Limit theorems for random dynamical systems using the spectral method

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Let $I = [0, 1]$ denote the unit interval equipped with Borel $\sigma$-algebra $\mathcal{B}$ and a Lebesgue measure $m$. We say that $T : I \to I$ is a **piecewise expanding map** if there exists a partition

$$0 = x_0 < x_1 < \ldots < x_{k-1} < x_k = 1$$

and $\alpha > 1$ such that:

1. restriction $T|_{(x_{i-1}, x_i)}$ is a $C^1$ function which can be extended to a $C^1$ function on $[x_{i-1}, x_i]$;

2. $|T'(x)| \geq \alpha$ for $x \in (x_{i-1}, x_i)$;

3. $g(x) = \frac{1}{|T'(x)|}$ is a function of bounded variation.
Deterministic setting

Let $T$ be a piecewise expanding map and consider the associated transfer operator $\mathcal{L}: L^1(m) \to L^1(m)$ by

$$\mathcal{L}f(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T'(y)|}.$$ 

We note that $\mathcal{L}$ doesn’t have good spectral properties as an operator on $L^1(m)$. However, it has as an operator on $BV$ (space of functions of bounded variation). More precisely, $\mathcal{L}: BV \to BV$ is a quasicompact operator. This means that it can be written as

$$\mathcal{L} = \sum_{i=1}^{k} \lambda_i \Pi_i + N,$$

where $\lambda_i$ are eigenvalues for $\mathcal{L}$, $|\lambda_i| = r(\mathcal{L}) = 1$, each $\Pi_i$ is a
projections onto an one-dimensional subspace of $BV$, 
$\Pi_i N = N\Pi_i = 0$ and $r(N) < 1$. Some important consequences:

1. there exist an **absolutely continuous invariant measure** for $T$, i.e. 1 is an eigenvalue of $\mathcal{L}$ with a positive eigenvector;

2. under some additional assumptions acim is **unique** and **mixing**; we denote it by $\mu$ (from now on we assume that this is the case);

3. we have exponential **decay of correlation** and **limit laws** (central limit theorem, local central limit theorem, large deviations, almost sure invariance principle...
Central limit theorem

Assume that $\phi: I \to \mathbb{R}$ bounded observable in $BV$ such that

$$\int_{[0,1]} \phi \, d\mu = 0.$$ For each $n \in \mathbb{N}$, let

$$S_n = \sum_{k=0}^{n-1} \phi \circ T^k.$$

**Theorem (Rousseau–Egele, 1983)**

We have that

$$\lim_{n \to \infty} \int_{[0,1]} \frac{S_n^2}{n} = \sigma^2,$$

where

$$\sigma^2 = \int_{[0,1]} \phi^2 \, d\mu + 2 \sum_{n=1}^{\infty} \int_{[0,1]} \phi(\phi \circ T^n) \, d\mu < \infty.$$

If $\sigma^2 > 0$, then $\frac{S_n}{\sqrt{n}}$ converges in distribution to $N(0, \sigma^2)$. 
Theorem

If $\sigma^2 > 0$, then there exists $\delta > 0$ and a strictly convex, continuous and nonnegative function $c : (-\delta, \delta) \to \mathbb{R}$ which vanishes only at 0 such that

$$
\lim_{n \to \infty} \frac{1}{n} \log \mu(S_n > n\varepsilon) = -c(\varepsilon), \quad \text{for } \varepsilon \in (0, \delta).
$$
Ideas of the proofs

We define

\[ \mathcal{L}^\theta(g) = \mathcal{L}(e^{\theta \phi} g), \quad \text{for } g \in BV \text{ and } \theta \in \mathbb{C}. \]

Since \( \theta \mapsto \mathcal{L}^\theta \) is analytic, for \( \theta \) sufficiently close to 0,

\[ \mathcal{L}^\theta = \omega(\theta)\Pi(\theta) + N(\theta), \]

where \( \Pi(\theta) \) is a projection of rank 1 and \( r(N(\theta)) < |\omega(\theta)|. \)

