[ \text{Im } P^1_n = \left\{ x \in X : \sup_{m \geq n} ||A(m, n)x|| < +\infty \right\}. \]
Furthermore, \( \text{Im } P^2_n \) consists of all \( x \in X \) for which there exists a sequence \( (x_m)_{m \leq n} \subset X \) such that
\[ \sup_{m \leq n} ||x_m|| < +\infty \text{ and } x_m = A(m, k)x_k \text{ for } k \leq m \leq n. \]
Let
\[ l^2 = \left\{ x = (x_n)_{n \in \mathbb{Z}} \subset X : \sum_{n=-\infty}^{\infty} ||x_n|| < +\infty \right\}. \]
It is straightforward to verify that \( l^2 \) is a Hilbert space with respect to the scalar product
\[ (x, y) = \sum_{n \in \mathbb{Z}} \langle x_n, y_n \rangle, \quad x = (x_n)_{n \in \mathbb{Z}}, \quad y = (y_n)_{n \in \mathbb{Z}} \in l^2. \]
We will need the following classical result.

**Theorem 2.4.** Let \( (A_m)_{m \in \mathbb{Z}} \) be a sequence of bounded operators on \( X \). The following statements are equivalent:
1. the sequence \( (A_m)_{m \in \mathbb{Z}} \) admits an exponential dichotomy;
2. for each \( y = (y_n)_{n \in \mathbb{Z}} \in l^2 \) there exists a unique \( x = (x_n)_{n \in \mathbb{Z}} \in l^2 \) such that
\[ x_{n+1} - A_n x_n = y_{n+1}, \quad \forall n \in \mathbb{Z}. \]

3. Characterization of exponential trichotomies on \( Z \)

**Theorem 3.1.** Assume that the sequence \( (A_m)_{m \in \mathbb{Z}} \subset B(X) \) admits an exponential trichotomy and that there exists \( C > 0 \) such that
\[ ||A_m|| \leq C, \quad \text{for } m \in \mathbb{Z}. \]
Then, there exist sequences \( (S^i_m)_{m \in \mathbb{Z}}, i = 1, 2 \) of bounded, self-adjoint and invertible operators on \( X \), sequences \( (H^i_m)_{m \in \mathbb{Z}}, (H^i_m)_{m \in \mathbb{Z}}, i = 1, 2 \) of self-adjoint operators in \( B(X) \) and constants \( K, \delta, \omega_1 > 0 \) and \( \omega_2 < 0 \) such that the following inequalities hold for each \( m \in \mathbb{Z} \) and \( i \in \{1, 2\} \):
1. \[ ||S^i_m|| \leq K \quad \text{and} \quad ||(S^i_m)^{-1}|| \leq K; \]
Proof. Take $\omega_1 \in (c, d)$ and consider the sequence $B_m = e^{\omega_1}A_m$. The cocycle associated to the sequence $(B_m)_{m \in \mathbb{Z}}$ is given by

$$B(m, n) = e^{\omega_1(m-n)}A(m, n).$$

It follows from (2.2) and (2.3) that

$$\|B(m, n)P_1^1\| \leq De^{-(d-\omega_1)(m-n)},$$

for $m \geq n$ and that

$$\|B(m, n)P_2^1\| \leq De^{-(b+\omega_1)(n-m)},$$

and

$$\|B(m, n)P_3^1\| \leq De^{-(\omega_1-c)(n-m)},$$

for $m \leq n$. By (3.6) and (3.7), we have

$$\|B(m, n)(P_2^1 + P_3^1)\| \leq 2De^{-\min(b+\omega_1, \omega_1-c)(n-m)}, \quad \text{for } m \leq n.$$

Let

$$\lambda = \min(d - \omega_1, b + \omega_1, \omega_1 - c) > 0.$$

It follows from (3.5) and (3.8) that

$$\|B(m, n)P_1^1\| \leq De^{-\lambda(m-n)}, \quad \text{for } m \geq n,$$

and

$$\|B(m, n)(P_2^1 + P_3^1)\| \leq 2De^{-\lambda(n-m)} \quad \text{for } m \leq n.$$

Choose an arbitrary $\rho \in (0, \lambda)$ and set

$$S_m^1 = \sum_{k \geq m} (B(k, m)P_1^1)^*B(k, m)P_1^1e^{2(\lambda-\rho)(k-m)}
- \sum_{k < m} (B(k, m)(I - P_1^1))^*B(k, m)(I - P_1^1)e^{2(\lambda-\rho)(m-k)}.$$

