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[5532] First Galley Proofs

NEW IMPROVEMENT OF THE CONVERSE JENSEN INEQUALITY

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Abstract. We give a new refinement of the converse Jensen inequality for linear functionals as well as improvements of some related results. Especially, we give two refinements of the converse Hölder inequality for functionals and a refinement of the converse Minkowski inequality for functionals. Application on the quasi-arithmetic and power mean is given.

1. Introduction

Let I be an interval in $\mathbb{R}$ and $f: I \to \mathbb{R}$ a convex function on I. If $\boldsymbol{x} = (x_1, \ldots, x_n)$ is any n-tuple in I^n and $\boldsymbol{p} = (p_1, \ldots, p_n)$ a nonnegative n-tuple such that $P_n = \sum_{i=1}^n p_i > 0$, then the well known *Jensen's inequality*

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \leqslant \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) \tag{1}$$

holds (see [6] or for example [12, p. 43]). If f is strictly convex then (1) is strict unless $x_i = c$ for all $i \in \{j : p_j > 0\}$.

Strongly related to Jensen's inequality is the so called *converse Jensen inequality* (see [10] or for example [11, p. 9])

$$\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) \leqslant \frac{M-\bar{x}}{M-m} f(m) + \frac{\bar{x}-m}{M-m} f(M), \tag{2}$$

which holds when $f: I \to \mathbb{R}$ is a convex function on I, $[m,M] \subset I$, $-\infty < m < M < +\infty$, $\boldsymbol{p}, \boldsymbol{x}$ are as in (1) and $\bar{x} = \frac{1}{P_n}\sum_{i=1}^n p_i x_i$. If f is strictly convex then (2) is strict unless $x_i \in \{m, M\}$ for all $i \in \{j : p_j > 0\}$.

Let E be a nonempty set and L a *linear class* of functions $f: E \to \mathbb{R}$ having the properties:

(L1) $(\forall f, g \in L)(\forall a, b \in \mathbb{R})\ \ af + bg \in L$;

(L2) $\mathbf{1} \in L$ (that is if $(\forall t \in E)\, f(t) = 1$ then $f \in L$).

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In other words L is a subspace of the vector space $\mathbb{R}^E$ over $\mathbb{R}$ containing $\mathbf{1}$.

In some cases we will also need to equip our linear class L with an aditional property denoted by (L3):

(L3) $(\forall f,g\in L)\,(\min\{f,g\}\in L\wedge \max\{f,g\}\in L)$ (*lattice* property).

Obviously, $\left(\mathbb{R}^E,\leqslant\right)$ (with standard ordering) is a lattice. It can also be easily verified that a subspace $X\subseteq\mathbb{R}^E$ is a lattice if and only if $x\in X$ implies $|x|\in X$. This is a simple concequence of the fact that for every $x\in X$ the functions $|x|$, x^- and x^+ can be defined by

$$|x|(t)=|x(t)|,\quad x^+(t)=\max\{0,x(t)\},\ \ x^-(t)=-\min\{0,x(t)\},\qquad t\in E,$$

and

$$x^++x^-=|x|,\qquad x^+-x^-=x,$$

$$\min\{x,y\}=\frac{1}{2}(x+y-|x-y|),\quad \max\{x,y\}=\frac{1}{2}(x+y+|x-y|). \tag{3}$$

We consider *positive linear functionals* $A\colon L\to\mathbb{R}$, or in other words we assume:

(A1) $(\forall f,g\in L)\,(\forall a,b\in\mathbb{R})\ \ A(af+bg)=aA(f)+bA(g)$ (linearity)

(A2) $(\forall f\in L)\,(f\geqslant 0\longrightarrow A(f)\geqslant 0)$ (positivity).

If additionally the condition $A(1)=1$ is satisfied, we say that A is a *positive normalized linear functional*.

In [7] (or see for example [12, p. 47]) we can find the following generalization of Jensen's inequality for convex functions which involves positive normalized linear functionals. Inequality (4) is known as *Jessen's inequality* for convex functions.

THEOREM 1. *Let L satisfy L1, L2 on a nonempty set E and let A be a positive normalized linear functional. If f is a continuous convex function on an interval $I\subset\mathbb{R}$ then for all $g\in L$ such that $f(g)\in L$ we have $A(g)\in I$ and*

$$f(A(g))\leqslant A(f(g)). \tag{4}$$

Accordingly to Theorem 1 Beesack and Pečarić gave in [1] (or see [12, p. 98]) the following generalization of the converse Jensen inequality.

THEOREM 2. *Let L and A be as in Theorem 1. If $f\colon[m,M]\to\mathbb{R}$ is convex then for all $g\in L$ such that $f(g)\in L$ the inequality*

$$A(f(g))\leqslant\frac{M-A(g)}{M-m}f(m)+\frac{A(g)-m}{M-m}f(M) \tag{5}$$

holds.

REMARK 1. The right hand side of (5) is an increasing function in M and a decreasing function in m. This follows by writing it in the form

$$f(m)+(A(g)-m)\frac{f(M)-f(m)}{M-m}=f(M)-(M-A(g))\frac{f(M)-f(m)}{M-m}$$

and noting that $m\leqslant A(g)\leqslant M$, while both functions $m\mapsto\frac{f(M)-f(m)}{M-m}$ and $M\mapsto\frac{f(M)-f(m)}{M-m}$ are increasing by the convexity of ϕ.

In the same paper [1] (or see [12, p. 100–101]) the authors also proved the following theorem.

THEOREM 3. *Let L, A and g be as in Theorem* 1 *and let $f\colon [m,M]\to\mathbb{R}$ be a differentiable function.*

(i) If f' is strictly increasing on $[m,M]$ then

$$A(f(g))\leqslant\lambda+f(A(g)) \tag{6}$$

for some λ satisfying $0<\lambda<(M-m)(\mu-f'(m))$, where

$$\mu=\frac{f(M)-f(m)}{M-m}.$$

More precisely, λ may be determined as follows: Let $\tilde{x}$ be the (unique) solution of the equation $f'(x)=\mu$. Then

$$\lambda=f(m)+\mu(\tilde{x}-m)-f(\tilde{x})$$

satisfies (6).