CLT \((d\mu = f \, dm)\): for \( t \in \mathbb{R} \) we have that

\[
\lim_{n \to \infty} \int_{[0,1]} e^{itS_n/\sqrt{n}} \, d\mu = \lim_{n \to \infty} \int_{[0,1]} (\mathcal{L}^{it/\sqrt{n}})^n(f) \, dm = \lim_{n \to \infty} \omega(it/\sqrt{n})^n = e^{-t^2\sigma^2/2}.
\]
LDP:

we first show that $\omega'(0) = 0$ and $\omega''(0) = \sigma^2$ and then that

$$\lim_{n \to \infty} \frac{1}{n} \log \int_{[0,1]} e^{\theta S_n} d\mu = \Lambda(\theta),$$

where $\Lambda(\theta) = \log \omega(\theta)$, for $\theta \in \mathbb{R}$ sufficiently close to 0.
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and assume that \(\sigma : \Omega \rightarrow \Omega\) is invertible transformation that preserves \(\mathbb{P}\). Furthermore, assume that \(\mathbb{P}\) is ergodic. We now take the collection \(T_\omega, \omega \in \Omega\) of piecewise expanding maps. By \(\mathcal{L}_\omega\) we denote the transfer operator associated to \(T_\omega\). For \(\omega \in \Omega\) and \(n \in \mathbb{N}\), set

\[
T^n_\omega = T_{\sigma^{n-1}} \circ \ldots \circ T_{\sigma} \circ T_\omega
\]

and

\[
\mathcal{L}^n_\omega = \mathcal{L}_{\sigma^{n-1}} \circ \ldots \circ \mathcal{L}_{\sigma} \circ \mathcal{L}_\omega.
\]

The associated skew-product transformation \(\tau : \Omega \times I \rightarrow \Omega \times I\) is given by \(\tau(\omega, x) = (\sigma \omega, T_\omega x)\).
We assume that:

1. there exists $K > 0$ such that $\|L_\omega\| \leq K$ for $\mathbb{P}$-a.e. $\omega \in \Omega$;
2. there exists $N \in \mathbb{N}$ and measurable $\alpha^N, \beta^N : \Omega \to (0, \infty)$ with
\[
\int_\Omega \log \alpha^N(\omega) \, d\mathbb{P}(\omega) < 0
\]
such that for any $f \in BV$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,
\[
\|L_\omega^N f\|_{BV} \leq \alpha^N(\omega)\|f\|_{BV} + \beta^N(\omega)\|f\|_1;
\]
3. there exist $D, \lambda > 0$ such that $\|L_\omega^n f\|_{BV} \leq De^{-\lambda n}\|f\|_{BV}$ for $f \in BV$, $\int f \, dm = 0$, $n \in \mathbb{N}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$;
4. there exists $N \in \mathbb{N}$ such that for any $a > 0$ and sufficiently large $n \in \mathbb{N}$, there is $c > 0$ such that\[
\text{essinf } L_\omega^n f \geq c\|f\|_1,
\]
for $\mathbb{P}$-a.e. $\omega \in \Omega$, $f \in C_a := \{f \in BV : f \geq 0 \text{ and } \text{var}(f) \leq a\|f\|_1\}$.  

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Limit theorems
Then, there exists a unique acim (w.r.t. $\mathbb{P} \times m$) $\mu$ for $\tau$ such that $\pi_*\mu = \mathbb{P}$, where $\pi: \Omega \times I \to \Omega$ is a projection. We can regard $\mu$ as a collection of fiber measures $\mu_\omega$, $\omega \in \Omega$ on $I$:

$$\int_{\Omega \times I} \phi(\omega, x) \, d\mu = \int_{\Omega} \int_{I} \phi(\omega, x) \, d\mu_\omega(x) \, d\mathbb{P}(\omega).$$

We consider observables $\phi: \Omega \times I \to \mathbb{R}$ such that

$$\text{esssup}_{(\omega, x)} |\phi(\omega, x)| < \infty \quad \text{and} \quad \text{esssup}_\omega \text{var}(\phi(\omega, \cdot)) < \infty.$$

Moreover, we assume that

$$\int_{[0,1]} \phi(\omega, \cdot) \, d\mu_\omega = 0, \quad \omega \in \Omega.$$

We form Birkhoff sums

$$S_n(\omega, x) = \sum_{i=0}^{n-1} (\phi \circ \tau^i)(\omega, x) = \sum_{i=0}^{n-1} \phi(\sigma^i \omega, T^i_\omega x).$$
We are interested in the **quenched** type of limit theorems i.e. those that give an information about the asymptotic behaviour of Birkhoff sums w.r.t. to $\mu_\omega$ for "typical" $\omega$.