It follows from (3.9) and (3.10) that

$$\langle S_m^1x, x \rangle \leq \sum_{k \geq m} \|B(k, m)P_1^1x\|^2e^{2(\lambda-\rho)(k-m)}
+ \sum_{k < m} \|B(k, m)(I - P_1^1)x\|^2e^{2(\lambda-\rho)(m-k)}
\leq \sum_{k \geq m} D^2e^{-2\lambda(k-m)}e^{2(\lambda-\rho)(k-m)}\|x\|^2
+ \sum_{k < m} 4D^2e^{-2\lambda(m-k)}e^{2(\lambda-\rho)(m-k)}\|x\|^2
= D^2 \left( \sum_{k \geq m} e^{-2\rho(k-m)} + 4 \sum_{k < m} e^{-2\rho(m-k)} \right) \|x\|^2
= K\|x\|^2,$$
for each $m \in \mathbb{Z}$, where
\[ K = D^2 \left( \sum_{k \geq m} e^{-2\rho(k-m)} + 4 \sum_{k < m} e^{-2\rho|m-k|} \right) < +\infty. \]

Obviously, $S_m^1$ is self-adjoint and thus
\[ \|S_m^1\| = \sup_{\|x\|=1} |(S_m^1x,x)| \leq K, \quad \forall m \in \mathbb{Z}. \]

Hence, we have obtained the first inequality in (3.2) for $i = 1$. Furthermore, we have
\[
B_m^*S_{m+1}^1B_m - S_m^1 = (e^{-2(\lambda-\rho)} - 1) \sum_{k \geq m} \left( B(k, m)P_m^1 \right)^* B(k, m)P_m^1 e^{2(\lambda-\rho)(k-m)} + \left( 1 - e^{2(\lambda-\rho)} \right) \sum_{k < m} \left( B(k, m)(I-P_m^1) \right)^* B(k, m)(I-P_m^1) e^{2(\lambda-\rho)(m-k)} - e^{2(\lambda-\rho)}(P_m^1)^*P_m^1 - e^{2(\lambda-\rho)}(I-P_m^1)^*(I-P_m^1).
\]

Since $e^{-2(\lambda-\rho)} - 1 < 0$ and $1 - e^{2(\lambda-\rho)} < 0$, we obtain that
\[
B_m^*S_{m+1}^1B_m - S_m^1 \leq -e^{-2(\lambda-\rho)}(P_m^1)^*P_m^1 - e^{2(\lambda-\rho)}(I-P_m^1)^*(I-P_m^1) \leq -e^{-2(\lambda-\rho)}(P_m^1)^*P_m^1 + (I-P_m^1)^*(I-P_m^1).
\]
Furthermore, we have
\[
2\langle (p_m^1)^* p_m^1 + (I - p_m^1)^* (I - p_m^1) \rangle_{x, x} = 2\|p_m^1 x\|^2 + 2\|(I - p_m^1) x\|^2 \\
\geq \|p_m^1\|^2 + 2\|p_m^1\| \cdot \|(I - p_m^1) x\| + \|(I - p_m^1) x\|^2 \\
= (\|p_m^1 x\|^2 + \|(I - p_m^1) x\|)^2 \\
\geq \|x\|^2,
\]
for each \(x \in X\) which implies that
\[
-e^{-2(\lambda - \rho)}((p_m^1)^* p_m^1 + (I - p_m^1)^* (I - p_m^1)) \leq -\frac{1}{2} e^{-2(\lambda - \rho)} I.
\]
Consequently,
\[
B_m^* S_{m+1} \mathcal{B}_m - S_m^1 \leq -\frac{1}{2} e^{-2(\lambda - \rho)} I, \quad \forall m \in \mathbb{Z},
\]
and we conclude that (3.3) holds for \(i = 1\) with
\[
H_m^1 = S_m^1 - e^{2\omega_{1}} A_m^* S_{m+1} A_m, \quad \text{and} \quad \delta = \frac{1}{2} e^{-2(\lambda - \rho)} > 0. \tag{3.12}
\]
We now wish to establish (3.4) for \(i = 1\). Let us define the operator \(T : l^2 \to l^2\) by
\[
(T x)_n = B_{n-1} x_{n-1}, \quad \text{for} \ x = (x_n)_{n \in \mathbb{Z}} \in l^2. \tag{3.13}
\]
It follows readily from (3.1) that \(T\) is well-defined and bounded linear operator.

**Lemma 3.2.** \(T^* : l^2 \to l^2\) is given by
\[
(T^* x)_n = B_n^* x_{n+1}, \quad \forall n \in \mathbb{Z}, \quad \text{and} \quad x = (x_n)_{n \in \mathbb{Z}} \in l^2.
\]

**Proof.** We define a linear operator \(G : l^2 \to l^2\) by
\[
(G x)_n = B_n^* x_{n+1}, \quad \forall n \in \mathbb{Z}, \quad \text{and} \quad x = (x_n)_{n \in \mathbb{Z}} \in l^2.
\]
Then, for every \(x = (x_n)_{n \in \mathbb{Z}}\) and \(y = (y_n)_{n \in \mathbb{Z}} \in l^2\) we have that
\[
\langle G x, y \rangle = \sum_{n \in \mathbb{Z}} \langle (G x)_n, y_n \rangle \\
= \sum_{n \in \mathbb{Z}} \langle B_n^* x_{n+1}, y_n \rangle \\
= \sum_{n \in \mathbb{Z}} \langle x_{n+1}, B_n y_n \rangle \\
= \sum_{n \in \mathbb{Z}} \langle x_{n+1}, (Ty)_{n+1} \rangle \\
= \langle x, Ty \rangle,
\]
which implies that \(G = T^*\). \qed

It follows from (3.9) and (3.10) that the sequence \((B_n)_{n \in \mathbb{Z}}\) admits an exponential dichotomy. Hence, by Theorem 2.4 we have that the operator \(I - T\) is invertible. However, we can also obtain the following stronger conclusion.