(ii) If f' is strictly decreasing on $[m,M]$ then

$$f(A(g))\leqslant\lambda+A(f(g)) \tag{7}$$

for some λ satisfying $0<\lambda<(M-m)(f'(m)-\mu)$, where μ is defined as in (i). More precisely, for $\tilde{x}$ defined as in (i) we have that

$$\lambda=f(\tilde{x})-f(m)-\mu(\tilde{x}-m)$$

satisfies (7).

In [2] (or see [12, p. 101]) Beesack and Pečarić gave a generalization of Theorem 3 which at the same time presents a generalization of Knopp's inequality for convex functions [9].

THEOREM 4. *Let L and A be as in Theorem 1. Let $f\colon [m,M]\to\mathbb{R}$ be a convex function and J an interval in $\mathbb{R}$ such that $f([m,M])\subset J$. If $F\colon J\times J\to\mathbb{R}$ is increasing in the first variable then for all $g\in L$ such that $f(g)\in L$ the following inequality holds*

$$\begin{aligned} F\left(A\left(f\left(g\right)\right),f\left(A\left(g\right)\right)\right) &\leqslant \max_{x\in[m,M]} F\left(\frac{M-x}{M-m}f\left(m\right)+\frac{x-m}{M-m}f\left(M\right),f\left(x\right)\right) \\ &= \max_{\theta\in[0,1]} F\left(\theta f\left(m\right)+\left(1-\theta\right)f\left(M\right),f\left(\theta m+\left(1-\theta\right)M\right)\right). \end{aligned} \tag{8}$$

Furthermore, the right-hand side of (8) is an increasing function of M and a decreasing function of m.

In article [8] the following improvement of Theorem 2 was given.

THEOREM 5. *Let L satisfy L1, L2, L3 on a nonempty set E and let A be a positive normalized linear functional. If f is a convex function on $[m,M]$ then for all $g\in L$ such that $f(g)\in L$ we have $A(g)\in[m,M]$ and*

$$A\left(f\left(g\right)\right)\leqslant \frac{M-A\left(g\right)}{M-m}f\left(m\right)+\frac{A\left(g\right)-m}{M-m}f\left(M\right)-A\left(\tilde{g}\right)\delta_f, \tag{9}$$

where

$$\tilde{g}=\frac{1}{2}\mathbf{1}-\frac{\left|g-\frac{m+M}{2}\mathbf{1}\right|}{M-m},\quad \delta_f=f\left(m\right)+f\left(M\right)-2f\left(\frac{m+M}{2}\right).$$

2. New improvements

Throughout this section without further noticing when using $[m,M]$ we assume that $-\infty<m<M<\infty$.

Let $r_n(v)$ be defined recursively by

$$\begin{aligned} r_0(v) &= \min\{v,1-v\} \\ r_n(v) &= \min\{2r_{n-1}(v),1-2r_{n-1}(v)\} \end{aligned}$$

for $0\leqslant v\leqslant 1$. It has been shown in [3] that

$$r_n(v)=\begin{cases} 2^n v-k+1\ , & \frac{k-1}{2^n}\leqslant v\leqslant \frac{2k-1}{2^{n+1}}, \\ k-2^n v\ , & \frac{2k-1}{2^{n+1}}<v\leqslant \frac{k}{2^n}, \end{cases}$$

for $k=1,2,\ldots,2^n$.

It has been shown (see [3]) that if N is a nonnegative integer and f is convex on $[0,1]$, then

$$(1-v)f(0)+vf(1)\geqslant f(v)+\sum_{n=0}^{N-1} r_n(v)\sum_{k=1}^{2^n}\Delta_f(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(v) \tag{10}$$

where

$$\Delta_f(n,k) = f\left(\frac{k-1}{2^n}\right) + f\left(\frac{k}{2^n}\right) - 2f\left(\frac{2k-1}{2^{n+1}}\right),$$

and χ represents the characteristic function of the corresponding interval. If $N = 0$ then sum is zero, that is we have convexity.

In the paper [4] previous relation is extended to hold for an arbitrary interval. Following result is given.

LEMMA 1. *Let N be a nonnegative integer and let f be convex on $[a,b]$. Then*

$$(1-v)f(a) + vf(b) \geqslant f((1-v)a+vb) + \sum_{n=0}^{N-1} r_n(v) \sum_{k=1}^{2^n} \Delta_f(a,b,n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(v) \tag{11}$$

where

$$\Delta_f(a,b,n,k) = f\left(\frac{(2^n-k+1)a+(k-1)b}{2^n}\right) + f\left(\frac{(2^n-k)a+kb}{2^n}\right) \\ -2f\left(\frac{(2^{n+1}-2k+1)a+(2k-1)b}{2^{n+1}}\right),$$

and χ represents the characteristic function of the corresponding interval.

Next theorem is our main result.

THEOREM 6. *Let L satisfy L1, L2, L3 on a nonempty set E and let A be a positive normalized linear functional. If f is a convex function on $[m,M]$ then for all $g \in L$ such that $f(g) \in L$ we have $A(g) \in [m,M]$ and*

$$\frac{M-A(g)}{M-m}f(m) + \frac{A(g)-m}{M-m}f(M) \\ \geqslant A(f(g)) + \sum_{n=0}^{N-1}\sum_{k=1}^{2^n} \Delta_f(m,M,n,k) A\left(\left(r_n \cdot \chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{g-m}{M-m}\right)\right) \tag{12}$$

where

$$\Delta_f(m,M,n,k) = f\left(\frac{(2^n-k+1)m+(k-1)M}{2^n}\right) + f\left(\frac{(2^n-k)m+kM}{2^n}\right) \\ -2f\left(\frac{\left(2^{n+1}-2k+1\right)m+(2k-1)M}{2^{n+1}}\right),$$