Previous work:

1. Kifer, 1998: quenched limit theorems but not with spectral method (main example: random subshifts of finite type);
2. Aimino-Nicol-Vaienti, 2014: spectral method but the base space is assumed to be a Bernoulli shift (piecewise expanding maps);
Related work on **sequential dynamics**: Bakhtin, Conze-Raugi, Conze-Le Borgne-Roger, Nandori-Szasz-Varju.
Assume that \((\Omega, \mathcal{F}, \mathbb{P}, \sigma)\) is an ergodic m.p.s. where \(\Omega\) is a Borel subset of a separable, complete metric space. Furthermore, let \(B\) be a Banach space and \(\mathcal{L} = \mathcal{L}_\omega\), \(\omega \in \Omega\) a family of bounded linear operators on \(B\) such that the map \(\omega \mapsto \mathcal{L}_\omega\) is Borel-measurable. Then, for a.e. \(\omega \in \Omega\), the following limits exist

\[
\Lambda(\mathcal{L}) := \lim_{n \to \infty} \frac{1}{n} \log \|\mathcal{L}^n_\omega\| \quad \text{and} \quad \kappa(\mathcal{L}) := \lim_{n \to \infty} \frac{1}{n} \log \text{ic}(\mathcal{L}^n_\omega),
\]

where \(\text{ic}(\mathcal{L}^n_\omega) = \inf\{ r > 0 : \mathcal{L}^n_\omega(B(0, 1)) \text{ can be covered with finitely many balls of radius } r \}\).
If $\kappa(L) < \Lambda(L)$, then there exists $1 \leq l \leq \infty$ and a sequence of Lyapunov exponents

$$\Lambda(L) = \lambda_1 > \lambda_2 > \ldots > \lambda_l > \kappa(L) \quad (\text{if } 1 \leq l < \infty)$$

or

$$\Lambda(L) = \lambda_1 > \lambda_2 > \ldots \quad \text{and} \quad \lim_{n \to \infty} \lambda_n = \kappa(L) \quad (\text{if } l = \infty);$$

and for $\mathbb{P}$-almost every $\omega \in \Omega$ there exists a unique splitting (called the Oseledets splitting) of $B$ into closed subspaces

$$B = V(\omega) \oplus \bigoplus_{j=1}^{l} Y_j(\omega),$$
depending measurably on $\omega$ and such that:

1. For each $1 \leq j \leq l$, $\dim Y_j(\omega) < \infty$, $Y_j$ is equivariant i.e. $L_\omega Y_j(\omega) = Y_j(\sigma\omega)$ and for every $y \in Y_j(\omega) \setminus \{0\}$,
   \[
   \lim_{n \to \infty} \frac{1}{n} \log \| L_\omega^n y \| = \lambda_j.
   \]

2. $V$ is equivariant i.e. $L_\omega V(\omega) \subseteq V(\sigma\omega)$ and for every $v \in V(\omega)$,
   \[
   \lim_{n \to \infty} \frac{1}{n} \log \| L_\omega^n v \| \leq \kappa(L).
   \]
In order to be able to apply MET for our cocycle of transfer operators, we will require that: $\Omega$ is a Borel subset of a separable, complete metric space and that

$$
\text{the map } \omega \mapsto T_\omega \text{ has a countable range}
$$

We have $\dim Y_1(\omega) = 1$ and $Y_1(\omega) = \text{span}\{v_\omega^0\}$, where $d\mu_\omega = v_\omega^0 dm$.

We also form a twisted cocycle. More precisely, for $\omega \in \Omega$ and $\theta \in \mathbb{C}$, we define

$$
\mathcal{L}_\omega^\theta(h) = \mathcal{L}_\omega(e^{\theta \phi(\omega, \cdot)} h), \quad h \in BV.
$$
Theorem

For $\theta \in \mathbb{R}$, we have

$$\lim_{n \to \infty} \frac{1}{n} \log \int_{[0,1]} e^{\theta S_n(\omega, \cdot)} \, d\mu_\omega = \Lambda(\theta),$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ where $\Lambda(\theta)$ is a top Lyapunov exponent of the cocycle $L^\theta_\omega$, $\omega \in \Omega$. 
Regularity of $\Lambda$

Key points ($d\mu_\omega = \nu_\omega^0 \, dm$):