**Lemma 3.3.** The operator \(T\) is hyperbolic.
Proof. Take $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$. Since the sequence $(B_n)_{n \in \mathbb{Z}}$ admits an exponential dichotomy we conclude that the sequence $(\frac{1}{\lambda}B_n)_{n \in \mathbb{Z}}$ also admits a uniform exponential dichotomy. Hence, it follows from Theorem 2.4 that the operator

$$x = (x_n)_{n \in \mathbb{Z}} \mapsto \left( x_n - \frac{1}{\lambda}B_{n-1}x_{n-1} \right)_{n \in \mathbb{Z}},$$

is an invertible linear operator on $l^2$. Consequently, the operator

$$x = (x_n)_{n \in \mathbb{Z}} \mapsto (\lambda x_n - B_{n-1}x_{n-1})_{n \in \mathbb{Z}},$$

is also invertible on $l^2$ and thus $\lambda \notin \sigma(T)$. We conclude that

$$\sigma(T) \cap S^1 = \emptyset,$$

and therefore $T$ is hyperbolic. \hfill \qed

Moreover, we define an operator $W$: $l^2 \to l^2$ by

$$(Wx)_n = S^1_m x_n, \quad n \in \mathbb{Z}, \quad x = (x_n)_{n \in \mathbb{Z}} \in l^2. \quad (3.14)$$

It follows from the already established first inequality in (3.2) for $i = 1$ that $W$ is well-defined. Furthermore, by (3.12) and Lemma 3.2 we have that

$$T^*WT - W = -H, \quad \text{and} \quad H \geq \delta I, \quad (3.15)$$

where $I$ now denotes the identity operator on $l^2$ and where $H$ is a self-adjoint operator on $l^2$ given by

$$(Hx)_m = H^1_m x_m, \quad x = (x_m)_{m \in \mathbb{Z}} \in l^2.$$

By Theorem 2.1, $W$ is invertible. Finally, we have the following lemma.

**Lemma 3.4.** The operator $S^1_m$ is invertible for all $m \in \mathbb{Z}$ and there exists $K > 0$ such that the second inequality in (3.2) holds for $i = 1$.

**Proof.** We first establish invertibility of operators $S^1_m$. Suppose that $S^1_m v = 0$ for some $v \in X$. Define $x = (x_n)_{n \in \mathbb{Z}} \in l^2$ by $x_m = v$ and $x_n = 0$ for all $n \neq m$. Then, $Wx = 0$ and using invertibility of $W$ we conclude that $x = 0$. Hence, $v = 0$ and $S^1_m$ is injective for each $m \in \mathbb{Z}$.

Now take $v \in X$ and define $y = (y_n)_{n \in \mathbb{Z}} \in l^2$ by $y_m = v$ and $y_n = 0$ for all $n \neq m$. Since $W$ is invertible, there exists $x \in l^2$ such that $Wx = y$. Thus, $(Wx)_m = y_m$, whence $S^1_m x_m = y_m = v$ and $S^1_m$ is surjective.

Moreover, using the notation from the previous step, we have that

$$(S^1_m)^{-1} v = (W^{-1} y)_m,$$

and therefore

$$\|(S^1_m)^{-1} v\| = \|(W^{-1} y)_m\| \leq \|W^{-1} y\| \leq \|W^{-1}|| \cdot \|y\| = \|W^{-1}|| \cdot \|v\|.$$

Hence, $\|(S^1_m)^{-1} v\| \leq \|W^{-1}||$ for all $m \in \mathbb{Z}$ and the proof of the lemma is complete. \hfill \qed

On the other hand, using Theorem 2.2 and (3.15), we conclude that there exists $\delta > 0$ and a self-adjoint operator $H$ on $l^2$ such that

$$TW^{-1}T^* - W^{-1} = -\tilde{H}, \quad \text{and} \quad \tilde{H} \geq \delta I,$$

which easily implies that (3.4) holds for $i = 1$ with $\tilde{H}_m v = (\tilde{H} v)_m$ where $v = (v_n)_{n \in \mathbb{Z}}, v_m = v$ and $v_n = 0$ for $n \neq m$. 
We now construct operators $S^2_m$, $m \in \mathbb{Z}$. Take $\omega_2 \in (-b, -a)$ and consider the sequence $C_m = e^{\omega_2 A_m}$. The associated cocycle is given by

$$C(m, n) = e^{\omega_2(m-n)A(m, n)}.$$  

It follows from (2.2) and (2.3) that

$$\|C(m, n)P^1_n\| \leq De^{-(d-\omega_2)(m-n)},$$  

and

$$\|C(m, n)P^3_n\| \leq De^{-(a-\omega_2)(m-n)},$$  

for $m \geq n$ and that

$$\|C(m, n)P^2_n\| \leq De^{-(b+\omega_2)(n-m)},$$  

for $m \leq n$. By (3.16) and (3.17) we have that

$$\|C(m, n)(P^1_n + P^3_n)\| \leq 2De^{-\min[d-\omega_2, a-\omega_2](m-n)}, \quad \text{for } m \geq n.$$  