and χ represents the characteristic function of the corresponding interval

Proof. First observe that $f(g) \in L$ also means that the composition $f(g)$ is well defined, hence $g(E) \subseteq [m,M]$. Now we have $m \leqslant g \leqslant M$ and

$$m = A(m) \leqslant A(g) \leqslant A(M) = M.$$

If we put $a=m,\ b=M,\ x=(1-v)a+vb$ in Lemma 1 using

$$v=\frac{x-m}{M-m},\qquad 1-v=\frac{M-x}{M-m}$$

we get

$$\begin{aligned}&\frac{M-x}{M-m}f(m)+\frac{x-m}{M-m}f(M)\\ &\geqslant f(x)+\sum_{n=0}^{N-1}r_n\left(\frac{x-m}{M-m}\right)\sum_{k=1}^{2^n}\Delta_f(m,M,n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{x-m}{M-m}\right)\end{aligned}$$

Let $g\in L$ be such that $f(g)\in L$. Applying A to the above inequality with $x\leftrightarrow g(x)$ we obtain

$$\begin{aligned}&\frac{M-A(g)}{M-m}f(m)+\frac{A(g)-m}{M-m}f(M)\\ &\geqslant A(f(g))+\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_f(m,M,n,k)A\left(r_n\left(\frac{g-m}{M-m}\right)\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{g-m}{M-m}\right)\right)\end{aligned}$$

which is inequality (12) $\square$

REMARK 2. Under conditions of the Theorem 6 from positivity of A follows

$$A\left(\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\left(\frac{g-m}{M-m}\right)\right)\right)\geqslant 0,$$

and from Jensen's inequality follows

$$\Delta_f(m,M,n,k)\geqslant 0,$$

so Theorem 6 is an improvement of Theorem 2.

REMARK 3. If we write equation (12) in the following form

$$\begin{aligned}&\frac{M-A(g)}{M-m}f(m)+\frac{A(g)-m}{M-m}f(M)\\ &\geqslant A(f(g))+\Delta_f(m,M,0,1)\cdot A\left(r_0\left(\frac{g-m}{M-m}\right)\cdot\chi_{\left(\frac{0}{2^0},\frac{1}{2^0}\right)}\left(\frac{g-m}{M-m}\right)\right)\\ &\quad+\sum_{n=1}^{N-1}\sum_{k=1}^{2^n}\Delta_f(m,M,n,k)A\left(\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{g-m}{M-m}\right)\right)\end{aligned}$$

and notice

$$\Delta_f(m,M,0,1)=f(m)+f(M)-2f\left(\frac{m+M}{2}\right)$$

$$r_0\left(\frac{g-m}{M-m}\right)=\frac{1}{2}-\frac{\left|g-\frac{m+M}{2}\right|}{M-m}$$

$$\chi_{\left(\frac{0}{2^0},\frac{1}{2^0}\right)}\left(\frac{g-m}{M-m}\right)=\chi_{(0,1)}\left(\frac{g-m}{M-m}\right)=1$$

we have that Theorem 6 is an improvement of Theorem 5.

We use Theorem 6 to obtain refinements of other inequalities mentioned previously. First we give an improvement of Theorem 4 in the special case $F(x,y) = x - y$.

THEOREM 7. *Under the assumptions of Theorem* 6 *the following inequality holds*

$$\begin{aligned} & A(f(g)) - f(A(g)) \\ \leqslant & \max_{x\in[m,M]} \left\{ \frac{M-x}{M-m} f(m) + \frac{x-m}{M-m} f(M) - f(x) \right\} \\ & - \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} \Delta_f(m,M,n,k) A\left(\left(r_n \cdot \chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)} \right) \left(\frac{g-m}{M-m} \right) \right) \\ = & \max_{\theta\in[0,1]} \{\theta f(m) + (1-\theta) f(M) - f(\theta m + (1-\theta) M)\} \\ & - \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} \Delta_f(m,M,n,k) A\left(\left(r_n \cdot \chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)} \right) \left(\frac{g-m}{M-m} \right) \right). \end{aligned} \tag{13}$$

Proof. This is an immediate consequence of Theorem 6. The identity follows from the change of variable $\theta = (M-x)/(M-m)$ so that for $x \in [m,M]$ we have $\theta \in [0,1]$ and $x = \theta m + (1-\theta)M$. □

Next we give an improvement of Theorem 3. We will consider only the case when f' is strictly increasing (and therefore f convex) since an analogous result for f' strictly decreasing can be obtained in a similar way.

THEOREM 8. *Let L and A be as in Theorem* 6. *If $f\colon [m,M] \to \mathbb{R}$ is a differentiable function such that f' is strictly increasing on $[m,M]$ then for all $g \in L$ such that $f(g) \in L$ the inequality*

$$A(f(g)) \leqslant \lambda + f(A(g)) - \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} \Delta_f(m,M,n,k) A\left(\left(r_n \cdot \chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)} \right) \left(\frac{g-m}{M-m} \right) \right) \tag{14}$$

holds for some λ satisfying $0 < \lambda < (M-m)(\mu - f'(m))$, *where*

$$\mu = \frac{f(M) - f(m)}{M-m}.$$

More precisely, λ may be determined as follows: Let $\tilde{x}$ be the (unique) solution of the equation $f'(x) = \mu$. Then

$$\lambda = f(m) + \mu(\tilde{x} - m) - f(\tilde{x})$$

satisfies (14).