1. we construct the top space as $\nu_\omega^0 + \mathcal{W}^\theta(\omega, \cdot)$ where $\mathcal{W}^\theta$ is a (unique) solution of $F(\theta, \mathcal{W}) = 0$, where

$$F(\theta, \mathcal{W}) = \frac{\mathcal{L}^\theta_{\sigma^{-1}\omega}(\nu_{\sigma^{-1}\omega}^0 + \mathcal{W}(\sigma^{-1}\omega, \cdot))}{\int (\mathcal{L}^\theta_{\sigma^{-1}\omega}(\nu_{\sigma^{-1}\omega}^0 + \mathcal{W}(\sigma^{-1}\omega, \cdot))) \, dm} - \mathcal{W}(\omega, \cdot) - \nu_\omega^0,$$

where $\mathcal{W} \in S$ and

$$S := \{ \mathcal{W} : \Omega \times I \to \mathbb{C} : \mathcal{W}(\omega, \cdot) \in BV, \text{esssup}_\omega \| \mathcal{W}(\omega, \cdot) \|_{BV} < \infty \}.$$

2. for $\theta$ close to 0, the top Oseledets space of the twisted cocycle $\mathcal{L}^\theta_\omega$ is one-dimensional;

3. $\Lambda(\theta) = \int \log|\int e^{\theta \phi(\omega, \cdot)}(\nu_{\omega}^0 + \mathcal{W}^\theta(\omega, \cdot)) \, dm| \, d\mathbb{P}(\omega)$. 

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Limit theorems
Also, $\Lambda'(0) = 0$ and $\Lambda''(0) = \Sigma^2$, where $\Sigma^2$ is a variance.

**Theorem (Large deviation principle)**

Assume that $\Sigma^2 > 0$. Then, there exists $\varepsilon_0 > 0$ and a function $c: (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ which is nonnegative, continuous, strictly convex, vanishing only at 0 and such that

$$
\lim_{n \to \infty} \frac{1}{n} \log \mu_\omega(S_n(\omega, \cdot) > n\varepsilon) = -c(\varepsilon), \quad \text{for } 0 < \varepsilon < \varepsilon_0 \text{ and a.e. } \omega.
$$

We can also obtain CLT.

**Theorem (Central limit theorem)**

If $\Sigma^2 > 0$, we have that

$$
\lim_{n \to \infty} \int g(S_n(\omega, \cdot)/\sqrt{n}) \, d\mu_\omega = \int g \, dN(0, \Sigma^2),
$$

for $g$ continuous and bounded and a.e. $\omega \in \Omega$. 
We need to show that
\[
\lim_{n \to \infty} \int e^{i t \frac{S_n(\omega, \cdot)}{\sqrt{n}}} d\mu_\omega = e^{-\frac{t^2 \Sigma^2}{2}}, \quad \text{for a.e. } \omega \in \Omega.
\]

This follows by proving that:

1. \[
\lim_{n \to \infty} \int e^{i t \frac{S_n(\omega, \cdot)}{\sqrt{n}}} d\mu_\omega = \lim_{n \to \infty} \prod_{j=0}^{n-1} \lambda_{\sqrt{n}^j \omega},
\]

where
\[
\lambda^\theta_\omega = \int L^{\theta-1}_\sigma (v^{0}_{\sigma^{-1} \omega} + \mathcal{W}^\theta (\sigma^{-1} \omega, \cdot)) \, dm =: H(\theta, \mathcal{W}^\theta)(\omega);
\]

2. by Taylor expansion of \( \theta \to H(\theta, \mathcal{W}^\theta) \) around 0:
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \lambda_{\sqrt{n}^j \omega} = -\frac{t^2 \Sigma^2}{2}.
\]
Theorem

Assume that \( \Lambda(it) < 0 \) for \( t \in \mathbb{R} \setminus \{0\} \). Then, for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \) and every bounded interval \( J \subset \mathbb{R} \), we have

\[
\lim_{n \to \infty} \sup_{s \in \mathbb{R}} \left| \sum \sqrt{n} \mu_\omega(s + S_n \phi(\omega, \cdot) \in J) - \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2n\Sigma^2}} \right| = 0.
\]

Theorem

The following is equivalent:

1. \( \Lambda(it) < 0 \) for \( t \in \mathbb{R} \setminus \{0\} \);

2. the equation \( e^{it\phi(\omega, \cdot)} L^*_\omega \psi_{\sigma_\omega} = \gamma_{it} \psi_\omega \), where \( \gamma_{it} \in S^1 \),

   \( \psi_\omega \in BV^* \) has measurable solutions only for \( t = 0 \) (when \( \gamma^0_\omega = 1 \) and \( \psi_\omega = m \)).
Further developments

Our results include **piecewise expanding maps in higher dimension**:

D. Dragičević, G. Froyland, C. González-Tokman and S. Vaienti: *A spectral approach for quenched limit theorems for random expanding dynamical systems*, Communications in Mathematical Physics, in press.

**Work in progress**: random composition of hyperbolic diffeomorphisms.