Let

$$\lambda' = \min[d-\omega_2, a-\omega_2, b+\omega_2] > 0.$$  

It follows from (3.18) and (3.19) that

$$\|C(m, n)(P^1_n + P^3_n)\| \leq 2De^{-\lambda'(m-n)}, \quad \text{for } m \geq n,$$  

and

$$\|C(m, n)P^2_n\| \leq De^{-\lambda'(n-m)}, \quad \text{for } m \leq n.$$  

Choose an arbitrary $\rho' \in (0, \lambda')$ and set

$$S^2_m = \sum_{k \geq m} (C(k, m)(I-P^2_m))^* C(k, m)(I-P^2_m) e^{2(\lambda'-\rho')(k-m)} - \sum_{k < m} (C(k, m)P^2_m)^* C(k, m)P^2_m e^{2(\lambda'-\rho')(m-k)}.$$  

Using (3.20) and (3.21), one can now repeat previous arguments (working with $C_m$ instead of $B_m$ and $S^2_m$ instead of $S^1_m$) and show that (3.2), (3.3) and (3.4) hold for $i = 2$ too. This completes the proof of the theorem. \qed

We now establish the converse of Theorem 3.1.

**Theorem 3.5.** Let $(A_m)_{m \in \mathbb{Z}} \subset B(X)$ be a sequence of operators such that (3.1) holds with some $C > 0$. Furthermore, suppose that there exist sequences $(S^1_m)_{m \in \mathbb{Z}}$, $i = 1, 2$ of bounded, self-adjoint and invertible operators on $X$, sequences $(H^i_m)_{m \in \mathbb{Z}}$, $(\tilde{H}^i_m)_{m \in \mathbb{Z}}$, $i = 1, 2$ of self-adjoint operators in $B(X)$ and constants $K_1, K_2, K_3, \omega_1 > 0$ and $\omega_2 < 0$ such that (3.2), (3.3) and (3.4) hold for each $m \in \mathbb{Z}$ and $i \in \{1, 2\}$. Then, the sequence $(A_m)_{m \in \mathbb{Z}}$ admits an exponential trichotomy.

**Proof.** We define a sequence of operators $(B_m)_{m \in \mathbb{Z}}$ by $B_m = e^{\omega_1 A_m}$, $m \in \mathbb{Z}$. Furthermore, let $T$ and $W$ be defined as in (3.13) and (3.14). It follows from (3.1) and (3.2) that $T$ and $W$ are well-defined. Moreover, the second inequality in (3.2) (for $i = 1$) implies that $W$ is invertible and that the inverse is given by

$$(W^{-1}x)_n = (S^1_n)^{-1}x_n, \quad n \in \mathbb{Z}, \quad x = (x_n)_{n \in \mathbb{Z}} \in l^2.$$  

We note that (3.3) and (3.4) imply that

$$T^*W - W = -H, \quad \text{and} \quad TW^{-1}T^* - W^{-1} = -\tilde{H},$$  

(3.22)
where 
\[(Hx)_m = H_m^1 x_m, \quad \text{and} \quad (\hat{H}x)_m = \hat{H}_m^1 x_m, \quad \text{for} \ x = (x_m)_{m \in \mathbb{Z}} \in l^2.\]

By (3.3) and (3.4), \(H \geq \delta I\) and \(\hat{H} \geq \delta I\) and thus it follows from (3.22) and Theorem 2.2 that \(T\) is hyperbolic. In particular, operator \(1 - T\) is invertible on \(l^2\) and by Theorem 2.4, the sequence \((B_m)_{m \in \mathbb{Z}}\) admits an exponential dichotomy. Similarly, one can show that sequence \((C_m)_{m \in \mathbb{Z}}\) defined by \(C_m = e^{\omega_2 A_m}\) admits an exponential dichotomy. Hence, there exist projections \(P_m^{1}\) and \(P_m^{2}\) for \(m \in \mathbb{Z}\) satisfying
\[B_m P_m^{1} = P_{m+1}^{1} B_m, \quad C_m P_m^{2} = P_{m+1}^{2} C_m,\]
such that operators
\[B_m | \ker P_m^{1}: \ ker P_m^{1} \rightarrow ker P_{m+1}^{1}, \quad \text{and} \quad C_m | \ker P_m^{2}: \ ker P_m^{2} \rightarrow ker P_{m+1}^{2},\]
are invertible for \(m \in \mathbb{Z}\) and there exist constants \(\lambda, D > 0\) such that
\[\|B(m, n)P_m^{1}\| \leq De^{-\lambda(m-n)},\]  
\[\|C(m, n)P_m^{2}\| \leq De^{-\lambda(m-n)},\]  
(3.23)
(3.24)