Proof. By Theorem 7 we have

$$\begin{aligned} & A(f(g)) - f(A(g)) \\ \leqslant & \max_{x\in[m,M]} \phi(x;m,M,f) - \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} \Delta_f(m,M,n,k) A\left(\left(r_n \cdot \chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)} \right) \left(\frac{g-m}{M-m} \right) \right), \end{aligned} \tag{15}$$

where $\phi:\ [m,M]\to\mathbb{R}$ is defined by

$$\phi(x) := \phi(x;m,M,f) = \frac{(M-x)f(m)+(x-m)f(M)}{M-m} - f(x).$$

Observe that $\phi(m)=\phi(M)=0$ and

$$\phi'(x) = \frac{f(M)-f(m)}{M-m} - f'(x) = \mu - f'(x).$$

Since f is strictly convex ϕ' is strictly decreasing on $[m,M]$ and the equation $\phi'(x)=0$ (that is, $f'(x)=\mu$) holds for a unique $x=\tilde{x}\in(m,M)$. It follows that $\phi(x)\geqslant 0$ for all $x\in[m,M]$ with equality for $x\in\{m,M\}$. Consequently, the maximum value on the right hand side of (15) is attained at $x=\tilde{x}$ and for

$$\begin{aligned}\lambda &= \phi(\tilde{x}) = \frac{(M-\tilde{x})f(m)+(\tilde{x}-m)f(M)}{(M-m)} - f(\tilde{x})\\ &= f(m)+\mu(\tilde{x}-m)-f(\tilde{x})\end{aligned}$$

we have that

$$A(f(g)) \leqslant \lambda + f(A(g)) - \sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_f(m,M,n,k)A\left(\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{g-m}{M-m}\right)\right) \qquad \square$$

REMARK 4. Analogously as in Remark 3 we see that previous two results are improvements of Theorem 13 and Theorem 14 from [8] .

We present two more applications of our main result.

COROLLARY 1. *Let L and A be as in Theorem 6. If $g\in L$ is such that $\log g$ belongs to L and $g(E)\subseteq[m,M]\subset\mathbb{R}_+$ then*

$$A(g) \leqslant \exp(A(\log g))\frac{\exp S\left(\frac{M}{m}\right)}{\prod_{n=0}^{N-1}\prod_{k=1}^{2^n}\left(\frac{\left((2^{n+1}-2k+1)m+(2k-1)M\right)^2}{2((2^n-k+1)m+(k-1)M)((2^n-1)m+kM)}\right)^{A(R_n(m,M,n,k,g))}}, \tag{16}$$

where $S(\cdot)$ is Specht ratio and

$$R_n(m,M,n,k,g) = \left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{g-m}{M-m}\right).$$

Proof. This is a special case of Theorem 8 for $f=-\log$. In this case (14) becomes

$$-A(\log g) \leqslant \lambda - \log A(g) - \sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_{-\log}(m,M,n,k)A\left(\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{g-m}{M-m}\right)\right)$$

that is,

$$\begin{aligned}\exp\log A(g) &\leqslant \exp\left(A(\log g)+\lambda-\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_{-\log}(m,M,n,k)A\left(R_n(m,M,n,k,g)\right)\right)\\ &= \exp\left(A(\log g)\right)\frac{\exp\lambda}{\exp\left(\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_{-\log}(m,M,n,k)A\left(R_n(m,M,n,k,g)\right)\right)},\end{aligned}$$

where

$$\begin{aligned}\Delta_{-\log}(m,M,n,k) &= -\log\left(\frac{(2^n-k+1)m+(k-1)M}{2^n}\right)\\ &\quad -\log\left(\frac{(2^n-k)\,m+kM}{2^n}\right)+2\log\left(\frac{\left(2^{n+1}-2k+1\right)m+(2k-1)M}{2^{n+1}}\right)\\ &= \log\frac{\left((2^{n+1}-2k+1)m+(2k-1)M\right)^2}{2\left((2^n-k+1)m+(k-1)M\right)\left((2^n-1)m+kM\right)}\end{aligned}$$

$$\mu=\frac{\log m-\log M}{M-m},\quad \tilde{x}=-\frac{1}{\mu}=\frac{M-m}{\log M-\log m},$$

hence

$$\begin{aligned}\lambda &= -\log m+\mu\left(\tilde{x}-m\right)+\log\tilde{x}\\ &= \log\frac{\left(\frac{M}{m}\right)^{\frac{m}{M-m}}}{e\log\left(\frac{M}{m}\right)^{\frac{m}{M-m}}}=S\left(\frac{M}{m}\right),\end{aligned}$$

where $S(\cdot)$ is Specht ratio (see for example [5, p. 71]) defined by

$$S(h)=\frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}},\quad h\in\mathbb{R}_+\setminus\{1\}.$$

Considering all this we obtain (16). □

COROLLARY 2. *Let L and A be as in Theorem* 6. *If $p\in L$ is such that* $\log(p)$ *belongs to L and $p(E)\subseteq[m,M]\subset\mathbb{R}_+$ then*

$$\begin{aligned}A(p) &\leqslant \exp A(\log p)+\frac{M-m}{\log\frac{M}{m}}S\left(\frac{M}{m}\right)\\ &\quad -\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\left[\left(m^{2^n-k+1}M^{k-1}\right)^{\frac{1}{2^n}}+\left(m^{2^n-k}M^k\right)^{\frac{1}{2^n}}+2\left(m^{2^{n+1}-2k+1}M^{2k-1}\right)^{\frac{1}{2^{n+1}}}\right]\\ &\quad \times A\left(R_n(\log m,\log M,n,k,\log p)\right) \qquad (17)\end{aligned}$$

where $S(\cdot)$ is Specht ratio and R_n is defined as in Corollary 1.