for \(m \geq n\) and
\[\|B(m, n)Q_m^{1}\| \leq De^{-\lambda(n-m)},\]  
\[\|C(m, n)Q_m^{2}\| \leq De^{-\lambda(n-m)},\]  
(3.25)
(3.26)

for \(m \leq n\), where \(Q_m^{1} = I - P_m^{1}\).

**Lemma 3.6.** For each \(n \in \mathbb{Z}\), we have
\[\text{Im } P_n^{1} \subset \text{Im } P_n^{2}, \quad \text{and} \quad \text{Im } Q_n^{2} \subset \text{Im } Q_n^{1}.\]  
(3.27)

*Proof.* Take \(x \in \text{Im } P_n^{1}\). It follows from Proposition 2.3 that
\[\sup_{m \geq n} \|B(m, n)x\| < +\infty.\]

Furthermore, we have
\[\|C(m, n)x\| = e^{\omega_2 (m-n)} \|A(m, n)x\| = e^{(\omega_2 - \omega_1)(m-n)} \|B(m, n)x\|,\]

for \(m \geq n\). Since \(\omega_2 - \omega_1 < 0\), we have that
\[\sup_{m \geq n} \|C(m, n)x\| < +\infty.\]

Using Proposition 2.3 we conclude that \(x \in \text{Im } P_n^{2}\). The proof of the second inclusion in (3.27) is analogous. \(\square\)

**Lemma 3.7.** The map \(1 - P_n^{1} - Q_n^{2}\) is a projection for each \(n \in \mathbb{Z}\).

*Proof.* It follows from the previous lemma that
\[P_n^{1} Q_n^{2} = Q_n^{2} P_n^{1} = 0,\]

for \(n \in \mathbb{Z}\). Hence
\[(1 - P_n^{1} - Q_n^{2})^2 = (1 - 2P_n^{1} - 2Q_n^{2} + (P_n^{1})^2 + (Q_n^{2})^2 + P_n^{1} Q_n^{2} + Q_n^{2} P_n^{1})\]
\[= 1 - P_n^{1} - Q_n^{2},\]

and the conclusion in the lemma follows. \(\square\)
Lemma 3.8. For each \( n \in \mathbb{Z} \), we have
\[
\text{Im}(I - P_n^1 - Q_n^2) = \text{Im} P_n^2 \cap \text{Im} Q_n^1.
\]

Proof. Take \( x \in \text{Im} P_n^2 \cap \text{Im} Q_n^1 \). We have \( Q_n^2 x = P_n^1 x = 0 \) and thus,
\[
(I - P_n^1 - Q_n^2)x = x.
\]
This implies that \( x \in \text{Im}(I - P_n^1 - Q_n^2) \). Now take \( x \in \text{Im}(I - P_n^1 - Q_n^2) \). We have \( P_n^1 x = -Q_n^2 x \). Applying \( P_n^1 \), we obtain \( P_n^1 x = 0 \) and thus \( x \in \text{Im} Q_n^1 \). Similarly, \( x \in \text{Im} P_n^2 \) and so \( x \in \text{Im} P_n^2 \cap \text{Im} Q_n^1 \).

We proceed with the proof of the theorem. By Lemma 3.8, the cocycle \( A(m, n) \) is invertible along the ranges of projections \( \text{Id} - P_n^1 - Q_n^2 \). It follows from (3.23) that
\[
\|A(m, n)P_n^1\| \leq De^{-(\lambda + \omega_1)(m-n)}, \quad \text{for} \quad m \geq n. \tag{3.28}
\]
Similarly, by (3.26) we have
\[
\|A(m, n)Q_n^2\| \leq DD'e^{-(\lambda - \omega_2)(n-m)}, \quad \text{for} \quad m \leq n. \tag{3.29}
\]
Moreover, it follows from (3.24), (3.25) and Lemma 3.8 that for each \( x \in \text{Im}(\text{Id} - P_n^1 - Q_n^2) \), we have
\[
\|A(m, n)x\| \leq De^{-(\lambda + \omega_2)(m-n)}, \quad \text{for} \quad m \geq n \text{ and} \quad m \leq n. \tag{3.23}
\]
In addition, by (3.23) and (3.26) (applied for \( m = n \)),
\[
\|I - P_n^1 - Q_n^2\| \leq 3D, \quad \text{for} \quad n \in \mathbb{Z}.
\]
Hence,
\[
\|A(m, n)(I - P_n^1 - Q_n^2)\| \leq 3D^2e^{-(\lambda + \omega_2)(m-n)}, \quad \text{for} \quad m \geq n, \tag{3.30}
\]
and
\[
\|A(m, n)(I - P_n^1 - Q_n^2)\| \leq 3D^2e^{-(\lambda - \omega_1)(n-m)}, \quad \text{for} \quad m \leq n. \tag{3.31}
\]
It follows from (3.28), (3.29), (3.30) and (3.31) that the sequence \( (A_m)_{m \in \mathbb{Z}} \) admits an exponential trichotomy.

4. Stability of trichotomies under linear perturbations

In this section we use the results obtained in the previous section to establish in a simple manner the stability of exponential trichotomies under sufficiently small linear perturbations.

Theorem 4.1. Let \((A_m)_{m \in \mathbb{Z}}\) and \((B_m)_{m \in \mathbb{Z}}\) be sequences of bounded linear operators on \( X \) such that:

1. the sequence \((A_m)_{m \in \mathbb{Z}}\) admits an exponential trichotomy and there exists \( C > 0 \) such that (3.1) holds;
2. there exists \( \rho > 0 \) such that
\[
\|A_m - B_m\| \leq \rho, \quad \text{for} \quad m \in \mathbb{Z}. \tag{4.1}
\]
If \( \rho \) is sufficiently small, then the sequence \((B_m)_{m \in \mathbb{Z}}\) admits an exponential trichotomy.

Proof. We first note that it follows from (3.1) and (4.1) that \( \|B_m\| \leq C + \rho \), for all \( m \in \mathbb{Z} \). By Theorem 3.1, there exist sequences \((S_i^m)_{m \in \mathbb{Z}}, i = 1, 2\) of bounded, self-adjoint and invertible operators, sequences
5. Lyapunov sequences and nonlinear perturbations

We consider the nonlinear dynamics

\[ x_{m+1} = A_m x_m + f_m(x_m), \quad (5.1) \]

where \( f_m : X \to X, \ m \in \mathbb{Z} \) are continuous functions. We are going to show that if the linear part of the equation (5.1) admits an exponential trichotomy and if the nonlinear perturbation is sufficiently small then each solution of (5.1) has the property that the associated sequence obtained by projecting the solution on the stable subspace of our trichotomy is uniformly exponentially stable. The precise statement is given below.

**Theorem 5.1.** Assume that the sequence \( (A_m)_{m \in \mathbb{Z}} \) admits an exponential trichotomy and that the sequence \( (f_m)_{m \in \mathbb{Z}} \) satisfies:

**Proof.**
1. there exists $\rho > 0$ such that

$$\|f_m(x)\| \leq \rho \|x\|, \quad \text{for } m \in \mathbb{Z} \quad \text{and} \quad x \in X; \quad (5.2)$$

2. 

$$\rho^1_{m+1} f_m(x) = f_m(\rho^1_m x), \quad \text{for } m \in \mathbb{Z} \quad \text{and} \quad x \in X. \quad (5.3)$$

Then for sufficiently small $\rho$, there exists $L > 0$ and $\eta \in (0, 1)$ such that

$$\|\rho^1_n x_n\| \leq L \eta^{n-m} \|\rho^1_m x_m\|, \quad (5.4)$$

for $m \geq n$ and every solution $(x_m)_{m \in \mathbb{Z}}$ of $(5.1)$.

Proof. Consider operators $S^1_m$, $m \in \mathbb{Z}$ defined as in $(3.11)$. We define a sequence of functions $H_m$, $m \in \mathbb{Z}$ by

$$H_m(x) = \langle S^1_m x, x \rangle, \quad x \in X.$$  

Furthermore, let $u_m = \rho^1_m x_m$, $m \in \mathbb{Z}$. In the proof of Theorem 3.1 we have showed that (recalling that $B_m = e^{\omega_1 A_m}$)

$$B^*_m \rho^1_{m+1} B_m = e^{-2(\lambda-\rho)} \sum_{k \geq m} (B(k, m) \rho^1_m)^* B(k, m) \rho^1_m e^{2(\lambda-\rho)(k-m)} - e^{-2(\lambda-\rho)} (\rho^1_m)^* \rho^1_m$$

$$- e^{2(\lambda-\rho)} \sum_{k < m} (B(k, m)(I - \rho^1_m))^* B(k, m)(I - \rho^1_m) e^{2(\lambda-\rho)(m-k)}$$

$$- e^{2(\lambda-\rho)} (I - \rho^1_m)^* (I - \rho^1_m).$$

In particular, this implies that

$$e^{2\omega_1 A_m^* S^1_m A_m} \leq e^{-2(\lambda-\rho) S^1_m}, \quad \forall m \in \mathbb{Z}. \quad (5.5)$$