Proof. This is a special case of Theorem 8 for $f=\exp$, $g=\log p$, $m=\log m$ and $M=\log M$. In this case (14) becomes

$$\begin{aligned}
A(\exp\log p) &\leqslant \lambda+\exp A(\log p)\\
&\quad -\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_{\exp}(\log m,\log M,n,k)A\left(\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{\log p-\log m}{\log M-\log m}\right)\right)\\
&= \lambda+\exp A(\log p)\\
&\quad -\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_{\exp}(\log m,\log M,n,k)A\left(R_n(\log m,\log M,n,k,\log p)\right)
\end{aligned}$$

where

$$\begin{aligned}
&\Delta_{\exp}(\log m,\log M,n,k)\\
&=\exp\left(\frac{(2^n-k+1)\log m+(k-1)\log M}{2^n}\right)\\
&\quad+\exp\left(\frac{(2^n-k)\log m+k\log M}{2^n}\right)-2\exp\left(\frac{\left(2^{n+1}-2k+1\right)\log m+(2k-1)\log M}{2^{n+1}}\right)\\
&=\left(m^{2^n-k+1}M^{k-1}\right)^{\frac{1}{2^n}}+\left(m^{2^n-k}M^k\right)^{\frac{1}{2^n}}+2\left(m^{2^{n+1}-2k+1}M^{2k-1}\right)^{\frac{1}{2^{n+1}}}
\end{aligned}$$

$$\mu=\frac{M-m}{\log M-\log m},\quad \tilde{x}=\log\mu=\log\frac{M-m}{\log M-\log m},$$

hence

$$\begin{aligned}
\lambda &= \exp\log m+\mu\left(\tilde{x}-\log m\right)-\exp\tilde{x}\\
&= m+\frac{M-m}{\log M-\log m}\left(\log\frac{M-m}{\log M-\log m}-\log m-1\right)=\frac{M-m}{\log\frac{M}{m}}S\left(\frac{M}{m}\right).
\end{aligned}$$

Considering all this we obtain (17). □

REMARK 5. Analogously as in Remark 3 we see that previous two results are improvements of Corollary 2 and Corollary 3 from [8] .

2.1. Quasi-arthmetic means

Now we shall use previous results for refinement of inequalities order among quasi-arithmetic means and inequalities among power means.

Let $I=\langle m,M\rangle$, $-\infty\leqslant m<M\leqslant+\infty$ and let $\psi,\chi\colon I\to\mathbb{R}$ be continuous and strictly monotonic. Suppose that L satisfy conditions L1, L2, A be a positive normalized linear functional and $\psi(g),\chi(g)\in L$ for some $g\in L$. We define quasi-arithmetic mean with respect to the operator A and ψ by

$$M_\psi(g,A)=\psi^{-1}\left(A\left(\psi(g)\right)\right),\quad g\in L. \tag{18}$$

As a special case of the quasi-arithmetic mean we study power mean, defined for $r \in \mathbb{R}$ by

$$M^{[r]}(g,A) = \begin{cases} (A(g^r))^{\frac{1}{r}} & , r \neq 0 \\ \exp(A(\log g)) & , r = 0. \end{cases} \tag{19}$$

THEOREM 9. *Let $I = \langle m, M \rangle$, $-\infty \leqslant m < M \leqslant +\infty$ and let $\psi, \chi : I \to \mathbb{R}$ be continuous and strictly monotonic. Suppose that L satisfy conditions L1, L2, L3 A be a positive normalized linear functional. Then for every $g \in L$ such that $\psi(g), \chi(g) \in L$ we have*

$$\begin{aligned} &(\psi(M) - \psi(m))A(\chi(g)) - (\chi(M) - \chi(m))A(\psi(g)) \\ &\leqslant psi(M)\chi(m) - \chi(M)\psi(m) - (\psi(M) - \psi(m)) \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} \Delta_f(\psi(m), \psi(M), n, k) \\ &\times A\left(\left(r_n \cdot \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}\right)\left(\frac{\psi(g) - \psi(m)}{\psi(M) - \psi(m)}\right)\right) \end{aligned} \tag{20}$$

provided $\phi = \chi \circ \psi^{-1}$ is convex. The inequality in (20) *is reversed when ϕ is concave.*

Proof. If ψ is increasing on I, we have

$$\psi(m) \leqslant \psi(g(t)) \leqslant \psi(M), \text{ for every } t \in E.$$

Now from Theorem 6 we have (with m and M replaced by $\psi(m), \psi(M)$, f replaced by $\chi \circ \psi^{-1}$, and g replaced by $\psi(g)$)

$$\begin{aligned} &A\left(\chi \circ \psi^{-1}(\psi(g))\right) \\ &\leqslant \frac{\psi(M) - A(\psi(g))}{\psi(M) - \psi(m)}(\chi \circ \psi^{-1})(\psi(m)) + \frac{A(\psi(g)) - \psi(m)}{\psi(M) - \psi(m)}(\chi \circ \psi^{-1})(\psi(M)) \\ &\quad - \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} \Delta_f(\psi(m), \psi(M), n, k) A\left(\left(r_n \cdot \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}\right)\left(\frac{\psi(g) - \psi(m)}{\psi(M) - \psi(m)}\right)\right) \end{aligned}$$

which gives inequality (20). Proof when ψ is decreasing and when ϕ is concave goes similarly. □

In the case of power mean (19) result (20) (for $\psi(x) = x^r$, $\chi(x) = x^s$) gives improvement of Goldman's inequality for positive linear functionals:

$$\begin{aligned} &(M^r - m^r)\left(M^{[s]}(g,A)\right)^s - (M^s - m^s)\left(M^{[r]}(g,A)\right)^r \\ &\leqslant M^r m^s - M^s m^r - (M^r - m^r) \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} \Delta_f(m^r, M^r, n, k) \\ &\times A\left(\left(r_n \cdot \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}\right)\left(\frac{g^r - m^r}{M^r - m^r}\right)\right) \end{aligned} \tag{21}$$

for $0<r<s$ or $r<0<s$ and the inequality is reversed for $r<s<0$. For $r=0$ we obtain (result (20) for $\psi(x)=\log x$, $\chi(x)=x^s$):

$$\begin{aligned}&\log\frac{M}{m}\left(M^{[s]}(g,A)\right)^s-(M^s-m^s)\log\left(M^{[0]}(g,A)\right)\\ &\leqslant m^s\log M-M^s\log m-(\log M-\log m)\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_f(\log(m),\log(M),n,k)\\ &\times A\left(\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{\log\frac{g}{m}}{\log\frac{M}{m}}\right)\right).\end{aligned}$$

Following theorem is an improvement of Theorem 3.9 from [12].