Using $(5.3)$, we have

$$H_{m+1}(u_{m+1}) = \langle S^1_{m+1} u_{m+1}, u_{m+1} \rangle$$

$$= \langle S^1_{m+1} \rho^1_{m+1} x_{m+1}, \rho^1_{m+1} x_{m+1} \rangle$$

$$= \langle S^1_{m+1} \rho^1_{m+1} (A_m x_m + f_m(x_m)), \rho^1_{m+1} (A_m x_m + f_m(x_m)) \rangle$$

$$= \langle A_m S^1_{m+1} A_m u_m + f_m(u_m), A_m u_m + f_m(u_m) \rangle$$

$$= \langle A_m^* S^1_{m+1} A_m u_m, u_m \rangle + \langle S^1_{m+1} A_m u_m, f_m(u_m) \rangle$$

$$+ \langle S^1_{m+1} f_m(u_m), A_m u_m \rangle + \langle S^1_{m+1} f_m(u_m), f_m(u_m) \rangle.$$  

By $(3.1)$, $(3.2)$, $(5.2)$ and $(5.5)$,

$$H_{m+1}(u_{m+1}) \leq e^{-2(\lambda-\rho)-2\omega_1} H_m(u_m) + 2 ||S^1_{m+1}|| \cdot ||A_m u_m|| \cdot ||f_m(u_m)||$$

$$+ ||S^1_{m+1}|| \cdot ||f_m(u_m)||^2$$

$$\leq e^{-2(\lambda-\rho)-2\omega_1} H_m(u_m) + 2 K \rho ||u_m||^2 + K \rho^2 ||u_m||^2.$$  

Noting that $H_n(z) \geq ||z||^2$ for $z \in \text{Im} \rho^1_n$, and that $u_n \in \text{Im} \rho^1_n$, we conclude that

$$H_{m+1}(u_{m+1}) \leq \eta^2 H_m(u_m), \quad (5.6)$$

where

$$\eta^2 = e^{-2(\lambda-\rho)-2\omega_1} + 2 K \rho + K \rho^2.$$  

By choosing $\rho$ sufficiently small, we can achieve that $\eta \in (0, 1)$. Iterating $(5.6)$, we obtain that

$$H_n(u_n) \leq \eta^{2(n-m)} H_m(u_m), \quad \text{for } n \geq m.$$  

Since $||u_n||^2 \leq H_n(u_n)$ and $H_m(u_m) \leq K ||u_m||^2$, we conclude that $(5.4)$ holds with $L = \sqrt{K}$. □
6. Characterization of exponential trichotomies on $Z_0^+$

In this section we establish versions of the results of Section 3 for trichotomies on the half-line $Z_0^+$. For each $n \geq 0$, let

$$S(n) = \left\{ v \in X : \sup_{m \geq n} \|A(m, n)v\| < +\infty \right\}.$$  

We begin with the following version of Proposition 2.3 for dichotomies on $Z_0^+$. It turns out that in this case only the ranges of projections $P^1_n$ are uniquely determined (see [45]).

**Proposition 6.1.** We have $\text{Im} P^1_n = S(n)$, for each $n \geq 0$.

We are also able to say what happens with the ranges of projections $P^2_n$ (again we refer to [45]).

**Proposition 6.2.** Assume that the sequence $(A_m)_{m \geq 0}$ of bounded operators on $X$ admits an exponential dichotomy and let $Z$ be any closed subspace of $X$ such that $X = S \oplus Z$. Then, $(A_m)_{m \geq 0}$ admits an exponential dichotomy with respect to projections $P^1_m$, $m \geq 0$, $i = 1, 2$ such that $\text{Im} P^i_m = A(m, 0)Z$.

In other words, the unstable direction at time $m \geq 0$ can be an image under the action of the cocycle of any closed subspace $Z$ with the property that $X$ can be decomposed as a direct sum of $S(0)$ and $Z$.

We can now comment on our strategy of establishing versions of the results of Section 3 for trichotomies on the half-line $Z_0^+$. We first emphasize that one is not able to proceed using the same type of arguments as in the proofs of Theorems 3.1 and 3.5. The major reason for this is the fact that Theorem 2.4 is no longer valid when one passes to the case of dichotomies on the half-line and consequently dichotomies on $Z_0^+$ cannot be characterized in terms of the invertibility (hyperbolicity) of a single operator. For appropriate versions of Theorem 2.4 for dichotomies on the half-line we refer to [45]. In order to overcome this difficulty, we will apply a different strategy of extending dichotomies on $Z_0^+$ to dichotomies on $Z$ and applying already established results. The following result which is only a particular case of result established in [4] will be crucial.

**Theorem 6.3.** A sequence $(A_m)_{m \in \mathbb{Z}} \subset B(X)$ admits an exponential dichotomy on $Z$, if and only if there exist projections $P^+_m$ for $m \geq 0$ and projections $P^-_m$ for $m \leq 0$ such that:

1. $(A_m)_{m \geq 0}$ admits an exponential dichotomy on $Z^+_0$ with projections $P^+_m$;
2. $(A_m)_{m \leq 0}$ admits an exponential dichotomy on $Z^-_0$ with projections $P^-_m$;
3. $X = \text{Im} P^+_0 \oplus \text{Ker} P^-_0$.

The following is a version of Theorem 3.1 for dichotomies on $Z_0^+$.

**Theorem 6.4.** Assume that the sequence $(A_m)_{m \in \mathbb{Z}} \subset B(X)$ admits an exponential trichotomy and that there exists $C > 0$ such that

$$\|A_m\| \leq C, \quad \text{for } m \geq 0.$$  \hspace{1cm} (6.1)

Then, there exist sequences $(S^i_n)_{n \geq 0}$, $i = 1, 2$ of bounded, self-adjoint and invertible operators on $X$, constants $K, \delta, \omega_1 > 0$ and $\omega_2 < 0$ such that (3.2), (3.3) and (3.4) hold for each $m \geq 0$ and $i \in \{1, 2\}$.

**Proof.** As in the proof of Theorem 3.1, take an arbitrary $\omega_1 \in (c, d)$ and consider the sequence $B_m = e^{\omega_1 A_m}$, $m \geq 0$. It follows from (3.9) and (3.10) that the sequence $(B_m)_{m \geq 0}$ admits an exponential dichotomy with projections $P^+_m = P^1_m$. Choose an arbitrary hyperbolic operator $B \in B(X)$ such that both $\text{Im} P^+_0$ and $\text{Ker} P^-_0$ are invariant under $B$ and with the property that the spectrum of restriction $B|_{\text{Im} P^+_0}$ is contained in $\{ \lambda : |\lambda| < 1 \}$ while the spectrum of restriction $B|_{\text{Ker} P^-_0}$ is contained in $\{ \lambda : |\lambda| > 1 \}$. Furthermore, extend the sequence $(B_m)_{m \geq 0}$ to a sequence defined on the whole line by setting $B_m = B$ for $m < 0$. It follows from Theorem 6.3 that the sequence $(B_m)_{m \in \mathbb{Z}}$ admits an exponential dichotomy on $Z$. One can now proceed as in the proof of Theorem 3.1 and construct a sequence $(S^i_n)_{n \in \mathbb{Z}}$ of bounded, self-adjoint and invertible operators on $X$ satisfying (3.2), (3.3) and (3.4) for $i = 1$ and $m \in \mathbb{Z}$. In particular,
those inequalities hold for $m \geq 0$ which establishes the assertion of the theorem for $i = 1$. Similarly, one constructs the sequence $(S^2_m)_{m \geq 0}$.

The following is a converse of Theorem 6.4.

**Theorem 6.5.** Let $(A_m)_{m \geq 0} \subset B(X)$ be a sequence of operators such that (6.1) holds with some $C > 0$. Furthermore, suppose that there exist sequences $(S^i_m)_{m \geq 0}$, $i = 1, 2$ of bounded, self-adjoint and invertible operators on $X$, constants $K, \delta, \omega_1 > 0$ and $\omega_2 < 0$ such that the inequalities (3.2), (3.3) and (3.4) hold for each $m \geq 0$ and $i \in \{1, 2\}$. Then the sequence $(A_m)_{m \geq 0}$ admits an exponential trichotomy.

**Proof.** It is an easy exercise in the functional calculus for self-adjoint operators to show that there exists $A \in B(X)$ self-adjoint such that

$$e^{2\omega_1}A S^1_0 A - S^1_0 \leq -\delta I.$$ By Theorem 2.2, we also have that

$$e^{2\omega_1}A S^1_0 A - S^1_0 \leq -\delta I.$$ Setting $A_m = A$ and $S^1_n = S^1_0$ for $n < 0$, we conclude that for $i = 1$, the inequalities (3.2), (3.3) and (3.4) hold for every $m \in \mathbb{Z}$. Arguing as in the proof of Theorem 3.5, one concludes that the sequence $(e^{\omega_1} A_m)_{m \in \mathbb{Z}}$ admits an exponential dichotomy on $Z$ which readily implies that the sequence $(e^{\omega_1} A_m)_{m \geq 0}$ admits an exponential dichotomy on $Z^+_0$ with respect to projections $P^1_m$ and $Q^1_m = 1 - P^1_m$, $m \geq 0$. Similarly, the sequence $(e^{\omega_1} A_m)_{m \geq 0}$ admits an exponential dichotomy on $Z^-_0$ with respect to projections $P^2_m$ and $Q^2_m = \text{Id} - P^2_m$, $m \geq 0$. Using Proposition 6.1, one can argue as in the proof of Lemma 3.6 and obtain the first inclusion in (3.27). Moreover, it follows from Proposition 6.2 that we may assume that the second inclusion in (3.27) also holds. One can now proceed as in the proof of Theorem 3.5 and conclude that the sequence $(A_m)_{m \geq 0}$ admits an exponential trichotomy.

Using Theorems 6.4 and 6.5, one can now establish versions of Theorems 4.1 and 5.1 for trichotomies on $Z^+_0$ with the exact same proofs as in the case of trichotomies on $Z$.

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