THEOREM 10. *a) Let L satisfy L1, L2, L3 on a nonempty set E and let A be a positive normalized linear functional. Assume that f is convex on $I=[m,M]$ such that $f''(x)\geqslant 0$ with equality for at most isolated points of I (so that f is strictly convex on I). Assume further that either (i) $f(x)>0$ for all $x\in I$, or (i') $f(x)>0$ for $m<x<M$ with either $f(m)=0$, $f'(m)\neq 0$, or $f(M)=0$, $f'(M)\neq 0$, or (ii) $f(x)<0$ for all $x\in I$, or (ii') $f(x)<0$ for $m<x<M$ with precisely one of $f(m)=0$, $f(M)=0$. Then for all $g\in L$ such that $f(g)\in L$ (so that $m\leqslant g(t)\leqslant M$ for all $t\in E$),*

$$A(f(g))\leqslant\lambda f(A(g))-\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_f(m,M,n,k)A\left(\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{g-m}{M-m}\right)\right)\tag{22}$$

holds for some $\lambda>1$ in cases (i), (i') or $\lambda\in\langle 0,1\rangle$ in cases (ii), (ii'). More precisely, a value of λ (depending only on m,M,f) for (22) may be determined as follows: Define $\mu=\frac{f(M)-f(m)}{M-m}$. If $\mu=0$, let $x=\tilde{x}$ be the unique solution of the equation $f'(x)=0$ ($m<\tilde{x}<M$). Then $\lambda=\frac{f(m)}{f(\tilde{x})}$ satisfies (22). If $\mu\neq 0$, let $x=\tilde{x}$ be the unique solution

$$\mu f(x)-f'(x)(f(M)+\mu(x-m))=0.$$

Then $\lambda=\frac{\mu}{f(\tilde{x})}$ satisfies (22). Moreover, we have $m<\tilde{x}<M$ in the cases (i), (ii).

b) Let all the assumptions of a) hold except that f is concave on I with $f''(x)\leqslant 0$ with equality for at most isolated points in I. Then the reverse inequality in (22) holds, where λ is determined as in a). Furthermore, $\lambda>1$ holds if $f(x)<0$ on $\langle m,M\rangle$ and $0<\lambda<1$ if $f(x)>0$ on $\langle m,M\rangle$.

Proof. We omit the proof as it is the same as in [12] except instead of Theorem 2 we use Theorem 6. □

2.2. Refinements of the converse Hölder inequality

Using Lemma 1 now we give refinement of the converse Hölder inequality for functionals. In article [13] following refinement of the known converse Hölder inequality was given.

THEOREM 11. *Let A be a positive linear functional on a linear class L. Let $p \in \mathbb{R}$, $q = \frac{p}{p-1}$, and $w, f, g \geqslant 0$ on E with $wf^p, wg^q, wfg \in L$.*

Let m, M be such that $0 < m \leqslant f(x)g^{-q/p}(x) \leqslant M$ for $x \in E$.

If $p > 1$, then

$$A(wfg) \geqslant K(p,m,M)A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(wg^q) + \Delta(g^q, fg)N(p,m,M) \tag{23}$$
$$\geqslant K(p,m,M)A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(wg^q) \tag{24}$$

where

$$K(p,m,M) = |p|^{\frac{1}{p}}|q|^{\frac{1}{q}} \frac{(M-m)^{\frac{1}{p}}|mM^p - Mm^p|^{\frac{1}{q}}}{|M^p - m^p|}$$
$$N(p,m,M) = \frac{m^p + M^p - 2\left(\frac{m+M}{2}\right)^p}{M^p - m^p}$$

and

$$\Delta(g^q, fg) = A\left(w\left(\frac{M-m}{2}g^q - \left|fg - \frac{m+M}{2}g^q\right|\right)\right).$$

If $0 < p < 1$ and $A(wg^q) > 0$, or $p < 0$ and $A(wf^p) > 0$, then the reversed inequalities in (23) and (24) hold.

Using Lemma 1 we give improvement of previous theorem using the same idea as in Theorem 3 in [13], only with Lemma 1.

We will also use the AG inequality in the following form:

PROPOSITION 1. *Let a, b be positive real numbers. If α, β are positive real numbers such that $\alpha + \beta = 1$, then*

$$\alpha a + \beta b \geqslant a^{\alpha} b^{\beta}. \tag{25}$$

If $\alpha < 0$ or $\alpha > 1$, then the reversed inequality in (25) holds.

THEOREM 12. *Let L satisfy L1, L2, L3 on a nonempty set E and let A be a positive normalized linear functional. Let $p \in \mathbb{R}$, $q = \frac{p}{p-1}$, and $w, f, g \geqslant 0$ on E with $wf^p, wg^q, wfg \in L$.*

Let m, M be such that $0 < m \leqslant f(x)g^{-q/p}(x) \leqslant M$ for $x \in E$.

If $p > 1$, then

$$A(wfg) \geqslant K(p,m,M)A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(wg^q) \tag{26}$$
$$+ \frac{M-m}{M^p - m^p} \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} \Delta_{x^p}(m,M,n,k) A\left((wg^q) \cdot \left(r_n \cdot \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}\right)\left(\frac{fg^{-\frac{q}{p}} - m}{M-m}\right)\right)$$
$$\geqslant K(p,m,M)A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(wg^q) \tag{27}$$

where

$$K(p,m,M) = |p|^{\frac{1}{p}}|q|^{\frac{1}{q}} \frac{(M-m)^{\frac{1}{p}}|mM^p - Mm^p|^{\frac{1}{q}}}{|M^p - m^p|}.$$

If $0 < p < 1$ and $A(wg^q) > 0$, or $p < 0$ and $A(wf^p) > 0$, then the reversed inequalities in (26) and (27) hold.

Proof. Putting in Lemma 1 $1 - v = \alpha$, $v = \beta$ where α and β are positive real numbers such that $\alpha + \beta = 1$, $f(x) = x^p$, $p > 1$ we have following ineqality:

$$(\alpha x + \beta y)^p \leqslant \alpha x^p + \beta y^p - \sum_{n=0}^{N-1} r_n(\beta) \sum_{k=1}^{2^n} \Delta_{x^p}(x,y,n,k) \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}(\beta). \tag{28}$$

Let h be a function from L such that $0 < m \leqslant h(x) \leqslant M$ for $x \in E$, $m \neq M$, and define α and β as following:

$$\alpha(x) = \frac{M - h(x)}{M - m}, \quad \beta(x) = \frac{h(x) - m}{M - m}.$$

Obviously, $\alpha(x) + \beta(x) = 1$, $h(x) = \alpha(x)m + \beta(x)M$. Putting in (28): $x = m$, $y = M$, and above-defined $\alpha(x)$ and $\beta(x)$ we have

$$\begin{aligned} h^p(x) \leqslant & \frac{M - h(x)}{M - m} m^p + \frac{h(x) - m}{M - m} M^p \\ & - \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} \Delta_{x^p}(m,M,n,k) \left(r_n \cdot \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)} \right) \left(\frac{h(x) - m}{M - m} \right). \end{aligned}$$

Multiplying that inequality with $k(x) \geqslant 0$ and using linear functional A we obtain:

$$\begin{aligned} A(kh^p) \leqslant & \frac{m^p}{M - m}(MA(k) - A(kh)) + \frac{M^p}{M - m}(A(kh) - mA(k)) \\ & - \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} \Delta_{x^p}(m,M,n,k) A\left(k \cdot \left(r_n \cdot \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)} \right) \left(\frac{h - m}{M - m} \right) \right). \end{aligned}$$

Putting $h = fg^{-\frac{q}{p}}$, $k = wg^q$, where $\frac{1}{p} + \frac{1}{q} = 1$ after multiplying with $M - m$ we get

$$\begin{aligned} & (M - m)A(wf^p) + (mM^p - Mm^p)A(wg^q) \\ & \quad + (M - m) \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} \Delta_{x^p}(m,M,n,k) A\left((wg^q) \cdot \left(r_n \cdot \chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)} \right) \left(\frac{fg^{-\frac{q}{p}} - m}{M - m} \right) \right) \\ & \leqslant (M^p - m^p)A(wfg). \end{aligned} \tag{29}$$

Using inequality (25) with $\alpha = \frac{1}{p} > 0$, $\beta = \frac{1}{q} > 0$, $a = p(M-m)A(wf^p) \geqslant 0$ and $b = q(mM^p - Mm^p)A(wg^q) \geqslant 0$ we obtain:

$$\begin{aligned} & (M - m)A(wf^p) + (mM^p - Mm^p)A(wg^q) \\ & = \frac{p}{p}(M - m)A(wf^p) + \frac{q}{q}(mM^p - Mm^p)A(wg^q) \\ & \geqslant p^{\frac{1}{p}} q^{\frac{1}{q}} (M - m)^{\frac{1}{p}} (mM^p - Mm^p)^{\frac{1}{q}} A^{\frac{1}{p}}(wf^p) A^{\frac{1}{q}}(wg^q). \end{aligned} \tag{30}$$

Combining (29) and (30) we finally have

$$
\begin{aligned}
&p^{\frac{1}{p}}q^{\frac{1}{q}}(M-m)^{\frac{1}{p}}(mM^p-Mm^p)^{\frac{1}{q}}A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(wg^q)\\
&\quad+(M-m)\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_{x^p}(m,M,n,k)A\left((wg^q)\cdot\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{fg^{-\frac{q}{p}}-m}{M-m}\right)\right)\\
&\leqslant (M^p-m^p)A(wfg).
\end{aligned}
$$

If $p>1$, then $M^p-m^p>0$, and after dividing with M^p-m^p we get

$$
\begin{aligned}
&K(p,m,M)A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(wg^q)\\
&\quad+\frac{M-m}{M^p-m^p}\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_{x^p}(m,M,n,k)A\left((wg^q)\cdot\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{fg^{-\frac{q}{p}}-m}{M-m}\right)\right)\\
&\leqslant A(wfg).
\end{aligned}
$$

Other cases of exponent p follow as in article [13] and we omit proof for them. □

Closely related to the converse Hölder inequality is the converse Minkowski inequality. Following result considering the converse Minkowski inequality for functionals is attained in article [13].

THEOREM 13. *Suppose that the assumptions of Theorem* 11 *are satisfied. Then for* $p>1$

$$
\begin{aligned}
A^{\frac{1}{p}}(w(f+g)^p)&\geqslant K(p,m,M)\left(A^{\frac{1}{p}}(wf^p)+A^{\frac{1}{p}}(wg^p)\right)\\
&\quad+N(p,m,M)\frac{\Delta((f+g)^p,f(f+g)^{p-1})+\Delta((f+g)^p,g(f+g)^{p-1})}{A^{1-\frac{1}{p}}(w(f+g)^p)},
\end{aligned}
$$

and for $p<1$ $(p\neq 0)$ *the reversed inequality holds.*

Using the improvement of the converse Hölder inequality we can prove the following improvement of the previous result.

THEOREM 14. *Suppose that the assumptions of Theorem* 12 *are satisfied. Then for* $p>1$

$$
\begin{aligned}
&A^{\frac{1}{p}}(w(f+g)^p)\\
&\geqslant K(p,m,M)\left(A^{\frac{1}{p}}(wf^p)+A^{\frac{1}{p}}(wg^p)\right)+\frac{M-m}{M^p-m^p}\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_{x^p}(m,M,n,k)\\
&\times\frac{A\left((w(f+g)^p)\cdot\left(\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{\frac{f}{f+g}-m}{M-m}\right)\right)+\left(\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{\frac{g}{f+g}-m}{M-m}\right)\right)\right)}{A^{1-\frac{1}{p}}(w(f+g)^p)},
\end{aligned}
$$

and for $p<1$ *(* $p\neq 0$ *) the reversed inequality holds.*

Proof. Let $p > 1$. Writing $A(w(f+g)^p)$ as

$$A(w(f+g)(f+g)^{p-1}) = A(wf(f+g)^{p-1} + wg(f+g)^{p-1})$$

and using inequality (26) we obtain

$$\begin{aligned}
&A(w(f+g)^p)\\
&= A(wf(f+g)^{p-1}) + A(wg(f+g)^{p-1})\\
&\geqslant K(p,m,M)A^{\frac{1}{p}}(wf^p)A^{\frac{1}{q}}(w(f+g)^p) + \frac{M-m}{M^p-m^p}\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_{x^p}(m,M,n,k)\\
&\quad \times A\left(\left(w(f+g)^{q(p-1)}\right)\cdot\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{f(f+g)^{-\frac{q}{p}(p-1)}-m}{M-m}\right)\right)\\
&\quad + K(p,m,M)A^{\frac{1}{p}}(wg^p)A^{\frac{1}{q}}(w(f+g)^p) + \frac{M-m}{M^p-m^p}\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_{x^p}(m,M,n,k)\\
&\quad \times A\left(\left(w(f+g)^{q(p-1)}\right)\cdot\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{g(f+g)^{-\frac{q}{p}(p-1)}-m}{M-m}\right)\right)\\
&= K(p,m,M)A^{\frac{1}{q}}(w(f+g)^p)\left(A^{\frac{1}{p}}(wf^p)+A^{\frac{1}{p}}(wg^p)\right)\\
&\quad + \frac{M-m}{M^p-m^p}\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_{x^p}(m,M,n,k)A\left((w(f+g)^p)\cdot\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{\frac{f}{f+g}-m}{M-m}\right)\right)\\
&\quad + \frac{M-m}{M^p-m^p}\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_{x^p}(m,M,n,k)A\left((w(f+g)^p)\cdot\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{\frac{g}{f+g}-m}{M-m}\right)\right)\\
&= K(p,m,M)A^{\frac{1}{q}}(w(f+g)^p)\left(A^{\frac{1}{p}}(wf^p)+A^{\frac{1}{p}}(wg^p)\right)\\
&\quad + \frac{M-m}{M^p-m^p}\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_{x^p}(m,M,n,k)\\
&\quad \times A\left((w(f+g)^p)\cdot\left(\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{\frac{f}{f+g}-m}{M-m}\right)\right.\right.\\
&\quad \left.\left. + \left(\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{\frac{g}{f+g}-m}{M-m}\right)\right)\right)\right).
\end{aligned}$$

Dividing by $A^{\frac{1}{q}}(w(f+g)^p)$ we get desired result.

Similar proof holds for $p < 1$, $(p \neq 0)$. $\square$

REMARK 6. Analogously as in Remark 3 we see that Theorems 12 and 14 are improvements of Theorems 11 and 13, respectively.

We can give another version of converse Hölder inequality using Theorem 7 and substitutions as in Theorem 4.12 from [12].

THEOREM 15. *Let L satisfy L1, L2, L3 on a nonempty set E and let A be a positive normalized linear functional. Let $p>1$, $q=\frac{p}{p-1}$, and $w,f,g \geqslant 0$ on E with $wf^p, wg^q, wfg \in L$ and $A(wg^q)>0$.*

Let m,M be such that $0<m \leqslant f(x)g^{-q/p}(x) \leqslant M$ for $x \in E$.

If $p>1$, then

$$\begin{aligned}
&A^{p-1}(wg^q)A(wf^p)-A^p(wfg)\\
&\leqslant \max_{x\in[m,M]}\left\{\frac{M-x}{M-m}m^p+\frac{x-m}{M-m}M^p-x^p\right\}A^p(wg^q) \qquad (31)\\
&\quad -A^{p-1}(wg^q)\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_{x^p}(m,M,n,k)A\left(wg^q\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{fg^{-\frac{q}{p}}-m}{M-m}\right)\right)\\
&=\max_{\theta\in[0,1]}\left\{\theta m^p+(1-\theta)M^p-(\theta m+(1-\theta)M)^p\right\}A^p(wg^q) \qquad (32)\\
&\quad -A^{p-1}(wg^q)\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_{x^p}(m,M,n,k)A\left(wg^q\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{fg^{-\frac{q}{p}}-m}{M-m}\right)\right).
\end{aligned}$$

Proof. We set $A(f)=\frac{A(wf)}{A(w)}$ and get

$$\begin{aligned}
&\frac{A\left(wf\left(g\right)\right)}{A(w)}-f\left(\frac{A\left(wg\right)}{A(w)}\right)\\
&\leqslant \max_{x\in[m,M]}\left\{\frac{M-x}{M-m}f\left(m\right)+\frac{x-m}{M-m}f\left(M\right)-f\left(x\right)\right\}\\
&\quad -\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_f(m,M,n,k)\frac{A\left(w\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{g-m}{M-m}\right)\right)}{A(w)}
\end{aligned}$$

Putting $f(x)=x^p$, $p>1$, $g=fg^{-\frac{q}{p}}$, $w=wg^q$ we get

$$\begin{aligned}
&\frac{A\left(wf^p\right)}{A(wg^q)}-\frac{A^p\left(wfg\right)}{A^p(wg^q)}\\
&\leqslant \max_{x\in[m,M]}\left\{\frac{M-x}{M-m}m^p+\frac{x-m}{M-m}M^p-x^p\right\}\\
&\quad -\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\Delta_{x^p}(m,M,n,k)\frac{A\left(wg^q\left(r_n\cdot\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}\right)\left(\frac{fg^{-\frac{q}{p}}-m}{M-m}\right)\right)}{A(wg^q)}
\end{aligned}$$

Finally multiplying by $A^p(wg^q)$ we get inequality (31). Inequality (32) follows as in Theorem 7. $\square$

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